FAULT DETECTION IN UNCERTAIN LINEAR CONTINUOUS-TIME PERIODIC SYSTEMS

P. Zhang*, **S.X. Ding***, **G.Z. Wang**†, **D.H. Zhou**†, **T. Jeinsch**‡, **R. Noack**♯
*Inst. for Automatic Control and Complex Systems, University of Duisburg-Essen, Germany
Fax: (0049) 203 3792928 E-mail: pzhang@uni-duisburg.de
†Dept. of Automation, Tsinghua University, Beijing, P.R. China
‡AE-MOW, IAV GmbH, Germany
♯Faculty IEM, Fachhochschule Lausitz, Germany

Keywords: Fault detection; periodic systems; residual generation; uncertainty; robustness.

Abstract

This paper studies fault detection problem of linear continuous-time periodic systems. At first, an optimal periodic observer-based residual generator is derived for the case of no model uncertainty, which achieves the best compromise between the sensitivity to faults and the robustness to unknown disturbances. Then, in order to deal with model uncertainty, a post-filter is introduced which provides more design freedom and the residual generator is designed to approximate the best compromise character. Periodic Riccati differential equation and periodic differential LMI play an important role in the computation of the optimal solution.

1 Introduction

Model based fault detection (FD) technology has attracted much attention in the last years [5, 6]. It is well recognized that, in order to ensure a successful detection of faults, both the *robustness* of the FD system to unknown disturbances and its *sensitivity* to faults should be taken into consideration. In the context of linear time-invariant systems, a number of approaches have been proposed for the optimal design of FD systems [4, 5, 6]. In this paper, we are concerned with the FD problem of linear continuous-time periodic systems, which exist widely in different areas [1, 2]. As far as the authors know, this problem has not yet been handled in the literature.

Model uncertainty presents one of the main challenges in dealing with real systems. The development of periodic Bounded Real Lemma and periodic differential linear matrix inequality (LMI) technique stimulates a renewed high interest in the control and filtering problem of periodic systems [1]. Making use of this knowledge, we treat both polytopic uncertainty and norm-bounded uncertainty after deriving the optimal FD system for periodic systems without model uncertainty.

2 Preliminary

This section presents some preliminary results on linear continuous-time periodic systems, which play an important role in the subsequent analysis.

Given a periodic system described by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \qquad (1)$$

where $x \in \mathbf{R}^n, u \in \mathbf{R}^p, y \in \mathbf{R}^m$ denote the vectors of states, inputs and outputs, respectively, $A(t), B(t), C(t)$, $D(t)$ are real bounded continuous periodic matrices with period T, i.e. $\forall t$, $A(t+T) = A(t)$, etc. Under the assumption that the initial condition of the system is $x(0) = 0$, if system (1) is stable, i.e. all the characteristic multipliers of $A(t)$ belong to the open unit disk [2], then associated with it there is a linear bounded operator Π mapping $u(t) \in L_2(\mathbf{R}, \mathbf{R}^p)$ to $y(t) \in L_2(\mathbf{R}, \mathbf{R}^m)$. For the sake of convenience, we write $\Pi = (A(t), B(t), C(t), D(t))$. It is well-known that the adjoint of operator Π is $\Pi^* = (-A'(t), -C'(t), B'(t), D'(t))$. The operator associated with periodic systems with only direct through terms like $y(t) = D(t)u(t)$ is denoted simply by $\Pi = (D(t))$. If the operator Π associated to a stable periodic system satisfies $\Pi\Pi^* = I$, then Π is called a *co-inner*. If both the operator Π and its left inverse are associated to stable periodic systems, then Π is called a *co-outer*.

Lemma 1 Given an operator $\Pi = (A(t), B(t), C(t), D(t))$ associated to a stable periodic system. It is co-inner, if there exists a T-periodic symmetric positive definite matrix $Q(t)$ (i.e. $\forall t$, $Q(t) = Q'(t) > 0$, $Q(t) = Q(t+T)$), such that $\forall t$,

$$\begin{aligned} D(t)B'(t) + C(t)Q(t) &= 0 \\ D(t)D'(t) &= I \\ A(t)Q(t) + Q(t)A'(t) + B(t)B'(t) &= \dot{Q}(t) \end{aligned}$$

Lemma 2 Given an operator $\Pi = (A(t), B(t), C(t), D(t))$ associated to a stable periodic system, then Π can be factorized as $\Pi = \Pi_o\Pi_i$, where $\Pi_o = (A(t), -\bar{L}(t)\bar{W}^{-1}(t), -C(t), \bar{W}^{-1}(t))$ is the co-outer and $\Pi_i = (A(t) - \bar{L}(t)C(t), B(t) - \bar{L}(t)D(t), \bar{W}(t)C(t), \bar{W}(t)D(t))$ is the co-inner with $\bar{L}(t), \bar{W}(t)$ given by

$$\begin{aligned} \bar{L}(t) &= (Y(t)C'(t) + B(t)D'(t))(D(t)D'(t))^{-1} \\ \bar{W}(t) &= (D(t)D'(t))^{-1/2} \end{aligned}$$

and $Y(t) > 0$ is the stabilizing T-periodic solution to periodic Riccati differential equation (PRDE)

$$\dot{Y}(t) = A(t)Y(t) + Y(t)A'(t) + B(t)B'(t) - \bar{L}(t)D(t)D'(t)\bar{L}'(t)$$

Lemma 1 and 2 are proved by considering the cascade connection of Π and Π^*, as well as that of Π_o and Π_i, while Lemma 1 builds the basis of Lemma 2. Due to limitation of space, proofs are omitted here.

3 Problem formulation

The systems considered in this paper are described by

$$\begin{aligned}\dot{x}(t) &= (A(t) + \Delta A(t))x(t) + (B(t) + \Delta B(t))u(t)\\ &\quad +(E_d(t) + \Delta E_d(t))d(t) + (E_f(t) + \Delta E_f(t))f(t)\\ y(t) &= C(t)x(t) + D(t)u(t) + F_d(t)d(t) + F_f(t)f(t)\end{aligned} \quad (2)$$

where $x \in \mathbf{R}^n, u \in \mathbf{R}^{k_u}, y \in \mathbf{R}^m, d \in \mathbf{R}^{k_d}, f \in \mathbf{R}^{k_f}$ denote the vector of states, control inputs, measured outputs, unknown disturbances and faults, respectively, $A(t)$, $B(t), C(t), D(t), E_d(t), E_f(t), F_d(t), F_f(t)$ are known real bounded continuous T-periodic matrices of appropriate dimensions, $\Delta A(t), \Delta B(t), \Delta E_d(t), \Delta E_f(t)$ are unknown matrices representing parameter uncertainties. It is assumed that $(A(t), C(t))$ is observable at each time t and system (2) remains stable for all possible ΔA. Let $\Delta := \begin{bmatrix} \Delta A(t) & \Delta B(t) & \Delta E_d(t) & \Delta E_f(t) \end{bmatrix}$. The following three cases will be considered.

Case I *no model uncertainty*

$$\Delta = O \quad (3)$$

Case II *polytopic uncertainty*

$$\Delta = \sum_{i=1}^{v} \lambda_i(t) \begin{bmatrix} A_i(t) & B_i(t) & E_{di}(t) & E_{fi}(t) \end{bmatrix} \quad (4)$$

where $A_i(t), B_i(t), E_{di}(t), E_{fi}(t)$ are known continuous T-periodic matrices, $\lambda_1(t), \cdots, \lambda_v(t)$ are unknown periodic quantities satisfying $\lambda_i \geqslant 0,\ \sum_{i=1}^{v} \lambda_i = 1$.

Case III *norm-bounded uncertainty*

$$\Delta = M(t)\Delta_o(t) \begin{bmatrix} N_a(t) & N_b(t) & N_d(t) & N_f(t) \end{bmatrix} \quad (5)$$

where $M(t), N_a(t), N_b(t), N_d(t)$ and $N_f(t)$ are known continuous T-periodic matrices, $\Delta_o(t)$ is unknown but bounded by $\Delta'_o(t)\Delta_o(t) \leq I, \forall t$.

In this paper, we concentrate on the problem of residual generation. Our aim is to design an optimal residual generator for system (2), which is sensitive to the faults and robust to the unknown disturbances as well as the model uncertainty.

4 Optimal design of residual generator

In this section, at first we treat case I and develop an approach to design the optimal residual generator. Then based on it, we consider the cases with model uncertainty.

4.1 Case I: $\Delta = O$

A linear continuous-time periodic residual generator can be constructed as

$$\begin{aligned}\dot{\hat{x}}(t) &= A(t)\hat{x}(t) + B(t)u(t) + L(t)r_b(t)\\ \hat{y}(t) &= C(t)\hat{x}(t) + D(t)u(t),\ r_b(t) = y(t) - \hat{y}(t)\\ r(t) &= W(t)r_b(t)\end{aligned} \quad (6)$$

where $r_b \in \mathbf{R}^m$ denotes the output estimation error, $r \in \mathbf{R}^{k_r}$ the so-called residual signal, $L(t)$ the observer gain matrix, $W(t)$ the weighting matrix, $L(t)$ and $W(t)$ are T-periodic matrices to be determined.

Let $e(t) := x(t) - \hat{x}(t)$. It can be easily derived that the dynamics of residual generator (6) is governed by

$$\begin{aligned}\dot{e}(t) &= (A(t) - L(t)C(t))e(t)\\ &\quad + (E_d(t) - L(t)F_d(t))d(t) + (E_f(t) - L(t)F_f(t))f(t)\\ r(t) &= W(t)(C(t)e(t) + F_d(t)d(t) + F_f(t)f(t))\end{aligned} \quad (7)$$

To ensure the stability of the error dynamics, $L(t)$ should be chosen to make the characteristic multipliers of $A(t) - L(t)C(t)$ lie inside the open unit circle. Since in case of $\Delta = O$, the control inputs will not influence the dynamics of the residual generator, the problem now is how to choose $L(t)$ and $W(t)$ to enhance the robustness of the FD system to the disturbances without loss of the sensitivity to the faults.

Use operators $\Pi_{rd} = (A(t) - L(t)C(t), E_d(t) - L(t)F_d(t), W(t)C(t), W(t)F_d(t))$ and $\Pi_{rf} = (A(t) - L(t)C(t), E_f(t) - L(t)F_f(t), W(t)C(t), W(t)F_f(t))$ to denote the mapping from $d(t)$ and $f(t)$ to the residual $r(t)$, respectively. According to the theory of functional analysis, the induced norms of operators Π_{rd}, Π_{rf} defined by

$$\|\Pi_{rd}\| = \sup_{d \in \mathbf{L}_2 - \{0\}} \frac{\|r\|_2}{\|d\|_2}, \quad \|\Pi_{rf}\| = \sup_{f \in \mathbf{L}_2 - \{0\}} \frac{\|r\|_2}{\|f\|_2}$$

represent, respectively, the maximal possible influence of the unknown disturbances and the faults of unit energy on the residual. Let

$$J = \frac{\|\Pi_{rd}\|}{\|\Pi_{rf}\|}$$

Minimizing J means to reduce $\|\Pi_{rd}\|$ and increase $\|\Pi_{rf}\|$. Therefore, J can be taken as an index for evaluating the performance of the residual generator. As a result, the optimal design of the residual generator can be formulated as the following optimization problem

$$\min_{L(t), W(t)} J = \min_{L(t), W(t)} \frac{\|\Pi_{rd}\|}{\|\Pi_{rf}\|} \quad (8)$$

whose optimal solution is presented in Theorem 1.

Theorem 1 Given system (7) and assume that $(A(t), C(t))$ is observable at each time t, then $L_n(t)$ and $W_n(t)$ given by

$$\begin{aligned}L_n(t) &= (Y(t)C'(t) + E_d(t)F'_d(t))(F_d(t)F'_d(t))^{-1}\\ W_n(t) &= (F_d(t)F'_d(t))^{-1/2}\end{aligned} \quad (9)$$

solve optimization problem (8), where the T-periodic matrix $Y(t) > 0$ is the stabilizing solution to PRDE

$$\dot{Y}(t) = A(t)Y(t) + Y(t)A'(t) + E_d(t)E_d'(t) - L_n(t)Q(t)L_n'(t) \quad (10)$$

Proof: It can be easily shown that Π_{rd} and Π_{rf} are indeed the following cascade connections

$$\Pi_{rd} = \Pi_W \Pi_L \Pi_{yd},\ \Pi_{rf} = \Pi_W \Pi_L \Pi_{yf}$$

where $\Pi_W = (W(t))$, $\Pi_L = (A(t) - L(t)C(t), L(t), -C(t), I)$, $\Pi_{yd} = (A(t), E_d(t), C(t), F_d(t))$, $\Pi_{yf} = (A(t), E_f(t), C(t), F_f(t))$. Optimization problem (8) can be rewritten as

$$\min_{L(t),W(t)} J = \min_{L(t),W(t)} \frac{\|\Pi_W \Pi_L \Pi_{yd}\|}{\|\Pi_W \Pi_L \Pi_{yf}\|} \quad (11)$$

Before solving (11), we consider at first

$$\min_{\Pi_R} J_o = \min_{\Pi_R} \frac{\|\Pi_R \Pi_L \Pi_{yd}\|}{\|\Pi_R \Pi_L \Pi_{yf}\|} \quad (12)$$

Note that the difference between (12) and (11) lies in that $\Pi_R = (A_R(t), B_R(t), C_R(t), D_R(t))$, which is associated to an arbitrarily selectable stable periodic post-filter of order k_R, appears in place of $\Pi_W = (W(t))$. Apparently Π_W constitutes only a subclass of Π_R. Thus, for any given stabilizing $L(t)$,

$$\min_{\Pi_R} J_o \le \min_{\Pi_W} J \quad (13)$$

Since $\Pi_L \Pi_{yd} = (A(t) - L(t)C(t), E_d(t) - L(t)F_d(t), C(t), F_d(t))$ is stable, according to Lemma 2, $\Pi_L \Pi_{yd}$ has the co-inner-outer factorization

$$\Pi_L \Pi_{yd} = \Pi_{do} \Pi_{di}$$

where Π_{do} is the co-outer and Π_{di} is the co-inner. Because $\Pi_R \Pi_L \Pi_{yd}$ is a bounded Hilbert space operator, we have

$$\|\Pi_R \Pi_L \Pi_{yd}\| = \|\Pi_R \Pi_{do} \Pi_{di}\| = \|\Pi_R \Pi_{do}\|$$

Setting $\Pi_R = \bar{\Pi}_R \Pi_{do}^{-1}$, we get

$$J_o = \frac{\|\Pi_R \Pi_{do}\|}{\|\Pi_R \Pi_L \Pi_{yf}\|} = \frac{\|\bar{\Pi}_R\|}{\|\bar{\Pi}_R \Pi_{do}^{-1} \Pi_L \Pi_{yf}\|}$$

Because

$$\|\bar{\Pi}_R \Pi_{do}^{-1} \Pi_L \Pi_{yf}\| \le \|\bar{\Pi}_R\| \, \|\Pi_{do}^{-1} \Pi_L \Pi_{yf}\|$$

there is

$$J_o \ge \frac{1}{\|\Pi_{do}^{-1} \Pi_L \Pi_{yf}\|}$$

The equality is guaranteed if $\bar{\Pi}_R = (I)$. Thus $\Pi_{R,opt} = \Pi_{do}^{-1}$ solves optimization problem (12). Applying Lemma 2 and introducing relations (9)-(10), it yields

$$\Pi_{R,opt} = \Pi_{do}^{-1} = (A(t) - L_n(t)C(t), L(t) - L_n(t), W_n(t)C(t), W_n(t)) \quad (14)$$

Note that if $L(t)$ is chosen to be equal to $L_n(t)$, then $\Pi_{R,opt}$ reduces to $(W_n(t))$. Since for *any* stabilizing $L(t)$, there is

$$\Pi_{L_n} = \Pi_{temp} \Pi_L$$

where both $\Pi_{temp} = (A(t) - L_n(t)C(t), L(t) - L_n(t), C(t), I)$ and Π_{temp}^{-1} are associated to stable periodic systems, so

$$\left. \frac{\|\Pi_R \Pi_{L_n} \Pi_{yd}\|}{\|\Pi_R \Pi_{L_n} \Pi_{yf}\|} \right|_{\Pi_R = (W_n(t))} = \min_{\Pi_R} \frac{\|\Pi_R \Pi_{L_n} \Pi_{yd}\|}{\|\Pi_R \Pi_{L_n} \Pi_{yf}\|}$$

$$= \min_{\Pi_R} \frac{\|\Pi_R \Pi_{temp} \Pi_L \Pi_{yd}\|}{\|\Pi_R \Pi_{temp} \Pi_L \Pi_{yf}\|} = \min_{\hat{\Pi}_R} \frac{\|\hat{\Pi}_R \Pi_L \Pi_{yd}\|}{\|\hat{\Pi}_R \Pi_L \Pi_{yf}\|}$$

$$\le \min_{W(t)} \frac{\|\Pi_W \Pi_L \Pi_{yd}\|}{\|\Pi_W \Pi_L \Pi_{yf}\|}$$

where $\hat{\Pi}_R := \Pi_R \Pi_{temp}$ and the last inequality comes from (13). Thus

$$\frac{\|\Pi_W \Pi_L \Pi_{yd}\|}{\|\Pi_W \Pi_L \Pi_{yf}\|} \,|_{L(t)=L_n(t), \Pi_W = W_n(t)} \le \min_{L(t),W(t)} \frac{\|\Pi_W \Pi_L \Pi_{yd}\|}{\|\Pi_W \Pi_L \Pi_{yf}\|}$$

which shows that $L_n(t), W_n(t)$ defined by (9)-(10) is the optimal solution to optimization problem (8).

Remark 1 From the proof we see that the residual generator can indeed also be constructed as an observer with (any) stable error dynamics followed by a periodic post filter (14).

In summary, in case of $\Delta = 0$, the optimal residual generator for periodic systems can be carried out according to the following steps, either

- solve PRDE (10) for $Y(t)$;
- calculate $L_n(t), W_n(t)$ according to (9);
- construct the optimal residual generator of the form (6),

or

- choose a (any) periodic observer gain matrix $L(t)$ which makes the characteristic multipliers of $A(t) - L(t)C(t)$ lie inside the open unit disk;
- solve PRDE (10) for $Y(t)$;
- determine $L_n(t), W_n(t)$ according to (9);
- compute the optimal post-filter by (14);
- construct the optimal residual generator of the form

$$\begin{aligned} \dot{\hat{x}}(t) &= A(t)\hat{x}(t) + B(t)u(t) + L(t)r_b(t) \\ \hat{y}(t) &= C(t)\hat{x}(t) + D(t)u(t),\ r_b(t) = y(t) - \hat{y}(t) \\ \dot{x}_R(t) &= A_R(t)x_R(t) + B_R(t)r_b(t) \\ r(t) &= C_R(t)x_R(t) + D_R(t)r_b(t) \end{aligned} \quad (15)$$

It is worth noticing that in the optimal case (6) and (15) deliver the same optimal dynamics

$$\begin{aligned} \dot{e}_n(t) &= (A(t) - L_n(t)C(t))e_n(t) \\ &+ (E_d(t) - L_n(t)F_d(t))d(t) + (E_f(t) - L_n(t)F_f(t))f(t) \\ r_n(t) &= W_n(t)(C(t)e_n(t) + F_d(t)d(t) + F_f(t)f(t)) \end{aligned} \quad (16)$$

while due to the introduction of the post filter, residual generator (15) provides more design freedom.

4.2 Case II and Case III: $\Delta \neq O$

Note that the optimal residual (16) represents the best compromise between the sensitivity to the faults and the robustness to the disturbances in the case of no model uncertainty ($\Delta = O$). We hope that in the presence of model uncertainty ($\Delta \neq O$) the residual dynamics can approximate (16) to preserve the best compromise character.

To get more design freedom, we choose residual generator of the form (15), where $L(t)$, $\Theta_R(t) := \begin{bmatrix} A_R(t) & B_R(t) \\ C_R(t) & D_R(t) \end{bmatrix}$ are design parameters. Since $\Delta \neq O$, now the dynamics of residual generator (15) is also related to control inputs $u(t)$ and governed by

$$\begin{aligned}
\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} &= \begin{bmatrix} A(t)+\Delta A(t) & O \\ \Delta A(t) & A(t)-L(t)C(t) \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \\
&+ \begin{bmatrix} E_f(t)+\Delta E_f(t) & E_d(t)+\Delta E_d(t) & B(t)+\Delta B(t) \\ E_f(t)-L(t)F_f(t)+\Delta E_f(t) & E_d(t)-L(t)F_d(t)+\Delta E_d(t) & \Delta B(t) \end{bmatrix} \delta \\
r_b &= \begin{bmatrix} O & C(t) \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} F_f(t) & F_d(t) & O \end{bmatrix} \delta \\
\dot{x}_R &= A_R(t)x_R + B_R(t)r_b \\
r &= C_R(t)x_R + D_R(t)r_b
\end{aligned} \tag{17}$$

where the vector $\delta = \begin{bmatrix} f' & d' & u' \end{bmatrix}'$ collects the signals that have influence on the residual. The dynamics of residual generator (15) is stable, as long as system (2) is stable and $L(t)$ is so selected that the characteristic multipliers of $A(t) - L(t)C(t)$ belong to the open unit disk. Due to the presence of the post-filter, we just select $L(t)$ according to the stability condition. Define $\xi(t) = r(t) - r_n(t)$ to represent the difference between the residual $r(t)$ in case of model uncertainty and the optimal residual $r_n(t)$ in (16).

Our problem is thus formulated as to determine the parameters of the periodic post filter $\Theta_R(t)$, such that $\xi(t) \to 0$.

The whole system can be arranged into the form of a lower linear fractional connection of the following extended plant with the post filter

$$\begin{aligned}
\dot{X}_p(t) &= A_p(t)X_p(t) + B_p(t)\delta(t) \\
\xi(t) &= C_{1p}(t)X_p(t) + D_{1p}(t)\delta(t) + r(t) \\
r_b(t) &= C_{2p}(t)X_p(t) + D_{2p}(t)\delta(t)
\end{aligned} \tag{18}$$

where the extended state vector $X_p = \begin{bmatrix} x' & e' & e_n' \end{bmatrix}'$ and the coefficient matrices are respectively

$$\begin{aligned}
A_p(t) &= A_{po}(t) + \Delta A_p(t) \qquad (19) \\
B_p(t) &= B_{po}(t) + \Delta B_p(t) \\
A_{po}(t) &= \begin{bmatrix} A(t) & O & O \\ O & A(t)-L(t)C(t) & O \\ O & O & A(t)-L_n(t)C(t) \end{bmatrix} \\
C_{1p}(t) &= \begin{bmatrix} O & O & -W_n(t)C(t) \end{bmatrix} \\
C_{2p}(t) &= \begin{bmatrix} O & C(t) & O \end{bmatrix} \\
B_{po}(t) &= \begin{bmatrix} E_f(t) & E_d(t) & B(t) \\ E_f(t)-L(t)F_f(t) & E_d(t)-L(t)F_d(t) & O \\ E_f(t)-L_n(t)F_f(t) & E_d(t)-L_n(t)F_d(t) & O \end{bmatrix} \\
D_{1p}(t) &= \begin{bmatrix} -W_n(t)F_f(t) & -W_n(t)F_d(t) & O \end{bmatrix} \\
D_{2p}(t) &= \begin{bmatrix} F_f(t) & F_d(t) & O \end{bmatrix} \\
\Delta A_p(t) &= \begin{bmatrix} \Delta A(t) & O & O \\ \Delta A(t) & O & O \\ O & O & O \end{bmatrix} \\
\Delta B_p(t) &= \begin{bmatrix} \Delta E_f(t) & \Delta E_d(t) & \Delta B(t) \\ \Delta E_f(t) & \Delta E_d(t) & \Delta B(t) \\ O & O & O \end{bmatrix}
\end{aligned}$$

As is standard, the dynamics of the whole system with $\delta(t)$ as input and $\xi(t)$ as output is governed by

$$\begin{aligned}
\dot{X}_e(t) &= A_e(t)X_e(t) + B_e(t)\delta(t) \\
\xi(t) &= C_e(t)X_e(t) + D_e(t)\delta(t)
\end{aligned} \tag{20}$$

where $X_e(t) = \begin{bmatrix} X_p' & x_R' \end{bmatrix}'$,

$$\begin{aligned}
A_e(t) &= A_{eo}(t) + \Delta A_e(t) + \tilde{B}\Theta_R(t)\tilde{C}(t) \qquad (21) \\
B_e(t) &= B_{eo}(t) + \Delta B_e(t) + \tilde{B}\Theta_R(t)\tilde{D}_{21}(t) \\
C_e(t) &= C_{eo}(t) + \tilde{D}_{12}\Theta_R(t)\tilde{C}(t) \\
D_e(t) &= D_{1p}(t) + \tilde{D}_{12}\Theta_R(t)\tilde{D}_{21}(t) \\
A_{eo}(t) &= \begin{bmatrix} A_{po}(t) & O \\ O & O \end{bmatrix}, B_{eo}(t) = \begin{bmatrix} B_{po}(t) \\ O \end{bmatrix} \\
C_{eo}(t) &= \begin{bmatrix} C_{1p}(t) & O \end{bmatrix} \\
\tilde{B} &= \begin{bmatrix} O & O \\ I & O \end{bmatrix}, \ \tilde{C}(t) = \begin{bmatrix} O & I \\ C_{2p}(t) & O \end{bmatrix} \\
\tilde{D}_{21}(t) &= \begin{bmatrix} O \\ D_{2p}(t) \end{bmatrix}, \ \tilde{D}_{12} = \begin{bmatrix} O & I \end{bmatrix} \\
\Delta A_e(t) &= \begin{bmatrix} \Delta A_p(t) & O \\ O & O \end{bmatrix}, \ \Delta B_e(t) = \begin{bmatrix} \Delta B_p(t) \\ O \end{bmatrix}
\end{aligned}$$

Use operator $\Pi_{\xi\delta}$ to denote the mapping from $\delta(t)$ to $\xi(t)$. Then the norm defined by

$$\|\Pi_{\xi\delta}\| = \sup_{\delta \in \mathbf{L}_2 - \{0\}} \frac{\|\xi\|_2}{\|\delta\|_2}$$

describes how *well* the approximation is. As a result, the problem of selecting the optimal post filter can be reformulated as

$$\min_{\Theta_R(t)} \gamma \tag{22}$$

$$\|\Pi_{\xi\delta}\| = \sup_{\delta \neq o} \frac{\|\xi\|_2}{\|\delta\|_2} < \gamma \tag{23}$$

According to the periodic Bounded Real Lemma [1], for given $\gamma > 0$, (23) holds if and only if there exists a T-periodic matrix $P_e(t) > 0$ such that $\forall t$, the following periodic differential LMI holds

$$\begin{bmatrix} \Omega_{11}(t) & P_e(t)B_e(t) & C_e'(t) \\ B_e'(t)P_e(t) & -\gamma^2 I & D_e'(t) \\ C_e(t) & D_e(t) & -I \end{bmatrix} < 0 \tag{24}$$

$$\Omega_{11}(t) = \dot{P}_e(t) + A_e'(t)P_e(t) + P_e(t)A_e(t)$$

Depending on the forms of model uncertainty, $\Delta A_e(t)$ and $\Delta B_e(t)$ in (20) may take different forms. Hence, different techniques are suitable to handle the problem. So in the following we shall treat Case II and Case III separately.

Case II: Polytopic uncertainty

In this case, $\Delta A_e(t)$ and $\Delta B_e(t)$ are

$$\begin{aligned} \left[\begin{array}{cc} \Delta A_e(t) & \Delta B_e(t) \end{array}\right] &= \sum_{i=1}^{v} \lambda_i \left[\begin{array}{cc} A_{ei}(t) & B_{ei}(t) \end{array}\right] \\ A_{ei}(t) &= \left[\begin{array}{cc} A_{pi}(t) & O \\ O & O \end{array}\right],\ B_{ei}(t) = \left[\begin{array}{c} B_{pi}(t) \\ O \end{array}\right] \\ A_{pi}(t) &= \left[\begin{array}{ccc} A_i(t) & O & O \\ A_i(t) & O & O \\ O & O & O \end{array}\right] \\ B_{pi}(t) &= \left[\begin{array}{ccc} E_{fi}(t) & E_{di}(t) & B_i(t) \\ E_{fi}(t) & E_{di}(t) & B_i(t) \\ O & O & O \end{array}\right] \\ i &= 1, 2, \cdots, v \end{aligned} \tag{25}$$

Formula (24) holds if there exists a T-periodic $P_e(t) > 0$, such that for $i = 1, 2, \cdots v$ and $\forall t \in [0, T]$,

$$\left[\begin{array}{ccc} \Psi_{11}(t) & \Psi_{12}(t) & \Psi_{13}(t) \\ \Psi'_{12}(t) & -\gamma^2 I & \Psi_{23}(t) \\ \Psi'_{13}(t) & \Psi'_{23}(t) & -I \end{array}\right] < 0, \tag{26}$$

where

$$\begin{aligned} \Psi_{11} &= \dot{P}_e(t) + (A_{eo}(t) + A_{ei}(t) + \tilde{B}\Theta_R(t)\tilde{C}(t))' P_e(t) \\ &\quad + P_e(t)(A_{eo}(t) + A_{ei}(t) + \tilde{B}\Theta_R(t)\tilde{C}(t)) \\ \Psi_{12} &= P_e(t)(B_{eo}(t) + B_{ei}(t) + \tilde{B}\Theta_R(t)\tilde{D}_{21}(t)) \\ \Psi_{13} &= (C_{eo}(t) + \tilde{D}_{12}\Theta_R(t)\tilde{C}(t))' \\ \Psi_{23} &= (D_{1p}(t) + \tilde{D}_{12}\Theta_R(t)\tilde{D}_{21}(t))' \end{aligned}$$

Then, (26) can be re-written into

$$\Phi(t) + \Gamma(t)\Theta_R(t)\Lambda(t) + (\Gamma(t)\Theta_R(t)\Lambda(t))' < 0 \tag{27}$$

where

$$\begin{aligned} \Phi(t) &= \left[\begin{array}{ccc} \Phi_{11}(t) & \Phi_{12}(t) & C'_{eo}(t) \\ \Phi'_{12}(t) & -\gamma^2 I & D'_{1p}(t) \\ C_{eo}(t) & D_{1p}(t) & -I \end{array}\right] \\ \Phi_{11}(t) &= \dot{P}_e(t) + (A_{eo}(t) + A_{ei}(t))' P_e(t) \\ &\quad + P_e(t)(A_{eo}(t) + A_{ei}(t)) \\ \Phi_{12}(t) &= P_e(t)(B_{eo}(t) + B_{ei}(t)) \end{aligned}$$

$$\Gamma(t) = \left[\begin{array}{c} P_e(t)\tilde{B} \\ O \\ \tilde{D}_{12} \end{array}\right], \Lambda(t) = \left[\begin{array}{ccc} \tilde{C}(t) & \tilde{D}_{21}(t) & O \end{array}\right]$$

There exists a matrix $\Theta_R(t)$ so that (27) holds, iff [7]

$$\Gamma^{\perp}(t)\Phi(t)\Gamma^{\perp\prime}(t) < 0,\ \ \Lambda'^{\perp}(t)\Phi(t)\Lambda'^{\perp\prime}(t) < 0 \tag{28}$$

Assume that $P_e(t)$ and $P_e^{-1}(t)$ can be partitioned as

$$P_e(t) = \left[\begin{array}{cc} S(t) & Y(t) \\ Y'(t) & \hat{S}(t) \end{array}\right]$$

$$P_e^{-1}(t) = \left[\begin{array}{cc} Q(t) & X(t) \\ X'(t) & \hat{Q}(t) \end{array}\right] \tag{29}$$

(29) holds, iff

$$S(t) \geqslant Q^{-1}(t),\ rank(Q^{-1}(t) - S(t)) \leqslant k_R \tag{30}$$

Substituting (21) and (29) into (28), it turns out to be

$$\begin{aligned} &-\dot{Q}(t) + (A_{po}(t) + A_{pi}(t))Q(t) + Q(t)(A'_{po}(t) + A'_{pi}(t)) \\ &\quad + \gamma^{-2}(B_{po}(t) + B_{pi}(t))(B_{po}(t) + B_{pi}(t))' < 0 \end{aligned} \tag{31}$$

$$V_l(t) \left[\begin{array}{ccc} \digamma_{11}(t) & \digamma_{12}(t) & C'_{1p}(t) \\ \digamma'_{12}(t) & -\gamma^2 I & D'_{1p}(t) \\ C_{1p}(t) & D_{1p}(t) & -I \end{array}\right] V'_l(t) < 0 \tag{32}$$

$$\begin{aligned} \digamma_{11}(t) &= \dot{S}(t) + (A'_{po}(t) + A'_{pi}(t))S(t) \\ &\quad + S(t)(A_{po}(t) + A_{pi}(t)) \\ \digamma_{12}(t) &= S(t)(B_{po}(t) + B_{pi}(t)) \end{aligned}$$

where $V_l(t) = diag\{V(t), I\}$ and $V(t)$ denotes the base of the null space of matrix $\left[\begin{array}{cc} C_{2p}(t) & D_{2p}(t) \end{array}\right]'$. As a result, we get the following theorem.

Theorem 2 Given system (20) with uncertainty (25) and $\gamma > 0$, there exists a post-filter (15) of order k_R so that $A_e(t)$ stable and (23) holds, if there exist T-periodic matrices $S(t) > 0, Q(t) > 0$ such that for $i = 1, 2, \cdots, v$ and $\forall t \in [0, T]$, (30)-(32) hold.

Case III: Norm-bounded uncertainty

In this case, $\Delta A_e(t)$ and $\Delta B_e(t)$ are

$$\begin{aligned} \left[\begin{array}{cc} \Delta A_e(k) & \Delta B_e(k) \end{array}\right] &= M_e(t)\Delta_o(t) \left[\begin{array}{cc} N_{ae}(t) & N_{be}(t) \end{array}\right] \\ M_e(t) &= \left[\begin{array}{c} M_p(t) \\ O \end{array}\right],\ M_p(t) = \left[\begin{array}{c} M(t) \\ M(t) \\ O \end{array}\right] \\ N_{ae}(t) &= \left[\begin{array}{cc} N_{ap}(t) & O \end{array}\right],\ N_{be}(t) = N_{bp}(t) \\ N_{ap}(t) &= \left[\begin{array}{ccc} N_a(t) & O & O \end{array}\right] \\ N_{bp}(t) &= \left[\begin{array}{ccc} N_f(t) & N_d(t) & N_b(t) \end{array}\right] \end{aligned} \tag{33}$$

Formula (24) holds if there exists $P_e(t) > 0$ and a scalar $\varepsilon > 0$, such that $\forall t \in [0, T]$,

$$\left[\begin{array}{cccc} \hat{\Psi}_{11} & \hat{\Psi}_{12} & C'_e(t) & P_e(t)M_e(t) \\ \hat{\Psi}'_{12} & \hat{\Psi}_{22} & D'_e(t) & O \\ C_e(t) & D_e & -I & O \\ M'_e(t)P_e(t) & O & O & -\varepsilon I \end{array}\right] < 0 \tag{34}$$

where

$$\begin{aligned} \hat{\Psi}_{11}(t) &= \dot{P}_e(t) + P_e(t)(A_{eo}(t) + \tilde{B}\Theta_R(t)\tilde{C}(t)) \\ &\quad + (A_{eo}(t) + \tilde{B}\Theta_R(t)\tilde{C}(t))' P_e(t) + \varepsilon N'_{ae}(t)N_{ae}(t) \\ \hat{\Psi}_{12}(t) &= P_e(t)(B_{eo}(t) + \tilde{B}\Theta_R(t)\tilde{D}_{21}(t)) + \varepsilon N'_{ae}(t)N_{be}(t) \\ \hat{\Psi}_{22}(t) &= -\gamma^2 I + \varepsilon N'_{be}(t)N_{be}(t) \end{aligned}$$

Then, (34) can be written into the form of

$$\hat{\Phi}(t) + \hat{\Gamma}(t)\Theta_R(t)\hat{\Lambda}(t) + (\hat{\Gamma}(t)\Theta_R(t)\hat{\Lambda}(t))' < 0 \tag{35}$$

where

$$\hat{\Phi}(t) = \begin{bmatrix} \hat{\Phi}_{11} & \hat{\Phi}_{12} & C'_{eo}(t) & P_e(t)M_e(t) \\ \hat{\Phi}'_{12} & \hat{\Phi}_{22} & D'_{1p}(t) & O \\ C_{eo}(t) & D_{1p}(t) & -I & O \\ M'_e(t)P_e(t) & O & O & -\varepsilon I \end{bmatrix}$$

$$\hat{\Phi}_{11} = \dot{P}_e(t) + A'_{eo}(t)P_e(t) + P_e(t)A_{eo}(t) + \varepsilon N'_{ae}N_{ae}$$

$$\hat{\Phi}_{12} = P_e(t)B_{eo}(t) + \varepsilon N'_{ae}(t)N_{be}(t)$$

$$\hat{\Phi}_{22} = -\gamma^2 I + \varepsilon N'_{be}(t)N_{be}(t)$$

$$\hat{\Lambda}(t) = \begin{bmatrix} \tilde{C}(t) & \tilde{D}_{21}(t) & O & O \end{bmatrix}$$

$$\hat{\Gamma}(t) = \begin{bmatrix} \tilde{B}'P_e(t) & O & \tilde{D}'_{12} & O \end{bmatrix}'$$

There exists a matrix $\Theta_R(t)$ so that (35) holds, iff [7]

$$\hat{\Lambda}'^{\perp}(t)\hat{\Phi}(t)\hat{\Lambda}'^{\perp\prime}(t) < 0, \ \ \hat{\Gamma}^{\perp}(t)\hat{\Phi}(t)\hat{\Gamma}^{\perp\prime}(t) < 0 \qquad (36)$$

Assume that $P_e(t), P_e^{-1}(t)$ are partitioned as (29) and let

$$U(t) = Q^{-1}(t) \qquad (37)$$

Then (36) are equivalent to

$$\begin{bmatrix} \dot{U}(t) + A'_{po}(t)U(t) + U(t)A_{po}(t) + \varepsilon N'_{ap}(t)N_{ap}(t) & U(t)B_{po}(t) + \varepsilon N'_{ap}(t)N_{bp}(t) & U(t)M_p(t) \\ B'_{po}(t)U(t) + \varepsilon N'_{bp}(t)N_{ap}(t) & -\gamma^2 I + \varepsilon N'_{bp}(t)N_{bp}(t) & O \\ M'_p(t)U(t) & O & -\varepsilon I \end{bmatrix} < 0 \qquad (38)$$

$$\hat{V}_l(t) \begin{bmatrix} \hat{\digamma}_{11}(t) & \hat{\digamma}_{12}(t) & C'_{1p}(t) & \hat{\digamma}_{14}(t) \\ \hat{\digamma}'_{12}(t) & \hat{\digamma}_{22}(t) & D'_{1p}(t) & O \\ C_{1p}(t) & D_{1p}(t) & -I & O \\ \hat{\digamma}'_{14}(t) & O & O & -\varepsilon I \end{bmatrix} \hat{V}'_l(t) < 0 \qquad (39)$$

$$\hat{\digamma}_{11}(t) = \dot{S}(t) + S(t)A_{po}(t) + A'_{po}(t)S(t) + \varepsilon N'_{ap}(t)N_{ap}(t)$$

$$\hat{\digamma}_{12}(t) = S(t)B_{po}(t) + \varepsilon N'_{ap}(t)N_{bp}(t)$$

$$\hat{\digamma}_{14}(t) = S(t)M_p(t), \hat{\digamma}_{22}(t) = -\gamma^2 I + \varepsilon N'_{bp}(t)N_{bp}(t)$$

$$\hat{V}_l(t) = diag\{V(t), I, I\}$$

And due to (37), (29) holds, iff

$$S(t) \geqslant U(t), \ rank(U(t) - S(t)) \leqslant k_R \qquad (40)$$

Theorem 3 is thus obtained.

Theorem 3 Given system (20) with uncertainty (33) and $\gamma > 0$, there exists a post-filter (15) of order k_R so that $A_e(t)$ is stable and (23) holds, if there exist T-periodic matrices $S(t) > 0, U(t) > 0$ and a scalar $\varepsilon > 0$, such that $\forall t \in [0, T]$, (38)-(40) hold.

In summary, in case of $\Delta \neq 0$, the residua generators for periodic systems with polytopic (*norm-bounded*) uncertainty can be designed as follows:

- Choose a stabilizing T-periodic observer gain matrix $L(t)$;
- Build the matrices of the extended plant according to (19);
- Given $\gamma > 0$, solve (30)-(32) (*(38)-(40)*) for T-periodic symmetric positive definite matrices $S(t)$, $Q(t)$ (*S(t),U(t)*);
- Reduce or increase the value of γ, iterate the last step till the minimal γ is found and then write down the corresponding $S(t), Q(t)$ (*S(t),Q(t)=*U^{-1}*(t)*);
- To determine $P_e(t)$, at first calculate two full column rank matrices $X(t), Y(t)$ satisfying

$$S(t)Q(t) + Y(t)X'(t) = I$$

then solve the equation below for $P_e(t)$

$$\begin{bmatrix} S(t) & I \\ Y'(t) & O \end{bmatrix} = P_e(t) \begin{bmatrix} I & Q(t) \\ O & X'(t) \end{bmatrix} \qquad (41)$$

- Substituting $P_e(t)$ back into (27) (*(35)*) and solve it for $\Theta_R(t)$, optimization problem (22) is then solved.

5 Conclusion

In this paper, we have addressed the FD problem of linear continuous-time periodic systems. In case of no model uncertainty, the key point of designing residual generator with optimal sensitivity and robustness consists in introducing operators and their induced norms to formulate the optimization problem. It is further shown that the optimal solution is related to a PDRE. After that both polytopic uncertainty and norm-bounded uncertainty are taken into consideration by using the periodic differential LMI technique. To solve the PDRE and periodic differential LMIs, the algorithms given in [3] and [1] are available.

Acknowledgements This work was in part supported by European Community grant IST-2001-32122 IFATIS.

References

[1] S. Bittanti, F.A. Cuzzola. "Continuous-time periodic H_∞ filtering via LMI", *European Journal of Control*, vol. 7, pp. 2-16, (2001).

[2] S. Bittanti, P. Colaneri. "Periodic control", *Webster J (Ed.) John Wiley Encyclopaedia on Electrical and Electronic Engineering*, vol. 16, pp. 59-74, (1999).

[3] S. Bittanti, P. Colaneri, G. De Nicolao. "The periodic Riccati equation", in *The Riccati Equation (S. Bittanti, A.J. Laub, J.C. Willems (Eds.))*, pp. 126-162, Springer, (1991).

[4] S.X. Ding, P.M. Frank, E.L. Ding, T. Jeinsch. "A unified approach to the optimization of fault detection systems", *Int. J. of Adaptive Control and Signal Processing*, vol. 14, pp. 725-745, (2000).

[5] J.J. Gertler. *Fault Detection and Diagnosis in Engineering Systems*, Marcel Dekker, (1998).

[6] R.J. Patton, P.M. Frank, R.N. Clark. *Issues of Fault Diagnosis for Dynamic Systems*, Springer, (2000).

[7] R.E. Skelton, T. Iwasaki, K.M. Grigoriadis. *A Unified Algebraic Approach to Linear Control Design*, Taylor and Francis, (1998).