AN OBSERVER DESIGN AND SEPARATION PRINCIPLE FOR THE MOTION OF THE N-DIMENSIONAL RIGID BODY

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Abstract

In this note it is shown how the n-dimensional rigid body equation naturally leads to Hamilton's canonical equation and how this may be used for controller and observer designs by using the geometry of mechanical systems on manifolds avoiding the parameterization of Lie group $SO(n)$. Based on this approach, it is possible to focus on the intrinsic property of the system and to show closed-loop stability, a separation principle, which has been conjectured but not yet been shown.

1 Introduction

The problem of controlling the motion of rigid bodies and mechanical linkages has been studied extensively in control, aerospace and robotics literature and has applications ranging from pointing and slewing maneuvers of spacecraft to object manipulation. A large amount of research has been carried out on the rigid body's attitude control problem ([2], [3], [8]-[10], [18], [19]). It has been shown that passivity-based control, i.e. linear feedback of the position error and angular velocity with scalar gains, globally asymptotically stabilizes the origin of the closed-loop system (see [17], [19]). However, angular velocity is not always measured in practice. For instance, small satellites are not equipped with gyros, angular velocity sensors, in recent trends because gyros are generally expensive and are often prone to degradation or failure. For such cases, an angular velocity observer of a rigid body from orientation and torque measurements was proposed in [15], but the closed-loop stability was not proven. Alternatively, the passivity-based, angular velocity-free set-point controller has been proposed by [10], [18].

It is well-known that the attitude motion of a rigid body is represented by a set of two equations: (1) Euler's dynamic equation, which describes the time evolution of the angular velocity vector, and (2) the kinematic equation, which relates the time derivatives of the orientation angles and rotation group $SO(3)$ to the angular velocity vector. The important feature of the system is that its configuration space is $SO(3)$, which is not the Euclidean space but a manifold. Several parameterizations exist to represent the $SO(3)$, including three-parameter representations with singularity (e.g., Euler angles, Rodrigues parameters) and the four-parameter representation with an additional constraint without singularity (e.g., Euler parameters). Most research (for example, [18], [10], [15], [19]) commonly involves the choice of a preliminary parameterization of coordinates for the configuration manifold $SO(3)$. In [7], by contrast, a coordinate-free approach is proposed for a trajectory tracking problem via differential geometric techniques.

In this note we deal with the free rotation of n-dimensional rigid body about its center of mass on the Lie group $SO(n)$ in a coordinate-free framework by using the geometry of mechanical systems on manifolds. Avoiding the parameterization of the configuration space, it is possible to focus on the intrinsic property of the system. First, in section II, we give Hamilton's canonical equation of n-dimensional rigid bodies. Then, in section III, we consider a set-point control problem of driving an attitude to a steady-state target attitude, and an angular velocity observer is obtained as a generalization of the result in [15]. By taking errors of the plant and observer states as a ratio, the error dynamics also evolves on the same configuration space $SO(n)$. We remark that this is commonly observed in linear systems but not in nonlinear systems in general. Through this note, it is seen that the approach taken enables us to see the geometric structure of the observer. Finally, in section IV, we solve the remaining problem, whether or not the observer-based controller still stabilizes the origin of the closed-loop system (separation principle).

In section V, we develop the above discussion into the global stabilization.

2 Dynamics of the n-dimensional rigid body

In this section we introduce some notation and review some principal results on the kinematics and dynamics of the free rotation of a n-dimensional rigid body about a fixed point, then derive Hamilton's equation in canonical coordinates for that system. Almost all statements in this part are based on [1], [11], [14].

The problem under consideration is the free rotation of an n-dimensional rigid body about its center of mass, which we assume to be the origin in $\mathbb{R}^n$. "Free" means that there are no external forces, and "rigid" means that the distance between any two points of the body is unchanged during the motion. Consider two coordinate systems: the body coordinate system and the spatial coordinate system. Throughout this note, quantities expressed in the body coordinate system will be prescripted by B, while quantities expressed in the spatial coordinate system will be prescripted by S. Let $X_S(X_B, t) \in \mathbb{R}^n$ denote the position of the particle of the body in spatial coordinate at time t which was at $X_B \in \mathbb{R}^n$ at time zero ($X_S(X_B, 0) = X_B$); rigidity implies that $X_S(X_B, t) = q(t)X_B$, where $q(t) \in SO(n)$, the proper rotation group of $\mathbb{R}^n$, the $n \times n$ orthogonal matrices with determinant 1. $SO(n)$ is a Lie group, and that its Lie algebra is $\mathfrak{so}(n)$, the space of skew-symmetric $n \times n$ matrices with bracket $[\xi, \eta] = \xi\eta - \eta\xi$, $\xi, \eta \in \mathfrak{so}(n)$. The body and space velocity is

$$V_B(X_B, t) = -\frac{\partial X_B(X_S, t)}{\partial t} = q(t)^{-1}\dot{q}X_B(X_S, t)$$
$$V_S(X_S, t) = \frac{\partial X_S(X_B, t)}{\partial t} = \dot{q}(t)q(t)^{-1}X_S(X_B, t),$$

where we define $\omega_B(t) = T_e L_{q(t)^{-1}}(\dot{q}(t)),\ \ \omega_S(t) = T_I R_{q(t)^{-1}}(\dot{q}(t)) \in \mathfrak{so}(n)$, body and space angular velocity. Then ω_B, ω_S are left and right translations of $\dot{q}$ by $q^{-1} \in SO(n)$, and express $\dot{q}$ in body and space coordinates respectively (see Figure 1). Thus kinematic equation is

$$\frac{dq(t)}{dt} = q(t)\omega_B(t) = \omega_S(t)q(t). \tag{1}$$

Next we consider the dynamic equation. Assume that the mass distribution of the body is described by a compactly supported density measure $\rho_0(X_B)\, d^nX_B$. Thus, the kinetic energy of the motion is given by

$$K(X_B) = \frac{1}{2}\int_{\mathcal{B}} \rho_0(X_B)\|\omega_B(t)X_B\|^2\, d^nX_B.$$

For $\xi, \eta \in \mathfrak{so}(n)$, introducing the new inner product

$$\langle\langle \xi, \eta \rangle\rangle = \int \rho_0(X_B)\,(\xi X_B)^T(\eta X_B)\, d^nX_B,$$

the kinetic energy becomes

$$K(\omega_B) = \frac{1}{2}\langle\langle \omega_B, \omega_B \rangle\rangle.$$

Furthermore, introducing the following inner product on $\mathfrak{gl}(n, \mathbb{R})$

$$\langle A, B \rangle = \frac{1}{2}\mathrm{Trace}(A^TB), \quad A, B \in \mathfrak{gl}(n, \mathbb{R}),$$

and considering the moment of inertia

$$D = D^T = \int \rho_0(X_B)X_B{X_B}^T\, d^nX_B > 0,$$

D can be diagonalized by $q \in SO(n)$, namely, $D_0 = qDq^{-1}$. Therefore, there is a new orthonormal basis of $\mathbb{R}^n$, principal axis body coordinate, having the same orientation as the initial orientation that is determined by the mass distribution of the rigid body. In what follows we work in a principal axis body coordinate. A unique isomorphism $J_0 : \mathfrak{so}(n) \rightarrow \mathfrak{so}(n), s.t.\ \langle\langle \xi, \eta \rangle\rangle = \langle J_0(\xi), \eta \rangle$ is determined by

$$J_0(\xi) = D_0\xi + \xi D_0,\ D_0 = \mathrm{diag}(d_1, \cdots, d_n) > 0.$$

Thus, the kinetic energy of the rigid body motion becomes

$$K(\omega_B) = \frac{1}{2}\langle J_0(\omega_B), \omega_B \rangle. \tag{2}$$

Note that the Ad-invariant form $\langle\cdot,\cdot\rangle$ on $\mathfrak{so}(n)$ induces a left and right invariant Riemannian metric $\langle\cdot,\cdot\rangle$ on $G = SO(n)$. Then the metric defines a diffeomorphism $TG \rightarrow T^*G$ in a natural way;

$$(\cdot)^\flat : \nu_g \in T_gG \mapsto \nu_g^\flat = \langle \nu_g, \cdot \rangle \in T_g^*G.$$

Define $(\cdot)^\sharp := {(\cdot)^\flat}^{-1} : T^*G \rightarrow TG$. Using the equation (1), (2), Lagrangian becomes

$$L(q, \dot{q}) = \frac{1}{2}\langle J_0(q^T\dot{q}), q^T\dot{q} \rangle,$$

thus, the variable p canonically conjugate to q is given by the Legendre transformation

$$p = \frac{\partial L}{\partial \dot{q}} = (qJ_0(q^T\dot{q}))^\flat, \quad \text{or} \quad p^\sharp = qJ_0(q^T\dot{q}).$$

Therefore, the Hamiltonian is

$$H(q, p) = \frac{1}{2}\langle q^Tp^\sharp, J_0^{-1}(q^Tp^\sharp) \rangle. \tag{3}$$

We summarize:

Proposition 1 Hamilton's canonical equation for the free rotation of n-dimensional rigid body about its center of mass is

$$\Sigma_H : \begin{cases} H(q, p) = \frac{1}{2}\langle J_0^{-1}(q^Tp^\sharp), q^Tp^\sharp \rangle \\ \dot{q} = \left(\dfrac{\partial H}{\partial p^\sharp}\right)^\sharp = q\, J_0^{-1}(q^Tp^\sharp) \\ \dot{p^\sharp} = -\left(\dfrac{\partial H}{\partial q}\right)^\sharp = p^\sharp\, J_0^{-1}(q^Tp^\sharp). \end{cases} \tag{4}$$

Using left and right translation (see Figure 1), we get from (4) the rigid body equations for body and spatial coordinates

$$\Sigma_B : \begin{cases} K = \dfrac{1}{2}\langle \omega_B, J_0(\omega_B)\rangle \\ \dfrac{dq}{dt} = q\,\omega_B \\ \dfrac{dJ_0(\omega_B)}{dt} = [\,J_0(\omega_B),\, \omega_B\,], \end{cases} \tag{5}$$

$$\Sigma_S : \begin{cases} K = \dfrac{1}{2}\langle \omega_S, J_S(\omega_S)\rangle \\ \dfrac{dq}{dt} = \omega_S\, q \\ \dfrac{dJ_S(\omega_S)}{dt} = 0\,, \end{cases} \tag{6}$$

where $J_0(\omega_B) = T_I L_{q^{-1}} p^\sharp$, $J_S(\omega_S) = T_I R_{q^{-1}} p^\sharp \in \mathfrak{so}(n)$ denote the angular momentum in body and spatial coordinate, respectively, and $J_S(\xi_S) = \mathrm{Ad}_q(J_0(\xi)) = D_S\xi_S - {\xi_S}^T D_S \in \mathfrak{so}(n),\;\; D_S = qD_0q^T,\;\; J_S^{-1}(\xi_S) = \mathrm{Ad}_q(J_0^{-1}(\xi)) = E_S\xi_S - {\xi_S}^T E_S,\;\; E_S = qE_0q^T$, $\xi_S = \mathrm{Ad}_q\xi = q\xi q^T$, $E_0 > 0$.

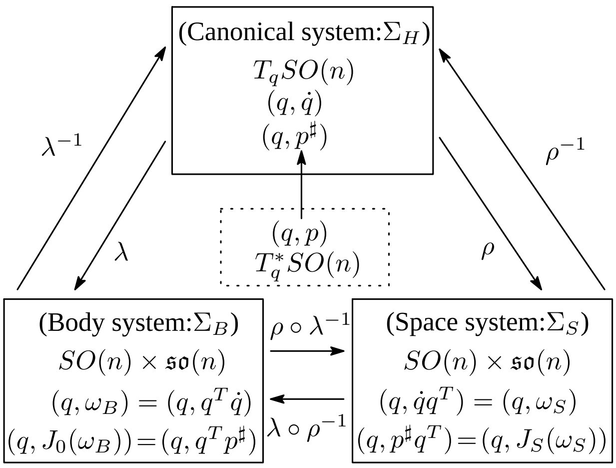

Figure 1: Three coordinate systems.

3 Controller and observer design

3.1 Stabilizing controller design

We consider the set-point control problem of driving the attitude (q, p) to a steady-state target attitude $(q_d, 0)$. The following theorem is well-known.

Theorem 1 ([10],[17]) For the rigid body control system

$$\Sigma_{HC} : \begin{cases} \dfrac{dq}{dt} = q\,J_0^{-1}(q^Tp^\sharp) \\ \dfrac{dp^\sharp}{dt} = p^\sharp\,J_0^{-1}(q^Tp^\sharp) + \tau_{HC} \end{cases} \tag{7}$$

the control law

$$\tau_{HC} = -k_v\dot q - k_p(q{q_d}^Tq - q_d), \tag{8}$$

with $k_p,\, k_v > 0$ asymptotically stabilizes the system.

Remark 1 Simple calculations show that the control law (8) can be rewritten in body and space coordinates, respectively. That is,

$$\Sigma_{BC} : \begin{cases} \dfrac{dq}{dt} = q\,\omega_B \\ \dfrac{dJ_0(\omega_B)}{dt} = [J_0(\omega_B), \omega_B] + \tau_{BC} \\ \tau_{BC} = -k_v\,\omega_B - k_p\,({q_d}^Tq - q^Tq_d) \end{cases} \tag{9}$$

in body coordinates. Also

$$\Sigma_{SC} : \begin{cases} \dfrac{dq}{dt} = \omega_S\, q \\ \dfrac{dJ_S(\omega_S)}{dt} = \tau_{SC} \\ \tau_{SC} = -k_v\,\omega_S - k_p\,(q{q_d}^T - q_dq^T) \end{cases} \tag{10}$$

in space coordinates.

Proof The closed-loop stability analysis uses the following Lyapunov function candidate

$$\begin{aligned} V_1 &= H(q,p) + k_pU(q) \\ &= \tfrac{1}{2}\langle q^Tp^\sharp,\, J_0^{-1}(q^Tp^\sharp)\rangle + k_p\langle I - q_d^Tq\,,\, I - q_d^Tq\rangle \end{aligned} \tag{11}$$

where the first term represents the kinetic energy and the second term represents the potential energy. We have

$$\frac{\partial U(q)}{\partial q}\cdot v = 2\langle q - q_d\,,\, v\rangle, \quad v \in T_qSO(n)$$

then, the derivative along the trajectories can be computed as

$$\begin{aligned} \dot V_1 &= \frac{\partial H}{\partial q}\cdot \dot q + \frac{\partial H}{\partial p^\sharp}\cdot \dot{p^\sharp} + k_p\frac{\partial U}{\partial q}\cdot \dot q \\ &= \frac{\partial H}{\partial p^\sharp}\cdot \tau_{HC} + k_p\frac{\partial U}{\partial q}\cdot \dot q \\ &= \langle \dot q\,,\, -k_v\dot q - k_p(q{q_d}^Tq - q_d)\rangle + 2k_p\langle q - q_d\,,\, \dot q\rangle \\ &= -k_v\langle \dot q\,,\, \dot q\rangle - k_p\langle q^T\dot q\,,\, {q_d}^Tq + q^Tq_d - 2I\rangle \\ &= -k_v\langle \dot q\,,\, \dot q\rangle \leqslant 0, \end{aligned}$$

since $\langle \xi, A\rangle = \frac{1}{2}\mathrm{Trace}(\xi^TA) = 0$ for all $A = A^T$, $\xi = -\xi^T \in \mathfrak{so}(n)$. Thus, LaSalle's Invariance Principle can be employed to complete the asymptotic stability proof. □

3.2 Observer design

We deal with the problem of obtaining the angular velocity ω (or angular momentum $J(\omega)$, conjugate momentum p) of an n-dimensional rigid body from orientation q and torque measurements τ only. This observer design generalizes that of [15] for n-dimensional rigid body in the Hamiltonian formulation.

Theorem 2 The n-dimensional rigid body observer for Hamiltonian control system (7) is

$$\Sigma_{HO}: \begin{cases} \dfrac{d\hat{q}}{dt} = \hat{q}\, J_0^{-1}(q^T \hat{p}^\sharp\, \hat{q}^T q) + u \\ \dfrac{d\hat{p}^\sharp}{dt} \hat{p}^\sharp\, J_0^{-1}(q^T \hat{p}^\sharp\, \hat{q}^T q) + v_H \end{cases} \tag{12}$$

where

$$\begin{aligned} u &= l_v \hat{q}\,(\hat{q}^T q - q^T \hat{q}) = l_v (q\hat{q}^T - \hat{q}q^T)\,\hat{q} \\ v_H &= \tau_H q^T \hat{q} + l_v \hat{p}^\sharp\,(\hat{q}^T q - q^T \hat{q}) \\ &\quad + l_p J_S^{-1}(q\hat{q}^T - \hat{q}q^T)\,\hat{q} \end{aligned}$$

and $l_p,\, l_v > 0$.

Remark 2 The observers for body and spatial coordinates system become

$$\Sigma_{BO}: \begin{cases} \dfrac{d\hat{q}}{dt} = \hat{q}\,\hat{\omega}_B + u \\ \dfrac{d(\hat{q}^T\hat{p}^\sharp)}{dt} = [\,\hat{q}^T\hat{p}^\sharp,\ \hat{\omega}_B] + v_B \\ v_B = \hat{q}^T q \tau_B q^T \hat{q} \\ \qquad + l_v[\,\hat{q}^T\hat{p}^\sharp,\ \hat{q}^T q - q^T\hat{q}\,] \\ \qquad + l_p \hat{q}^T q\, J_0^{-1}(\hat{q}^T q - q^T \hat{q})\, q^T \hat{q} \end{cases}$$

where $\hat{\omega}_B = J_0^{-1}(q^T \hat{p}^\sharp\, \hat{q}^T q)$, $\hat{q}^T\hat{p}^\sharp = \hat{q}^T q J_0(\hat{\omega}_B) q^T \hat{q}$, and

$$\Sigma_{SO}: \begin{cases} \dfrac{d\hat{q}}{dt} = (\hat{q}q^T \hat{\omega}_S q\hat{q}^T)\hat{q} + u \\ \dfrac{d(\hat{p}^\sharp\hat{q}^T)}{dt} = v_S \\ v_S = \tau_S + l_p J_S^{-1}(q\hat{q}^T - \hat{q}q^T) \end{cases}$$

where $\hat{\omega}_S = J_S^{-1}(\hat{p}^\sharp\hat{q}^T)$, $\hat{p}^\sharp\hat{q}^T = J_S(\hat{\omega}_S)$, respectively.

Proof The observer error evolution is governed by the following equations

$$\Sigma_{SE}: \begin{cases} \dfrac{d(q\hat{q}^T)}{dt} = J_S^{-1}(p^\sharp q^T - \hat{p}^\sharp \hat{q}^T) q\hat{q}^T \\ \qquad\qquad - l_v (q\hat{q}^T - \hat{q}q^T)\, q\hat{q}^T \\ \dfrac{d(p^\sharp q^T - \hat{p}^\sharp\hat{q}^T)}{dt} = -l_p J_S^{-1}(q\hat{q}^T - \hat{q}q^T). \end{cases} \tag{13}$$

Consider the Lyapunov function candidate V_2

$$\begin{aligned} V_2 = &\frac{1}{2}\big\langle p^\sharp q^T - \hat{p}^\sharp\hat{q}^T,\, p^\sharp q^T - \hat{p}^\sharp\hat{q}^T \big\rangle \\ &+\ l_p \langle I - q\hat{q}^T,\ I - q\hat{q}^T \rangle. \end{aligned} \tag{14}$$

Then, the time derivative of V_2 along the trajectories of the error system become

$$\begin{aligned} \dot{V}_2 &= \Big\langle p^\sharp q^T - \hat{p}^\sharp\hat{q}^T,\, \frac{d(p^\sharp q^T - \hat{p}^\sharp \hat{q}^T)}{dt} \Big\rangle \\ &\quad - 2l_p \Big\langle I - q\hat{q}^T,\, \frac{d(q\hat{q}^T)}{dt} \Big\rangle \\ &= -\big\langle p^\sharp q^T - \hat{p}^\sharp\hat{q}^T, l_p J_S^{-1}(q\hat{q}^T - \hat{q}q^T) \big\rangle \\ &\quad - 2l_p \big\langle \hat{q}q^T - I, J_S^{-1}(p^\sharp q^T - \hat{p}^\sharp\hat{q}^T) - l_v(q\hat{q}^T - \hat{q}q^T) \big\rangle \\ &= -l_p \big\langle J_S^{-1}(p^\sharp q^T - \hat{p}^\sharp\hat{q}^T), q\hat{q}^T + \hat{q}q^T - 2I \big\rangle \\ &\quad - 2 l_v l_p \big\langle I - \hat{q}q^T, q\hat{q}^T - \hat{q}q^T \big\rangle \\ &= -l_v l_p \| q\hat{q}^T - \hat{q}q^T \|^2 \leqslant 0 \end{aligned}$$

Because equation (12) is not autonomous, LaSalle's Invariance Principle cannot be applied. Instead, the asymptotic stability follows from Barbalat's lemma (see, e.g. [13]). □

Remark 3 If we write $x = q\hat{q}^T \in SO(n)$ and $\xi = J_S^{-1}(p^\sharp q^T - \hat{p}^\sharp\hat{q}^T) \in \mathfrak{so}(n)$, the observer error equations with $l_p = l_v = 0$ become

$$\begin{cases} \dfrac{dx}{dt} &= \xi\, x \\ \dfrac{d(J_S(\xi))}{dt} &= 0, \end{cases}$$

which corresponds to the rigid body equation in space coordinates. The stabilization of error dynamics is accomplished, first, adding the potential force $-l_p J_S^{-1}(x - x^T)$, and next, the dissipation $-l_v(x^2 - I)$. We note that the mechanism of stabilization of the error dynamics is quite similar to that of Theorem 1 and that it is possible to see this picture because we avoid parameterizations of $SO(n)$ using geometric mechanics.

4 Observer-based controller: separation principle

In this last section, it is shown that a separation principle-like property also holds for the nonlinear system considered in this note, that is, it is possible in the stabilizing control law (8), (9), (10) to replace $p^\sharp q^T$ by its estimation $\hat{p}^\sharp\hat{q}^T$.

Theorem 3 Consider the closed-loop system described by

$$\Sigma_{C+O}:\begin{cases}\dfrac{dq}{dt}=q\,J_0^{-1}(q^Tp^\sharp)\\ \dfrac{d\hat{q}}{dt}=\hat{q}\,J_0^{-1}(q^T\hat{p}^\sharp\,\hat{q}^Tq)+u\\ \dfrac{dp^\sharp}{dt}=p^\sharp\,J_0^{-1}(q^Tp^\sharp)+\tau'\\ \dfrac{d\hat{p}^\sharp}{dt}=\hat{p}^\sharp\,J_0^{-1}(q^T\hat{p}^\sharp\,\hat{q}^Tq)+v,\end{cases}\tag{15}$$

and that with the control law

$$\tau'=-k_v\,q\,J_0^{-1}(q^T\hat{p}^\sharp\,\hat{q}^Tq)-k_p\,(qq_d{}^Tq-q_d),\tag{16}$$

where $k_p,\,k_v,\,l_p,\,l_v\ >\ 0$. Then the equilibrium $(q,\hat{q},p^\sharp,\hat{p}^\sharp)=(q_d,q,0,p^\sharp)=(q_d,q_d,0,0)$ of the system (15) is asymptotically stable.

Proof First, let us prove that the estimated states exponentially converge to the real states. We augment the Lyapunov function (14) used in Section 3.2 as:

$$W_{2\varepsilon}=V_2-\frac{1}{4}\varepsilon\left\langle p^\sharp q^T-\hat{p}^\sharp\hat{q}^T\,,\,J_S^{-1}(q\hat{q}^T-\hat{q}q^T)\right\rangle.\tag{17}$$

Rewriting $\mu=p^\sharp q^T-\hat{p}^\sharp\hat{q}^T$, $\eta=q\hat{q}^T-\hat{q}q^T$, then the above becomes

$$\begin{aligned}W_{2\varepsilon}=&\frac{1}{8}\mathrm{Tr}\left\{\begin{bmatrix}\mu\\ I-q\hat{q}^T\end{bmatrix}^T\begin{bmatrix}I&\varepsilon E_S\\ \varepsilon E_S&2l_p\end{bmatrix}\begin{bmatrix}\mu\\ I-q\hat{q}^T\end{bmatrix}\right\}\\&+\frac{1}{8}\mathrm{Tr}\left\{\begin{bmatrix}\mu\\ I-\hat{q}q^T\end{bmatrix}^T\begin{bmatrix}I&-\varepsilon E_S\\ -\varepsilon E_S&2l_p\end{bmatrix}\begin{bmatrix}\mu\\ I-\hat{q}q^T\end{bmatrix}\right\}.\end{aligned}$$

In addition, by using Schur complement, we get

$$0<\varepsilon<\sqrt{\frac{l_p}{2\lambda_{\max}(E_0{}^2)}}\implies\frac{1}{2}V_2\leqslant W_{2\varepsilon}\leqslant\frac{3}{2}V_2,\tag{18}$$

The time derivative of $W_{2\varepsilon}$ along the trajectories of the closed-loop system is

$$\begin{aligned}\dot{W}_{2\varepsilon}=&\dot{V}_2-\frac{1}{4}\varepsilon\Big\{\big\langle\dot{\mu},J_S^{-1}(\eta)\big\rangle+\big\langle\mu,J_S^{-1}(\dot{\eta})\big\rangle\\&+\big\langle\mu,\dot{J}_S^{-1}(\eta)\big\rangle\Big\}\\=&-l_vl_p\langle\eta,\eta\rangle-\frac{1}{4}\varepsilon\Big\{-l_p\langle J_S^{-1}(\eta),J_S^{-1}(\eta)\rangle\\&+2\langle J_S^{-1}(\mu),(J_S^{-1}(\mu)-l_v\eta)q\hat{q}^T\rangle+\langle\mu,\dot{J}_S^{-1}(\eta)\rangle\Big\},\end{aligned}$$

where $\dot{J}_S^{-1}(\eta)=\dot{E}_S\eta+\eta\dot{E}_S$, $\dot{E}_S=\dot{q}E_0q^T+qE_0\dot{q}^T=J_S^{-1}(q^Tp^\sharp)E_S-E_SJ_S^{-1}(q^Tp^\sharp)=\dot{E}_S^T$. Moreover,

$$\begin{cases}2\lambda_{\min}(E_0)\langle\xi,\eta\rangle\leqslant\langle J_S^{-1}(\xi),\eta\rangle\leqslant2\lambda_{\max}(E_0)\langle\xi,\eta\rangle\\ \|I-q\|^2\leqslant3\Longrightarrow\|I-q\|\leqslant\|q-q^T\|\\ \|I-q\|^2\leqslant2(1-\cos\theta)\leqslant2\Longrightarrow\cos\theta\|\xi\|^2\leqslant\langle\xi,\xi q\rangle\end{cases}$$

we consider the neighborhood of equilibrium

$$\|I-q\hat{q}^T\|^2\leqslant2(1-\cos\theta)<2\tag{19}$$

$$l_v-\varepsilon\lambda_{\max}(E_0)^2>0.\tag{20}$$

Then, the above becomes

$$\begin{aligned}\dot{W}_{2\varepsilon}\leqslant&-l_p\big(l_v-\varepsilon\lambda_{\max}(E_0)^2\big)\|\eta\|^2-\frac{1}{2}\varepsilon\langle J_S^{-1}(\mu),J_S^{-1}(\mu)q\hat{q}^T\rangle\\&+\frac{1}{2}\varepsilon l_v\|J_S^{-1}(\mu)\|\|\eta\|-\frac{\varepsilon}{4}\frac{\lambda_{\min}(\dot{E}_S)}{\lambda_{\max}(E_0)}\langle J_S^{-1}(\mu),\eta\rangle\\\leqslant&-l_p\big(l_v-\varepsilon\lambda_{\max}(E_0)^2\big)\|\eta\|^2-\frac{1}{2}\varepsilon\cos\theta\|J_S^{-1}(\mu)\|^2\\&+\frac{1}{2}\varepsilon l_v\|J_S^{-1}(\mu)\|\|\eta\|+\frac{\varepsilon}{4}\frac{\lambda_{\min}(\dot{E}_S)}{\lambda_{\max}(E_0)}\|J_S^{-1}(\mu)\|\|\eta\|\\\leqslant&-\lambda_{\min}(P)\Big\{\|J_S^{-1}(\mu)\|^2+\|I-q\hat{q}^T\|^2\Big\}\\\leqslant&-\lambda_{\min}(P)\min\{8\lambda_{\min}(E_0)^2,1/l_p\}V_2\\\leqslant&-\frac{2}{3}\lambda_{\min}(P)\min\{8\lambda_{\min}(E_0)^2,1/l_p\}W_{2\varepsilon}\leqslant0\end{aligned}$$

where

$$P=\begin{bmatrix}\frac{1}{2}\varepsilon\cos\theta&-\frac{1}{4}\varepsilon\left(l_v+\frac{\lambda_{\min}(\dot{E}_S)}{2\lambda_{\max}(E_0)}\right)\\-\frac{1}{4}\varepsilon\left(l_v+\frac{\lambda_{\min}(\dot{E}_S)}{2\lambda_{\max}(E_0)}\right)&l_p\big(l_v-\varepsilon\lambda_{\max}(E_0)^2\big)\end{bmatrix},$$

$$l_pl_v>\varepsilon\left\{l_p\lambda_{\max}(E_0)^2+\frac{1}{8\cos\theta}\left(l_v+\frac{\lambda_{\min}(\dot{E}_S)}{2\lambda_{\max}(E_0)}\right)^2\right\}\tag{21}$$

for $P>0$. We summarize that if we choose ε small enough that conditions (18), (21) are satisfied, then the observer error converges to zero exponentially.

Finally, to complete the proof, choose

$$V_2(0)=\frac{1}{2}\|p^\sharp q^T-\hat{p}^\sharp\hat{q}^T\|^2+l_p\|I-q\hat{q}^T\|^2\ \Big|_{t=0}<2l_p,\tag{22}$$

then the observer error is exponentially stable. Consider the following Lyapunov function candidate

$$V_3=\frac{2\lambda_{\min}(P_o)}{k_v}V_1+W_{2\varepsilon},$$

and evaluate its derivative for Σ_{C+O}:

$$\begin{aligned}\dot{V}_3\leqslant&-\lambda_{\min}(P_o)\Big\{2\langle\,J_S^{-1}(p^\sharp q^T)\,,\,J_S^{-1}(\hat{p}^\sharp\hat{q}^T)\,\rangle\\&+\|J_S^{-1}(p^\sharp q^T-\hat{p}^\sharp\hat{q}^T)\|^2+\|I-q\hat{q}^T\|^2\Big\}\\=&-\lambda_{\min}(P_o)\Big\{\|J_S^{-1}(p^\sharp q^T)\|^2+\|J_S^{-1}(\hat{p}^\sharp\hat{q}^T)\|^2\\&+\|I-q\hat{q}^T\|^2\Big\}\leqslant0.\end{aligned}$$

Then, by LaSalle's Invariance Principle, it follows that the equilibrium of the closed-loop system Σ_{C+O} is asymptotically stable. □

5 Global stability

According to Milnor's theorem [12],[16], smooth vector fields on $TSO(n)$ cannot be globally asymptotically stable, and the argument thus far is local. However, if $n = 3$, with the analogy of [15], it can be shown that a slight modification of the last terms of τ_{HC}, v_H in (8), (12):

$$\begin{cases} -k_p q \dfrac{{q_d}^T q - q^T q_d}{\sqrt{1 + \text{Trace}(q {q_d}^T)}} & (\text{Trace}({q_d}^T q) \neq -1) \\ -k_p q n_1^\times & (\text{Trace}({q_d}^T q) = -1) \end{cases}$$

$$\begin{cases} \dfrac{l_p J_S^{-1}(q\hat{q}^T - \hat{q} q^T)\hat{q}}{\sqrt{1 + \text{Trace}(q\hat{q}^T)}} & (\text{Trace}(q\hat{q}^T) \neq -1) \\ l_p J_S^{-1}(n_2^\times)\hat{q} & (\text{Trace}(q\hat{q}^T) = -1) \end{cases}$$

achieves globally asymptotically stability in Theorem 1, 2 and 3, where $x^\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \in \mathfrak{so}(3)$ and n_1, n_2 are the normalized eigenvector with eigenvalue 1 of ${q_d}^T q$, $q\hat{q}^T$, respectively.

6 Conclusions

This note was devoted to the attitude control problem and design of an angular velocity observer for the motion of n-dimensional rigid body in the Hamiltonian formulation. Avoiding parameterizations of $SO(n)$, it was possible to reveal the geometric structure of the stabilizing controller and the angular velocity observer, and to demonstrate that the observer-based controller, the controller (8) with the observer (12), still stabilized the origin of the closed-loop system (separation principle).

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