# A BEHAVIORAL APPROACH TO CONTROL RELATED ANALYSIS OF DISCRETE LINEAR REPETITIVE PROCESSES

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## Abstract

Repetitive processes are a distinct class of 2D systems (i.e. information propagation occurs in two independent directions) of both systems theoretic and applications interest. The feature which makes them distinct from other classes of 2D systems is that information propagation in one of the two independent directions only occurs over a finite duration. Discrete linear repetitive processes are a very important sub-class and in this paper we study them in the behavioral setting. In particular, we show that the key systems properties of stability and controllability are naturally characterized in this setting by means of algebraic conditions on the matrices which define the model.

## 1 Introduction

The essential unique characteristic of a repetitive, or multipass, process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, dynamics of the next pass profile. This, in turn, leads to the unique control problem for these processes in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

To introduce a formal definition, let $m < +\infty$ be an integer denoting the pass length (assumed constant). Then in a repetitive process the pass profile $y_k(p),\ 0 \le p \le m$ generated on pass $k$ acts as a forcing function on, and hence contributes to, the dynamics of the new pass profile $y_{k+1}(p),\ 0 \le p \le m,\ k \ge 0$. The fact that the pass length is finite (where this is an intrinsic feature of the process dynamics and not a simplifying assumption), and hence information in this direction only occurs over a finite duration, is the key difference with other classes of 2D (discrete) linear systems and, in particular, those represented by the Roesser and Fornasini Marchesini state space models respectively (see, for example, the relevant references in[4]).

Physical examples of repetitive processes include long-wall coal cutting and metal rolling operations (see, for example,[2]). Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these so-called algorithmic applications of repetitive processes include classes of iterative learning control (ILC) schemes [1] and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle [3]. In the case of ILC for the linear dynamics case, the stability theory for differential and discrete linear repetitive processes is the essential basis for a rigorous stability/convergence analysis of such algorithms.

Attempts to control these processes using standard (or 1D) systems theory/algorithms fail in general precisely because such an approach ignores their inherent 2D systems structure. In particular, it ignores the fact that information propagation occurs from pass to pass and along a given pass, and that the pass initial conditions are reset before the start of each new pass. A rigorous stability theory for linear repetitive processes has been developed. This theory [4] is based on an abstract model in a Banach space setting which includes all such processes as special cases.

The purpose of this paper is to examine some of the key problems of repetitive process theory for the important sub-class of so-called discrete linear repetitive processes, which are of particular relevance in the ILC application area, within the framework of behavioral systems theory [6, 7]. In this framework an analysis of a dynamic system (repetitive process or otherwise) is developed using the most natural model of the system rather than in terms of models in which inputs outputs and states are pre-specified. This offers several advantages and allows transformations of all the variables of the system rather than being restricted to the three separate classes of inputs, outputs, or states.

The behavioral approach is inherently suitable for modeling repetitive processes. We show how the well known model of a discrete linear repetitive process leads to a hybrid (or a latent variable) representation of its behavior. In this case, considering the time history of variables at all the intermediate instances along a pass, the behavior turns out to have a finite number of manifest variables and has also a standard, termed 1D in repetitive process literature, character. Hence its kernel representation can be determined by standard methods of behavioral theory. We show in this paper that this kernel representation can be computed by simple linear algebraic procedures without recourse to polynomial methods. This further enables an analysis of asymptotic stability and controllability properties of these processes to be undertaken in much simpler ways than reported previously (see, for example, [5]).

We then derive the conditions for stability and controllability of the repetitive processes and also give results which are a significant first step towards a procedure for obtaining the stabilizing controller, which is a counterpart of the well known constant output feedback controller in standard systems theory. Further development of these results should make it possible to circumvent the difficulties arising out of 2D nature of these processes and lead to the development of a stabilization theory by, in effect, utilizing 1D analysis.

## 2 Background

A discrete linear repetitive process having a $K$-pass memory, i.e. when the previous $K$ pass profiles explicitly contribute to the current one, and having a constant pass length of $m$ is given by the dynamical equations,

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) \\ &+ \sum_{j=0}^{K-1} B_{0j} y_{k-j}(p) \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) \\ &+ \sum_{j=0}^{K-1} D_{0j} y_{k-j}(p) \end{aligned}$$

together with boundary conditions which incorporate the history of variables of the previous pass given by

$$x_{k+1}(0) = d_{k+1} + \sum_{j=0}^{m-1} K_j y_k(j) \tag{2}$$

where $d_k$ is a vector with constant entries on each pass, $k$ denotes the pass number or index, and $p$, $0 \leq p \leq m-1$ denotes an instant during a pass. The *unit pass* repetitive process has $K = 1$ and hence in this case we set $B_{00} = B_0$ and $D_{00} = D_0$. In this paper we shall consider the unit memory case only since the results for the $K$-pass memory case follow as natural generalizations.

A fundamental problem in the analysis of repetitive processes is to determine the conditions for stability, since the stability analysis for 1D systems does not transfer. In this paper we approach this problem by developing a hybrid model representing the behavior of processes described by the model given above (with $d_k = 0$) in which $y_k(p)$ and $u_k(p)$ are manifest variables while $x_k(p)$ are latent variables. In order to eliminate the independent variable $p$ on every pass we consider the behavior of all these variables for $0 \leq p \leq m-1$. Hence we consider variables $Y(k)$, $U(k)$ denoting the collection of all outputs $y_k(p)$ and inputs $u_k(p)$ respectively. Due to finiteness of $m$ the resulting behavior has finite number of manifest variables $W = \text{col}\,(Y,\ U)$ and whose dynamical behavior evolves with $k$ as the independent variable. This leads to an equivalence with existing theory [4] but a much more concise approach to deriving the conditions for stability than alternatives [5].

Next we treat the question of controllability of discrete linear repetitive processes in the behavioral framework. For this purpose we consider the kernel representation of the behavior of $W(k)$ which can be obtained by the use of the elimination theorem on the hybrid model referred to above. In the behavioral approach the variables $U(k)$ are just treated as a subset of manifest variables $W(k)$ and the controllability of the behavior is the existence of a patching trajectory $W(k)$ for two arbitrary trajectories in the behavior. We refer the reader to [6, 7] for details and the criteria for controllability in terms of a kernel representation in the standard case. We show that the criterion for controllability can be obtained much more easily in terms of a transformation of the manifest variables to obtain an isomorphic behavior. The behavioral approach to controllability subsumes the classical approach in which $u_k(p)$ are inputs and controllability amounts to driving the states (or outputs) to a desired pass profile [5]. Finally, we show that the criterion for controllability can be obtained using matrix computational methods as in the classical approach.

Given the controllability analysis of the behavior of the repetitive process, a natural next step is to consider the problem of closed loop eigenvalue assignment. In the case of repetitive processes the meaning of eigenvalues (or poles) as well as the concept of assignment of eigenvalues has to be redefined from first principles. This difficulty again stems from the inherent 2D nature of these processes. However the 1D equivalent model of the behavior of $W(k)$ referred above is a discrete time linear time invariant behavior in which the variables $Y(k)$ and $U(k)$ serve physically as collections of outputs and inputs respectively but are not differentiated mathematically. Hence obtaining a constant gain controller of the form $Y(k) = KU(k)$ is analogous to the classical constant gain output feedback and hence the exponents in the exponential solutions in the finite dimensional (or autonomous) behavior of variables $Y(k)$ after such a controller is incorporated, are an analogue of the eigenvalues of the closed loop repetitive process.

We in this paper we develop results which should, with further development, enable the assignment of the eigenvalues of the discrete linear repetitive processes considered here to be carried out once an analogous stabilizability condition is satisfied and the derivation of the controller can be completed within the framework of the 1D model. These results are unique (at this stage) in the analysis of discrete linear repetitive processes.

## 3 The behavioral models

In this section we develop the hybrid and kernel forms of models of the behavior of the repetitive processes considered in this wprk. A representation of the behavior of the unit memory version of (1) and (2) can be obtained by specifying the manifest variables. The resultant set of equations then gives a hybrid representation of the behavior. To define the manifest and latent variables we shall first consider the collections of variables at all instances of a pass to obtain a 1D behavior, and use the following notation

$$\begin{aligned} Y(k) &= \text{col}\,(y_0(p), y_1(p) \dots y_{(m-1)}(p)) \\ U(k) &= \text{col}\,(u_0(p), u_1(p) \dots u_{(m-1)}(p)) \\ X(k) &= \text{col}\,(x_0(p), x_1(p) \dots x_{(m-1)}(p)) \end{aligned}$$

Thus $Y(k)$, $U(k)$, $X(k)$ are vector valued variables of $m$ vector components of the variables $y_k(p)$, $u_k(p)$ and $x_k(p)$ respectively along the instances of the $k$th pass. Also let $W(k)$ denote the following vector,

$$W(k) = \begin{bmatrix} Y(k) \\ U(k) \end{bmatrix}$$

which we shall consider as the vector valued manifest variable. Then the equations (1) and (2) can be rewritten as (taking $d_k = 0$),

$$\left( \begin{bmatrix} 0 & \hat{B} \\ I & -\hat{D} \end{bmatrix} \sigma + \begin{bmatrix} \hat{K} & 0 \\ -\hat{D}_0 & 0 \end{bmatrix} \right) W(k) = \sigma \begin{bmatrix} \hat{Q} \\ \hat{C} \end{bmatrix} X(k) \tag{3}$$

where $\sigma$ denotes the difference operator $\sigma f(k) = f(k+1)$ on sequences $f(k)$. The matrices in the above equation are given by

$$\begin{aligned} \hat{D} &= \text{diag}\,\{D, D, \dots D\} \\ \hat{D}_0 &= \text{diag}\,\{D_0, D_0, \dots D_0\} \\ \hat{C} &= \text{diag}\,\{C, C, \dots C\} \end{aligned}$$

$$\hat{B} = \begin{bmatrix} 0 & & & & \\ B & 0 & & & \\ & B & 0 & & \\ & & \ddots & \ddots & \\ & & & B & 0 \end{bmatrix}$$

$$\hat{K} = \begin{bmatrix} K_0 & K_1 & \dots & \dots & K_{(m-1)} \\ B_0 & & & & \\ & B_0 & & & \\ & & \ddots & & \vdots \\ & & & B_0 & 0 \end{bmatrix}$$

$$\hat{Q} = \begin{bmatrix} I & & & & \\ -A & I & & & \\ & -A & & & \\ & & \ddots & \ddots & \\ & & & -A & I \end{bmatrix}$$

This completes the description of the latent variable (or hybrid) representation (3) of the behavior of the repetitive process.

This is a linear time invariant discrete time behavior with a finite number of manifest variables $W(k)$. To denote such a behavior in standard notation, observe that the manifest variables $W$ now take values in $\mathbb{R}^q$ where $q = m(N_y + N_u)$ where $N_y$, $N_u$ are the number of $y$ and $u$ variables respectively (and $m$ is the pass length). Hence this behavior is denoted by $\mathcal{L}(\mathbb{Z})$ or by the triple $(\mathbb{Z}, \mathbb{R}^q, \mathcal{B})$, or for ease of notation, by $\mathcal{B}$.

### 3.1 Fundamental theorems

We now discuss two of the important mathematical facts concerned with the above representation, the elimination theorem and the module behavior correspondence [8] which form fundamental pillars of the behavioral theory. As observed in the above model the matrices of the representation (3) are defined over the polynomial ring $\mathbb{R}[\sigma]$. Let the collection of all sequences $f(k), k = 0, 1 \dots, \; f(k) \in \mathbb{R}$ be denoted as $\mathcal{V}$. Then $\mathcal{V}$ is a module over the commutative ring $\mathbb{R}[\sigma]$ under the operation $\sigma f(k) = f(k+1)$. The solution trajectories of variables $W(k)$ and $X(k)$ are matrices whose entries are also such sequences. Hence the behavior of $W$ is also defined as a module over this ring.

This last fact means that all of the techniques of behavioral theory are applicable to the processes considered in this paper since the manifest and latent variables are defined over a finite Cartesian product of the module $\mathcal{V}$. The elimination theorem is stated next.

**Theorem 1** *Let $\mathcal{B}$ be a behavior whose trajectories belong to $\mathcal{V}$ and given by a latent variable representation*

$$R(\sigma)w(n) = M(\sigma)l(n)$$

*If $Q(\sigma)$ is a matrix whose rows generate the $\mathbb{R}[\sigma]$-module of relations of rows of the matrix $M(\sigma)$ then a kernel representation of $\mathcal{B}$ is given by*

$$Q(\sigma)R(\sigma)w(n) = 0$$

The elimination theorem establishes that a behavior over $\mathcal{V}$ represented by a latent variable representation has a kernel representation. We omit the proof of this theorem as it can be developed by, in effect, routine modifications to that for the well known discrete time case of behavioral systems over the ring $\mathbb{R}[\sigma, \sigma^{-1}]$ and trajectories defined over two sided infinite sequences [6]. Next we state the theorem on module behavior correspondence for behaviors in terms of matrices of kernel representations. (The notation $M(R(\sigma)$ denotes the $\mathbb{R}[\sigma]$-module generated by the rows of the matrix $R(\sigma)$).

**Theorem 2** *Let $R_i(\sigma)w_i(n) = 0$ be kernel representations of behaviors $\mathcal{B}_i$ respectively. Then $\mathcal{B}_1 = \mathcal{B}_2$ if, and only if, there is a unimodular matrix $U(\sigma)$ such that $R_1(\sigma) = U(\sigma)R_2(\sigma)$. Also $\mathcal{B}_1 \subset \mathcal{B}_2$ if, and only if, $M(R_2(\sigma)) \subset M(R_1(\sigma))$.*

We again omit the proof of this result as it follows routinely from the discrete time results of [8].

Using these results, we can compute the kernel representation of the behavior of $W(k)$ with the help of following lemma. We denote by rel $P(\sigma)$ the module of relations of the rows of a matrix $P(\sigma)$ over the polynomial ring $\mathbb{R}[\sigma]$ and denote by $< P(\sigma) >_r$ the module generated by its rows over this ring. When $P$ is a constant matrix we denote the vector spaces of relations of rows of $P$ and that generated by its rows over the real field by the same symbols.

**Lemma 1**

$$\text{rel} \begin{bmatrix} \hat{Q} \\ \hat{C} \end{bmatrix} = < [Q_0 \; I] >_r$$

*where*

$$Q_0 = \begin{bmatrix} -C & & & \\ -CA & -C & & \\ \vdots & \vdots & & \\ -CA^{m-1} & -CA^{m-2} & \dots & -C \end{bmatrix}$$

**Proof:** Follows by computing the relations for $m = 3$ and then by induction. The details are mainly computational and are omitted.

**Proposition 1** *The behavior of $W(k)$ has the kernel representation*

$$\left( \begin{bmatrix} \sigma I + (Q_0 \hat{K} - \hat{D}_0) & \sigma(Q_0 \hat{B} - \hat{D}) \end{bmatrix} \right) W(k) = 0 \quad (4)$$

*and the variables $U(k)$ are maximally free.*

**Proof.** The module of relations of the rows of the polynomial matrix $\sigma[\hat{Q}^T \; \hat{C}^T]^T$ is given by that of the relations of the constant matrix $[\hat{Q}^T \; \hat{C}^T]^T$. This linear algebraic computation is given by the above lemma. The result now follows on applying the elimination theorem to the latent variable model. Note that $W(k)^T = (Y(k)^T, U(k)^T)$. Since $\det(\sigma I + (Q_0 \hat{K} - \hat{D}_0)) \neq 0$ it follows that $U(k)$ are free variables.

We have thus obtained the hybrid and kernel representations of the behavior of the manifest variables $W$. In the next section we employ these to obtain the criteria for stability and controllability.

## 4 Stability and controllability analysis

The derivation of the kernel representation of the behavior of the repetitive system in terms of the manifest variable $W$ was carried out in the last section. This computation turned out to be greatly simplified due to the fact that the module of relations of the polynomial matrix on the right hand side of (3) has the same generators as that of the vector space of relations of rows of the constant matrix col $[\hat{Q} \; \hat{C}]$. Hence the cumbersome computation of relations of a polynomial matrix usually required in elimination of latent variables is completely avoided. We now develop the stability and controllability analysis of the system (1), (2).

### 4.1 Stability criterion

From a behavioral point of view, stability (asymptotic stability) of a behavior is characterized by uniform boundedness (asymptotic decay) of all of its trajectories. Hence such a notion is applicable only to behaviors which are autonomous i.e. those which do not have free (or input) variables since such variables can always be chosen to be unbounded. Hence it is necessary to define stability of repetitive processes described by (1) and (2) in terms of variables whose laws of motion determined by these equations are autonomous. A physically meaningful notion of stability in this context is given by

**Definition 1** A discrete repetitiveprocess described by (1) and (2) is *externally stable* if for $u_k(p) = 0$ for all $0 \leq p \leq m-1$ and $k = 1, 2, \dots$ the solutions $y_k(p)$ are uniformly bounded (tend to zero) for $k = 1, 2, \dots$ (for $k \to \infty$) under arbitrary initial conditions $y_0(p)$ for these variables for all $0 \leq p \leq m-1$.

Thus it is necessary to verify from the model (4) that the behavior of the variables $Y(k)$ when $U(k) = 0$ is autonomous and then determine the stability of the behavior under the conditions of zero $U(k)$. The kernel representation (4) shows that this is precisely the case i.e. when $U(k) = 0$ the resultant behavior of $Y(k)$ is represented by

$$(\sigma I + Q_0 \hat{K} - \hat{D}_0) Y(k) = 0 \quad (5)$$

Hence the presence of $\sigma I$ term shows that the matrix of the kernel representation of $Y(k)$ under the condition that $U(k) = 0$ is nonsingular and hence this behavior autonomous. We have thus established the external stability of a repetitive process described by (1) and (2) which is formally stated as follows.

**Theorem 3** *A discrete linear repetitive process described by (1) and (2) is externally stable if, and only if, the matrix $(\hat{D}_0 - \hat{Q}_0 \hat{K})$ has all its eigenvalues inside the open unit disc of the complex plane.*

**Proof.** From the kernel representation (4) of the behavior it follows that the laws of the behavior of $Y(k)$ under initial conditions and zero inputs $U(k) = 0$ for all $k$ are given by choosing $U(k) = 0$ in (4). These are precisely given by (5). Clearly the presence of $\sigma I$ term shows that this behavior is autonomous. The solution this system is given by

$$Y(k) = (-1)^k (Q_0 \hat{K} - \hat{D}_0)^k Y(0)$$

The theorem now follows readily from the above expression.

The above stability criterion was also obtained in [5] however the above proof is much simpler due to the direct approach facilitated by notions of behavioral theory. In particular, it is a simpler proof of the property termed asymptotic stability by [5] which has a well defined physical meaning and is the essential requirement of any physically relevant application of repetitive process theory. Moreover, we will show later that the behavioral setting has (with further development) the potential to enable the design of a stabilizing control law — a key task which has proved to be very difficult to formulate in repetitive process theory (and other classes of 2D systems) beyond obtaining conditions for stability under a control law.

### 4.2 Controllability analysis

We now take up investigation of controllability of discrete linear repetitive processes from a behavioral point of view. For completeness of exposition we first recall the property of controllability of a behavior which appears quite distinct from the classical notion of controllability in (1D) state space models. Let $\mathcal{B}$ denote a (1D) linear time invariant behavior in the space $\mathcal{V}^q$ with manifest variables $W$. Recall that this space is a module over the polynomial ring $\mathbb{R}[\sigma]$ and from the fundamental theory of behaviors there exists a kernel representation of $\mathcal{B}$ of the form $R(\sigma)W(k) = 0$. The behavioral interpretation of controllability for such a system is given by

**Definition 2** The behavior $\mathcal{B}$ is said to be *controllable* if for any two trajectories $W_1(k)$, $W_2(k)$ in $\mathcal{B}$ there exists a time $n$ and a trajectory $W(k)$ in $\mathcal{B}$ such that

$$\begin{aligned} W(k) &= W_1(k) \quad \text{for } k = 0, 1, \ldots, n \\ &= W_2(k) \quad \text{for } k \geq (n+1) \end{aligned}$$

From the above definition it follows that the behavioral interpretation of controllability does not depend on any concept of state. It is for this reason that the behavioral concept of controllability is useful even when there is no state space representation available. A very valuable fact for behavioral theory is that the behavioral interpretation of controllability when applied to a (1D) state space system turns out to be the same as that of the classical concept of controllability of state from the inputs [7]. In terms of a kernel representation a criterion for controllability of a behavior is given by the following.

**Proposition 2** *Consider a behavior $\mathcal{B}$ given by a kernel representation $R(\sigma)W(k) = 0$ in which the matrix $R(\sigma)$ has $r$ rows all of which are linearly independent over the ring $\mathbb{R}[\sigma]$. Then $\mathcal{B}$ is controllable if, and only if, there is no complex number $\lambda$ such that* rank $R(\lambda) < r$.

We omit the proof of this result as it can be easily constructed in a very similar manner to the well known proof in the discrete time case where the polynomial ring is $\mathbb{R}[\sigma, \sigma^{-1}]$ and the space of sequences $\mathcal{V}$ consists of all sequences over $\mathbb{Z}$, rather than only the nonnegative integers considered in this paper [6]. Note, however, that the above criterion can pose a numerical hurdle in its implementation since it depends on the existence (or a lack) of zeros of the polynomial matrix $R(\sigma)$. Although such computations can be performed using polynomial division and the theory of polynomial matrix triangularization, such polynomial computational tasks are considered to be exhaustive from point of view numerical stability and floating point error accumulation, especially for large dimensions and degrees of matrices $R(\sigma)$. Hence matrix computational procedures are preferred in practice.

We can now immediately apply the above criterion to the kernel representation (4) of our discrete linear repetitive process. In our problem we can in fact derive a criterion for controllability for the repetitive process which involves purely linear algebraic computations instead of polynomial computations due to the first order nature of our kernel representation (4).

Introduce now the notation, $M_0 = Q_0\hat{K} - \hat{D}_0$ and $N_0 = Q_0\hat{B} - \hat{D}$. Also we shall now use the classical state space terminology of calling a pair of matrices $(A, B)$ controllable if they satisfy the following.

**Definition 3** A pair of real matrices $(A \;\; B)$, where $A$ square and $B$ has the same number of rows as $A$ is said to be controllable provided rank$[B \;\; AB \;\; \ldots A^{n-1}B] = n$.

**Theorem 4** *The behavior of $W(k)$ represented by (4) is controllable if, and only if, the pair $(M_0, \; M_0N_0)$ is controllable.*

**Proof.** The kernel representation (4) is of the form

$$(\sigma I + M_0)Y(k) + \sigma N_0 U(k) = 0$$

Observe that the matrix $R(\sigma)$ in this representation has full row rank. Moreover the above representation is also of the form

$$(\sigma[I \;\; N_0] + [M_0 \;\; 0]) \begin{bmatrix} Y(k) \\ U(k) \end{bmatrix} = 0$$

Note that

$$\ker[I \;\; N_0] = \operatorname{im} \begin{bmatrix} -N_0 \\ I \end{bmatrix}$$

Consider the nonsingular transformation

$$Y(k) = -N_0 Z^1(k) + Z^2(k) \qquad U(k) = Z^1(k) \tag{6}$$

of the manifest variables $W(k)$. Then it can be shown that the behavior of $W$ is controllable if, and only if, $Z^1$, $Z^2$ is controllable since the above transformation is nonsingular. The behavior of these new variables is obtained by substituting the transformation in the kernel representation, which gives

$$(\sigma I + M_0)Z^2(k) - (M_0N_0)Z^1(k) = 0 \tag{7}$$

The result thus follows on using the well known classical controllability criterion of state space theory.

A major advantage of the above result is now clear. The criterion of controllability of (4) is stated in terms of the controllability of a pair of matrices in its representation instead of in terms of the roots of the polynomial matrix in the kernel representation. Hence the above criterion can be computed using matrix computations and is equivalent to that of Theorem 1 of [5].

In physical terms, the property of controllability characterized here is equivalent to the existence of an admissible input sequence which will drive the process to produce a pre-specified pass profile on a pre-specified pass number — see [2] for a discussion of why this is physically meaningful for repetitive processes. As with other classes of 2D linear systems, it is also essential to note that controllability is not a single concept (see [5] and the relevant cited references).

In the next section we shall utilize the representation (7) to provide some original results on the critical problem of finding a stabilizing controller for the repetitive processes considered.

## 5 Stabilization and eigenvalue assignment

The problem of stabilizing discrete linear repetitive processes described by (1) and (2) is now considered. We aim to solve this problem in the output feedback form as this is most desirable from a practical point of view. Consider the behavior of the manifest variables $W(k)$ which consists of the variables $Y(k)$ and $U(k)$ which are traditionally output and input variables of the repetitive process. This behavior is given by the kernel representation (4). A stabilizing controller for this behavior is defined as follows.

**Definition 4** *Let $\mathcal{B}$ be the behavior of $W$ represented by (4). A behavior $\mathcal{B}_c$ represented by $R_c(\sigma)W_c(k) = 0$ is said to be a stabilizing controller for $\mathcal{B}$ if 1) the number of variables $W_c$ are same as that of $W$ and 2) the behavior $\mathcal{B} \cap \mathcal{B}_c$ is asymptotically stable. If such a behavior $\mathcal{B}_c$ exists we call $\mathcal{B}$ stabilizable.*

These conditions imply that $W_c$ can be partitioned in the variables $Y_c$, $U_c$ of same dimensions as that of $Y$, $U$ respectively and that the behavior represented by

$$\begin{bmatrix} R(\sigma) \\ R_c(\sigma) \end{bmatrix} W(k) = 0$$

is autonomous and asymptotically stable and moreover we have

**Definition 5** *Let $W$ be partitioned as* col $(Y \ U)$ *in which $U$ is a maximal family of free variables (or inputs) and let $W_c$ be manifest variables of the stabilizing controller with the partition* col $(Y_c \ U_c)$. *Then $\mathcal{B}_c$ is said to be a regular stabilizing controller if $Y_c$ are free variables in the behavior $\mathcal{B}_c$.*

In the notation of the Theorem 4 of the last section we can first establish the stabilizability of $\mathcal{B}$ as follows. Recall that in the classical notion of stabilizability, a pair of matrices $(A \ B)$ ($A$ square and $B$ with same number of rows as in $A$) is said to be stabilizable if there exists a matrix $F$ such that $A + BF$ is Hurwitz i.e. has all its eigenvalues inside the unit circle.

**Proposition 3** *$\mathcal{B}$ is stabilizable if the pair $(M_0, \ M_0N_0)$ is stabilizable.*

**Proof.** As in the case of Theorem 4 of the last section, consider the behavior of $Z^1$ and $Z^2$ given by (7). If the pair $(M_0, \ M_0N_0)$ is stabilizable, there exists $F$ such that the law $Z^1 = FZ^2$ stabilizes the behavior, i.e. the behavior

$$\begin{bmatrix} M_0 & -M_0N_0 \\ -F & I \end{bmatrix} \begin{bmatrix} Z^2 \\ Z^1 \end{bmatrix} = 0$$

is asymptotically stable. However the transformation (6) is nonsingular, hence substituting $Z^1$, $Z^2$ in terms of $Y$, $U$ it follows that there is a behavior $\mathcal{B}_c$ such that $\mathcal{B} \cap \mathcal{B}_c$ is nonsingular and asymptotically stable.

Although the above proposition gives a sufficient condition for stabilizability of the behavior of $W$ it is as yet not clear whether there is a regular stabilizing controller. This problem is addressed is currently under investigation and will be reported on in due course.

## 6 Conclusions

It is shown in this paper that the class of discrete linear repetitive processes can be fruitfully investigated using the ideas of behavioral systems theory. Criteria for stability and controllability are derived in a direct manner using behavioral concepts. These conditions are also expressed in terms of algebraic conditions reminiscent of the classical state space theory. Our major conclusion is that the analysis and control of linear repetitive processes in the behavioral setting is a potentially very powerful approach. This paper has reported the first substantial progress in this respect.

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