CHAOS MAY BE AN OPTIMAL PLAN

Markos Papageorgiou

Dynamic Systems and Simulation Laboratory, Department of Production Engineering and Management,
Technical University of Crete, University Campus, 73100 Chania, GREECE
E-mail: markos@dssl.tuc.gr
Fax: +30-28210-37584

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Abstract

An interpretation of some chaotic systems as the result of optimal decisions is presented. First, a generalised discrete-time two-person game is introduced that may be solved by Dynamic Programming. Then, a specific game of this type is formulated whose optimal solution transforms an originally linear discrete-time system into a well-known discrete-time chaotic system. Finally, a particular continuous-time optimal control problem is formulated whose optimal feedback solution transforms an originally linear continuous-time system into a well-known continuous-time chaotic system.

1 Introduction

Chaotic systems have attracted the attention of control engineers trying to eliminate the chaotic behaviour by application of suitable feedback laws [3]. On the other hand, it appears curious enough to consider the transformation of simple, deterministic, non-chaotic systems into chaotic systems via optimal feedback resulting from suitably defined optimisation problems. The possibility of interpreting chaotic systems as optimally controlled systems or optimal games has been known in Economic Science since the mid-80's [1,2].

This paper defines a generalised two-person game (section 2) that may be solved by Dynamic Programming. A specific game of this type is then formulated (section 3) whose optimal solution transforms an originally linear discrete-time system into a well-known discrete-time chaotic system. Moreover, a particular continuous-time optimal control problem is formulated whose optimal feedback solution transforms an originally linear continuous-time system into a well-known continuous-time chaotic system (section 4). In brief, both discrete-time and continuous-time chaotic systems may appear as the result of optimal decision processes.

2 A Generalized Two-Person Game

Consider a discrete-time deterministic two-person game characterized by a state vector $\mathbf{x}(k)$, with time index k = 0,1,...,K, taking values from a discrete or continuous set X_k. Consider a disjoint division of X_k, such that $X_k^1 \cup X_k^2 = X_k$ and $X_k^1 \cap X_k^2 = \emptyset$. If $\mathbf{x}(k) \in X_k^1$, then player 1 is making a decision, while if $\mathbf{x}(k) \in X_k^2$, then player 2 is making a decision. The evolution of the state may be described by a difference equation

$$\mathbf{x}(k+1) = \mathbf{f}_k[\mathbf{x}(k), \mathbf{u}(k)]\,,\ \mathbf{x}(0) = \mathbf{x}_0 \tag{1}$$

where $\mathbf{u}(k)$ is the decision vector to be selected from an admissible region $\mathbf{u}(k) \in U_k^1[\mathbf{x}(k)]$ or $\mathbf{u}(k) \in U_k^2[\mathbf{x}(k)]$ if players 1 or 2, respectively, are making the decision. An objective function

$$J = \theta[\mathbf{x}(K)] + \sum_{k=0}^{K-1} \varphi_\kappa[\mathbf{x}(k), \mathbf{u}(k)] \tag{2}$$

should be minimized or maximized by players 1 or 2, respectively.

The introduced game differs from common two-person games (e.g. chess) because the player to make a decision at time k, depends upon the current state $\mathbf{x}(k)$ rather than on strict alternation. Common two-person games may be expressed in a generalized form as above by augmenting their state vector with a new state variable $\tilde{x}(k) \in \{0,1\}$ using the state equation

$$\tilde{x}(k+1) = 1 - \tilde{x}(k). \tag{3}$$

Then, defining $\tilde{X}^1 = \{1\}, \tilde{X}^2 = \{0\}$, players 1 or 2 are making a decision if $\tilde{x}(k) \in \tilde{X}^1$ or $\tilde{x}(k) \in \tilde{X}^2$, respectively, which imposes a strict alternation.

The introduced generalized two-person game may be solved, like common two-person games, via Dynamic Programming (min-max procedure), the only difference being that, at each stage k, a minimization or maximization may take place according to the current decision maker 1 or 2, respectively, rather than in a strictly alternating order.

Several generalizations, including a stochastic version, of the formulated game may be introduced.

3 Chaos as an Optimal Game

Consider a specific generalised two-person game with state $x(k)\in R$, $X^1 = \{x \mid x \le \alpha/2\beta\}$, $X^2 = \{x \mid x>\alpha/2\beta\}$, where α, β are non-negative parameters, and state equation

$$x(k+1) = Ax(k) + u(k) \tag{4}$$

with A>0. The objective function at each stage is composed as follows. First, there is a standard objective function

$$\varphi_s[x(k), u(k)] = \frac{1}{2}u(k)^2 + \frac{A\alpha}{2\beta}x(k) - \frac{1}{2}(A^2-\alpha A+1)x(k)^2 - \frac{A\beta}{3}x(k)^3. \tag{5}$$

Furthermore, there is an additional objective function

$$\varphi_a[x(k)] = \frac{A}{12\beta^2}[a^2 - 4\beta x(k)]^{3/2} \tag{6}$$

that is added to or subtracted from the standard objective if player 1 or 2, respectively, was the decision maker at stage k–1. To be conform with the game formulation of section 2, we augment the system state via

$$y(k+1) = \text{sign}\,[x(k) - \frac{\alpha}{2\beta}] \tag{7}$$

where sign(η) equals 1 or –1 if η>0 or η≤0, respectively, and we obtain an augmented state $\mathbf{x} = [x\ y]^T$. The objective function (2) is then defined via

$$\varphi[\mathbf{x}(k), u(k)] = \varphi_s[x(k), u(k)] - y(k) \cdot \varphi_a[x(k)]. \tag{8}$$

For the final cost we assume

$$\theta[\mathbf{x}(K)] = \frac{A\alpha}{2\beta}x(K) - \frac{1}{2}x(K)^2 - y(K)\varphi_a[x(K)]. \tag{9}$$

The admissible decision region is common for both players and is defined by

$$u(k) \le \frac{\alpha^2}{4\beta} - Ax(k) \tag{10}$$

which avoids for the square root argument in (6) to become negative. This completes the problem formulation.

To solve this problem, one may provide the Bellman-equation

$$V_k[\mathbf{x}(k)] = \underset{u\in U}{\text{opt}}\ \{\varphi[\mathbf{x}(k), u(k)] + V_{k+1}[\mathbf{x}(k+1)]\} \tag{11}$$

with terminal condition

$$V_K[\mathbf{x}(K)] = \theta[\mathbf{x}(K)] \tag{12}$$

where $V_k[\mathbf{x}(k)]$ is the optimal objective function to go and opt in (11) corresponds to min or max if $x(k)\in X^1$ or $x(k)\in X^2$, respectively.

For k = K–1, equation (11) yields with (4), (7), (12)

$$\begin{aligned} V_{K-1}[\mathbf{x}(K-1)] = \underset{u\in U}{\text{opt}}\ \{ & \frac{1}{2}u(K-1)^2 + \frac{A\alpha}{2\beta}x(K-1) \\ & - \frac{1}{2}(A^2-\alpha A+1)x(K-1)^2 \\ & - \frac{A\beta}{3}x(K-1)^3 - y(K-1)\varphi_a[x(K-1)] \\ & + \frac{A\alpha}{2\beta}[Ax(K-1)+u(K-1)] \\ & - \frac{1}{2}[Ax(K-1)+u(K-1)]^2 \\ & - \text{sign}[x(K-1)-\frac{\alpha}{2\beta}]\frac{A}{12\beta^2} \\ & [\alpha^2-4\beta[Ax(K-1)+u(K-1)]]^{3/2}\}. \end{aligned} \tag{13}$$

Setting the derivative w.r.t. u(K–1) of the term under optimization equal to zero, one obtains after some calculations the unique solution

$$u(K-1) = (\alpha-A)x(K-1) - \beta x(K-1)^2 \tag{14}$$

that may be readily shown to satisfy the control constraint (10). Taking the second derivative w.r.t. u(K–1) of the term under optimization, one obtains the term

$$-\text{sign}[x(K-1)-\frac{\alpha}{2\beta}]\cdot A\cdot[\alpha^2-4A\beta x(K-1)-4\beta u(K-1)]^{-1/2}. \tag{15}$$

Substituting u(K–1) from (14), the term in (15) becomes

$$A[\alpha-2\beta x(K-1)]^3 \tag{16}$$

which is nonnegative if $x(K-1)\in X^1$ and negative if $x(K-1)\in X^2$. Hence, the solution (14) is a common optimal policy for both players, because it will minimize or maximize the term under optimization in (13) according to the first or second player, respectively, being the decision maker.

Substituting (14) into (13) yields after some calculations

$$V_{K-1}[\mathbf{x}(K-1)] = \frac{A\alpha^3}{12\beta^2}+\frac{A\alpha}{2\beta}x(K-1)-\frac{1}{2}x(K-1)^2$$
$$-y(K-1)\frac{A}{12\beta^2}[\alpha-4\beta x(K-1)]^{3/2}. \qquad (17)$$

Because V_{K-1} in (17) is identical to V_K (see (9), (12)) except for a constant term, the Bellman-equation will provide an identical optimal feedback for k = K–2, and in fact for all previous stages k = K–3,..., 0. Hence the optimal policy for both players in this game reads

$$u(k) = (\alpha-A)x(k)-\beta x(k)^2, \quad k=0,\dots, K-1. \qquad (18)$$

Substituting (18) into (4), one obtains the optimal state evolution

$$x(k+1) = \alpha x(k) - \beta x(k)^2 \qquad (19)$$

which is a well-known [4] discrete-time system that is stable for values $\alpha=\beta$ less than about 3.3, periodic for $\alpha=\beta$ less than about 3.6 and higher than about 3.3, and chaotic for $\alpha=\beta$ higher than about 3.6. Hence, for the corresponding range of α, β values, the formulated generalized two-person game transforms the initial linear system (4) into an optimally controlled chaotic system.

4 Chaos as an Optimally Controlled System

Consider the continuous-time linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{u} \qquad (20)$$

with state vector $\mathbf{x}\in R^3$, $\mathbf{x}^T = [x\ y\ z]$, control vector $\mathbf{u}\in R^3$, and time-invariant, diagonal state matrix $\mathbf{A}\in R^{3\times3}$, $\mathbf{A} = \mathbf{diag}(a_1, a_2, a_3)$. We look for an optimal feedback law for (20) that minimizes the cost criterion

$$J = \theta[\mathbf{x}(T)] + \int_0^T \varphi[\mathbf{x}(t), \mathbf{u}(t)]dt \qquad (21)$$

where

$$\varphi(\mathbf{x},\mathbf{u}) = \frac{1}{2}(\frac{b}{a}u_1^2+u_2^2+u_3^2) + u_2xz - u_3xy$$
$$+\frac{1}{2}\frac{b}{a}(a^2+ab-a_1^2)x^2+\frac{1}{2}(1+ab-a_2^2)y^2$$
$$+\frac{1}{2}(c^2-a_3^2)z^2-b(a+1)xy$$
$$+\frac{1}{2}x^2z^2+\frac{1}{2}x^2y^2 - bx^2z + (1+a_2-a_3-c)\,xyz \qquad (22)$$

and

$$\theta(\mathbf{x})=\frac{1}{2}\frac{b}{a}(a_1+a)x^2+\frac{1}{2}(1+a_2)y^2+\frac{1}{2}(c+a_3)z^2-bxy \qquad (23)$$

where a, b, c have positive constant values.

The Hamilton-Jacobi-Bellman equation for this problem reads

$$\frac{\partial V}{\partial t}+\min_{\mathbf{u}}\{\varphi(\mathbf{x},\mathbf{u})+\frac{\partial V^T}{\partial \mathbf{x}}(\mathbf{A}\mathbf{x}+\mathbf{u})\}=0 \qquad (24)$$

with the boundary condition

$$V[\mathbf{x}(T),T] = \theta[\mathbf{x}(T)] \qquad (25)$$

where $V(\mathbf{x},t)$ is the optimal cost function to go.

We claim that $V(\mathbf{x},t)$ is given by

$$V(\mathbf{x},t)=V(\mathbf{x})=$$
$$\frac{1}{2}\frac{b}{a}(a_1+a)x^2+\frac{1}{2}(1+a_2)y^2+\frac{1}{2}(c+a_3)z^2-bxy \qquad (26)$$

which immediately verifies (25).

Substituting (26) in (24) and setting the derivative w.r.t. **u** of the term under minimization equal to zero, one obtains after some calculations the unique solution

$$u_1 = -(a_1+a)\,x + ay$$
$$u_2 = bx - (1+a_2)\,y - xz \qquad (27)$$
$$u_3 = -(a_3+c)z + xy.$$

This is a minimizing feedback law because the Hessian matrix w.r.t. **u** of the term under minimization in (24) is clearly positive definite. Substitution of (26), (27) may be seen, after many calculations, to verify (24), hence (27) is a unique optimal feedback for the formulated optimal control problem.

Substituting the optimal feedback (27) into the system equation (20) yields

$$
\begin{aligned}
\dot{x} &= a(y - x) \\
\dot{y} &= bx - y - xz \\
\dot{z} &= -cz + xy
\end{aligned} \tag{28}
$$

which is a well-known [4] continuous-time chaotic system (the Lorenz strange attractor). Thus, the formulated optimal control problem transforms an originally linear continuous-time system into the chaotic Lorenz system, that may therefore be viewed as an optimally controlled system.

5 Conclusion

The paper provided two examples of simple systems that are rendered chaotic via optimal control. In the first example, the original discrete-time system is represented as a two-person game while in the second example three interconnected continuous-time state-space equations are used. At present there is no obvious application of this idea other than the curiosity of the fact that some chaotic dynamic systems can be understood as optimally controlled systems by appropriate selection of an objective function.

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