ADAPTIVE CHAOTIC SYNCHRONIZATION THROUGH DECENTRALIZED EXTENDED KALMAN-TYPE OBSERVERS

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Abstract

This paper deals with the problem of adaptive observer-based synchronization of chaotic dynamical systems. The proposed approach consists in estimating the states and the unknown parameters in a separate way to reduce computational requirements. A rigorous stability analysis is performed and a judicious parametrization of the algorithm is proposed to enlarge the bassin of attraction with high tracking ability. The results are illustrated by numerical examples of adaptive synchronization of *Chua*'s system with application to the problem of encoded signal transmission in secure communications.

1 Introduction

Synchronization of dynamical chaotic systems has received growing attention during the last decade due to its potential applications to secure communications problems. From a control theory point of view, chaos synchronization can be considered as an observer design problem in the sense that the receiver system is an observer of the chaotic transmitter system [1, 2, 3]. Few results, however, has been established to deal with the problem of synchronizing chaotic systems when the designer of the receiver doesn't know some of the parameters of the transmitter, this problem is referred to adaptive (non linear) observers [4, 5] or adaptive synchronization [6], the lack of knowledge of some parameters may corrupt the synchronization and even break it. On the other hand, adaptive synchronization is an essential topic in secure communications when parameter modulation technique is used for message transmission, this approach consists in encoding the information message into the chaotic transmitter and the receiver should be able to recover the message (see [7] and the references inside for an overview of decoding observer-based techniques).

In this contribution, we propose a useful and decentralized algorithm for the states and parameters estimation of a large class of chaotic transmitters in the deterministic context. We show that the proposed technique is equivalent to the global extended Kalman observer when the state vector is completed by the unknown parameters. Some connections between the basin of attraction of the observer and the design of weighting arbitrary matrices are established. Performance of the proposed approach will be shown trough numerical simulations on the classical *Chua*'s circuit with applications in secure communications.

2 The decentralized adaptive synchronization algorithm

Consider a chaotic discrete-time transmitter system written in the general form :

$$x_{k+1} = f(x_k, y_{1k}, ..., y_{qk}, \theta) \tag{1a}$$

$$y_{ik} = C_i^x x_k \qquad i = 1, \ldots, q \tag{1b}$$

where $x_k \in \mathbb{R}^n$ is the state vector of the transmitter, f is a differentiable nonlinear function, y_{ik} is the i^{th} transmitted chaotic signal, C_i^x is the i^{th} row of a known real matrices $C^x \in \mathbb{R}^{q \times n}$ and $\theta = \theta_k = \theta_{k+1} \in \mathbb{R}^p$ is an unknown constant parameter vector that possibly contains the encoding messages in case of secure communications.

To achieve adaptive chaotic synchronization, the objective is to design an observer-based receiver for transmitter system (1a)-(1b) that provides both estimates $\hat{x}_k$ and $\hat{\theta}_k$ of the unknown transmitter states and parameters respectively based on the transmission of one or more driving signals y_{ik},$i = 1, \ldots, q$.

An intuitive approach consists in using nonlinear estimation techniques on the augmented system where the state vector has been extended with the unknown parameters. However, in addition to the large computational requirements, we notice that this approach is very sensitive to initial conditions with some divergence problems occur when the initialization is not sufficiently close to the actual values. In order to avoid the previous problems, we propose to estimate the state and the parameter vectors in a separate way. Therefore the dynamic of the receiver is :

$$\hat{x}_{k+1} = l(\hat{x}_k, y_{1k}, ..., y_{qk}, \hat{\theta}_k) \tag{2}$$

$$\hat{\theta}_{k+1} = m(\hat{x}_k, y_{1k}, ..., y_{qk}, \hat{\theta}_k) \tag{3}$$

and we want to design the functions $m(.)$ and $l(.)$ in order to ensure local asymptotic convergence :

$$\lim_{k\to\infty}(x_k - \hat{x}_k) = 0 \tag{4}$$

$$\lim_{k\to\infty}(\theta - \hat{\theta}_k) = 0 \tag{5}$$

for all $(x_0 - \hat{x}_0) \in \mathcal{B}_0^x$ and $(\theta_0 - \hat{\theta}_0) \in \mathcal{B}_0^\theta$ where $\mathcal{B}_0^x$ and $\mathcal{B}_0^\theta$ are large enough open subsets ($\mathcal{B}^x = \{x \;:\; \|x\| \leq \varepsilon^x,\ \varepsilon^x \geq 0\}$).

Using standard Kalman theory, the decoupled states and parameters estimation algorithm that we propose is of the following form :

Stage 1 : states estimation

$$\hat{x}_{k+1} = \hat{x}_{k+1/k} + K_{k+1}^x e_{k+1} \tag{6}$$

$$K_{k+1}^x = \Phi_{k+1}\Delta_{k+1} \tag{7}$$

$$P_{k+1/k}^x = \begin{pmatrix} F_k^x & F_k^\theta \end{pmatrix}\begin{pmatrix} P_k^x & P_k^{x\theta} \\ P_k^{x\theta^T} & P_k^\theta \end{pmatrix}\begin{pmatrix} F_k^{x^T} \\ F_k^{\theta^T} \end{pmatrix} + Q_k^x \tag{8}$$

$$P_{k+1}^x = P_{k+1/k}^x - K_{k+1}^x \Phi_{k+1}^T \tag{9}$$

Stage 2 : parameters estimation

$$\hat{\theta}_{k+1} = \hat{\theta}_k + K_{k+1}^\theta e_{k+1} \tag{10}$$

$$K_{k+1}^\theta = \Psi_{k+1}\Delta_{k+1} \tag{11}$$

$$P_{k+1/k}^\theta = P_k^\theta + Q_k^\theta \tag{12}$$

$$P_{k+1}^\theta = P_{k+1/k}^\theta - K_{k+1}^\theta \Psi_{k+1}^T \tag{13}$$

where

$$\hat{x}_{k+1/k} = f(\hat{x}_k, \hat{\theta}_k, y_{1k}, ..., y_{qk}, u_k) \tag{14}$$

$$\Delta_{k+1} = (C^x P_{k+1/k}^x C^{x^T} + R_{k+1})^{-1} \tag{15}$$

$$\Phi_{k+1} = P_{k+1/k}^x C^{x^T} \tag{16}$$

$$\Psi_{k+1} = P_{k+1/k}^{x\theta^T} C^{x^T} \tag{17}$$

$$P_{k+1/k}^{x\theta} = F_k^x P_k^{x\theta} + F_k^\theta P_k^\theta + Q_k^{x\theta} \tag{18}$$

$$P_{k+1}^{x\theta} = P_{k+1/k}^{x\theta} - K_{k+1}^x \Psi_{k+1}^T \tag{19}$$

with

$$F_k^x = \left.\frac{\partial f(x_k, y_{1k}, ..., y_{qk}, \theta)}{\partial x_k}\right|_{\hat{x}_k, \hat{\theta}_k} \tag{20}$$

$$F_k^\theta = \left.\frac{\partial f(x_k, y_{1k}, ..., y_{qk}, \theta)}{\partial \theta}\right|_{\hat{x}_k, \hat{\theta}_k} \tag{21}$$

$$e_{k+1} = y_{k+1} - C^x \hat{x}_{k+1/k} = C^x(x_{k+1} - \hat{x}_{k+1/k}) \tag{22}$$

We set at the initialization $P_0^x = \mu^x I_n$, $P_0^\theta = \mu^\theta I_q$ and $P_0^{x\theta} = 0$ where μ^x and μ^θ are positive scalars, in particular, large for bad initializations. Hereafter, Q_k^x, $Q_k^{x\theta}$, Q_k^θ and R_{k+1} – which usually correspond to system and measurement noise covariance matrices in a stochastic context – will be used as free instrumental matrices whose choices are crucial to control stability and rate of convergence of the algorithm.

3 Convergence analysis

Before we give the main result in this section, we show first that the proposed separate-bias observer is equivalent to the global state and bias estimation algorithm obtained from the Extended Kalman Observer (EKO) where the states are augmented by the bias vector. Indeed, (7), (11), (15), (16) and (17) may be written as :

$$\begin{pmatrix} K_{k+1}^x \\ K_{k+1}^\theta \end{pmatrix} = \begin{pmatrix} P_{k+1/k}^x & P_{k+1/k}^{x\theta} \\ P_{k+1/k}^{x\theta^T} & P_{k+1/k}^\theta \end{pmatrix}\begin{pmatrix} C^{x^T} \\ 0 \end{pmatrix}\left(\begin{pmatrix} C^x & 0 \end{pmatrix} \times \begin{pmatrix} P_{k+1/k}^x & P_{k+1/k}^{x\theta} \\ P_{k+1/k}^{x\theta^T} & P_{k+1/k}^\theta \end{pmatrix}\begin{pmatrix} C^{x^T} \\ 0 \end{pmatrix} + R_{k+1}\right)^{-1} \tag{23}$$

and by the use of (8), (12) and (18) we have :

$$\begin{pmatrix} P_{k+1/k}^x & P_{k+1/k}^{x\theta} \\ P_{k+1/k}^{x\theta^T} & P_{k+1/k}^\theta \end{pmatrix} = \begin{pmatrix} F_k^x & F_k^\theta \\ 0 & I_q \end{pmatrix}\begin{pmatrix} P_k^x & P_k^{x\theta} \\ P_k^{x\theta^T} & P_k^\theta \end{pmatrix} \times \begin{pmatrix} F_k^x & F_k^\theta \\ 0 & I_q \end{pmatrix}^T + \begin{pmatrix} Q_k^x & Q_k^{x\theta} \\ Q_k^{x\theta^T} & Q_k^\theta \end{pmatrix} \tag{24}$$

also from (9), (13) and (19) we deduce that :

$$\begin{pmatrix} P_{k+1}^x & P_{k+1}^{x\theta} \\ P_{k+1}^{x\theta^T} & P_{k+1}^\theta \end{pmatrix} = \left(\begin{pmatrix} I_n & 0 \\ 0 & I_q \end{pmatrix} - \begin{pmatrix} K_{k+1}^x \\ K_{k+1}^\theta \end{pmatrix}\begin{pmatrix} C^x & 0 \end{pmatrix}\right) \times \begin{pmatrix} P_{k+1/k}^x & P_{k+1/k}^{x\theta} \\ P_{k+1/k}^{x\theta^T} & P_{k+1/k}^\theta \end{pmatrix} \tag{25}$$

By using the notations $K_{k+1} = \begin{pmatrix} K_{k+1}^x \\ K_{k+1}^\theta \end{pmatrix}$, $P_{k+1} = \begin{pmatrix} P_{k+1}^x & P_{k+1}^{x\theta} \\ P_{k+1}^{x\theta^T} & P_{k+1}^\theta \end{pmatrix}$, $P_{k+1/k} = \begin{pmatrix} P_{k+1/k}^x & P_{k+1/k}^{x\theta} \\ P_{k+1/k}^{x\theta^T} & P_{k+1/k}^\theta \end{pmatrix}$, $F_k = \begin{pmatrix} F_k^x & F_k^\theta \\ 0 & I_q \end{pmatrix}$, $C = \begin{pmatrix} C^x & 0 \end{pmatrix}$ and $Q_k = \begin{pmatrix} Q_k^x & Q_k^{x\theta} \\ Q_k^{x\theta^T} & Q_k^\theta \end{pmatrix}$, it's easy to verify that (23), (24) and (25) are the propagation equations of the augmented EKO.

In the following we show, by the use of the exact linearization technique presented in [8], that the design of the instrumental matrices, in particular Q_k^x and Q_k^θ, plays a central role to improve tracking and rate of convergence of the proposed separate-bias observer. In the rest of the paper and without loss of generality, we set $Q_k^{x\theta} = 0$.
Let us introduce the error vectors $\tilde{x}_{k+1}$, $\tilde{x}_{k+1/k}$ and $\tilde{\theta}_{k+1}$:

$$\tilde{x}_{k+1} = x_{k+1} - \hat{x}_{k+1} \tag{26}$$

$$\tilde{x}_{k+1/k} = x_{k+1} - \hat{x}_{k+1/k} \tag{27}$$

$$\tilde{\theta}_{k+1} = \theta_{k+1} - \hat{\theta}_{k+1} = \theta - \hat{\theta}_{k+1} \tag{28}$$

and consider the following candidate Lyapunov function V_{k+1} :

$$V_{k+1} = \begin{pmatrix} \tilde{x}_{k+1} \\ \tilde{\theta}_{k+1} \end{pmatrix}^T P_{k+1}^{-1}\begin{pmatrix} \tilde{x}_{k+1} \\ \tilde{\theta}_{k+1} \end{pmatrix} \tag{29}$$

The goal is to point out conditions so that V_{k+1} is a decreasing sequence.

First of all, we introduce an unknown diagonal matrix β_k to parametrize all the errors due to the first order linearization technique of the nonlinear function f [8]. We have then the following exact relation :

$$\tilde{x}_{k+1/k} = \beta_k (F_k^x \tilde{x}_k + F_k^\theta \tilde{\theta}_k) \quad (30)$$

instead of the following approximation usually used in the literature :

$$\tilde{x}_{k+1/k} \approx F_k^x \tilde{x}_k + F_k^\theta \tilde{\theta}_k \quad (31)$$

which are correct only at the neighborhood of the actual trajectories.

The unknown diagonal matrix β_k depends on how far $\tilde{x}_k$ and $\tilde{\theta}_k$ are from zeros. In the following, (30) will be used in the Lyapunov function in order to evaluate the propagation errors and to point out connections between convergence of the proposed observer and the linearisation errors. From (23) to (25), we have :

$$\begin{aligned} K_{k+1} &= P_{k+1} C^T R_{k+1}^{-1} \\ &= P_{k+1/k} C^T \left(C P_{k+1/k} C^T + R_{k+1} \right)^{-1} \quad (32) \\ P_{k+1}^{-1} &= P_{k+1/k}^{-1} + C^T R_{k+1}^{-1} C \quad (33) \end{aligned}$$

Using (32), (6) and (10), the Lyapunov function V_{k+1} (29) becomes :

$$\begin{aligned} V_{k+1} &= \begin{pmatrix} \tilde{x}_{k+1/k} \\ \tilde{\theta}_k \end{pmatrix}^T P_{k+1}^{-1} \begin{pmatrix} \tilde{x}_{k+1/k} \\ \tilde{\theta}_k \end{pmatrix} \\ &- \begin{pmatrix} \tilde{x}_{k+1/k} \\ \tilde{\theta}_k \end{pmatrix}^T C^T R_{k+1}^{-1} e_{k+1} - e_{k+1}^T R_{k+1}^{-1} C \begin{pmatrix} \tilde{x}_{k+1/k} \\ \tilde{\theta}_k \end{pmatrix} \\ &+ e_{k+1}^T R_{k+1}^{-1} C P_{k+1} C^T R_{k+1}^{-1} e_{k+1} \quad (34) \end{aligned}$$

Using (33), (30), (22) and the special structure of $C = \begin{pmatrix} C^x & 0 \end{pmatrix}$ we obtain :

$$\begin{aligned} V_{k+1} = &\begin{pmatrix} \tilde{x}_k \\ \tilde{\theta}_k \end{pmatrix}^T F_k^T \check{\beta}_k \left(F_k P_k F_k^T + Q_k \right)^{-1} \check{\beta}_k F_k \begin{pmatrix} \tilde{x}_k \\ \tilde{\theta}_k \end{pmatrix} \\ &+ e_{k+1}^T (-R_{k+1}^{-1} + R_{k+1}^{-1} C^x P_{k+1}^x C^{x^T} R_{k+1}^{-1}) e_{k+1} \quad (35) \end{aligned}$$

with the extended matrix $\check{\beta}_k = \begin{pmatrix} \beta_k & 0 \\ 0 & I_q \end{pmatrix}$.

The Lyapunov sequence $\{V_k\}_{k=1,\ldots}$ is a decreasing one if there exists a positive scalar $\zeta \in]0,1[$ so that :

$$V_{k+1} - V_k \leq -\zeta V_k, \quad (36)$$

or equivalently :

$$\begin{aligned} V_{k+1} - (1-\zeta) V_k = &\begin{pmatrix} \tilde{x}_k \\ \tilde{\theta}_k \end{pmatrix}^T \big(F_k^T \check{\beta}_k \times \\ &\left(F_k P_k F_k^T + Q_k \right)^{-1} \check{\beta}_k F_k - (1-\zeta) V_k \big) \begin{pmatrix} \tilde{x}_k \\ \tilde{\theta}_k \end{pmatrix} \\ &+ e_{k+1}^T (-R_{k+1}^{-1} + R_{k+1}^{-1} C^x P_{k+1}^x C^{x^T} R_{k+1}^{-1}) e_{k+1} \quad (37) \end{aligned}$$

We notice that a sufficient condition to ensure (37) consists in verifying the following couple of inequalities :

$$F_k^T \check{\beta}_k \big(F_k P_k F_k^T + Q_k \big)^{-1} \check{\beta}_k F_k \leq (1-\zeta) P_k^{-1} \quad (38)$$

$$R_{k+1}^{-1} C^x P_{k+1}^x C^{x^T} \leq I_q \quad (39)$$

In the following theorem we give sufficient conditions to ensure asymptotic convergence of the proposed adaptive observer.

Theorem 1 *If we assume that :*

i. The augmented – when the states are completed by the parameters vector – linearized system along the estimator's trajectory is N-locally observable, i.e. there exist a finite integer $N \geq 1$ and positive real numbers γ_1 and γ_2 such that, for all $k \geq N-1$, we have :

$$\gamma_1 I_{n+q} \leq \mathcal{O}_{[k-N+1,k]}^T \operatorname{diag}(R_{k-N+1}^{-1}, \ldots, R_k^{-1}) \times \mathcal{O}_{[k-N+1,k]} \leq \gamma_2 I_{n+q} \quad (40)$$

where $0 < \gamma_1, \gamma_2 < \infty$ and,

$$\mathcal{O}_{[k-N+1,k]} = \begin{pmatrix} C \\ C F_{k-N+1} \\ \vdots \\ C F_{k-1} F_{k-2} \ldots F_{k-N+1} \end{pmatrix} \quad (41)$$

for all $\begin{pmatrix} \hat{x}_k \\ \hat{\theta}_k \end{pmatrix} \in \mathcal{N}_{(x_k,\theta)}$ which denotes a neighborhood of $\begin{pmatrix} x_k \\ \theta_k \end{pmatrix}$,

ii. F_k^x, F_k^θ are uniformly bounded matrices and ${F_k^x}^{-1}$ exists,

iii. The instrumental matrix $\beta_k \in \mathbb{R}^{n\times n}$ satisfies :

$$\bar{\beta}_k \leq \left(\frac{(1-\zeta)\sigma_{\min}(F_k P_k F_k^T + Q_k)}{\sigma_{\max}(F_k^T)\sigma_{\max}(P_k)\sigma_{\max}(F_k)} \right)^{1/2} = \Pi_k^\beta \quad (42)$$

with $0 < \bar{\beta}_k = \max_{i=1,\ldots,n} (|\beta_{ik}|, 1) < \infty$,

iv. The weighting matrix R_{k+1} is chosen as :

$$R_{k+1} = \lambda C^x P_{k+1/k}^x C^{x^T}, \qquad \lambda \geq 1 \quad (43)$$

then (6)-(22) is a decentralized adaptive observer for (1a)-(1b) and :

$$\lim_{k\to\infty} (x_k - \hat{x}_k) = 0 \quad \textit{and} \quad \lim_{k\to\infty} (\theta - \hat{\theta}_k) = 0$$

$\sigma_{\max}$ and $\sigma_{\min}$ denote the maximum and minimum singular values respectively.

Proof : First of all, hypothesis *ii.* and the local observability assumption (40) are introduced to assure that the matrix P_k is bounded from above and below, for more details see [9], while hypothesis *iii.* and *iv.* lead to obtain a decreasing Lyapunov function $\{V_k\}_{k=1,\dots}$. Indeed, using the fact that β_k is diagonal matrices and from (42), we obtain :

$$\left(\sigma_{\max}(\breve{\beta}_k)\right)^2 \leq \frac{(1-\zeta)\sigma_{\min}(P_k^{-1})}{\sigma_{\max}(F_k^T)\sigma_{\max}((F_kP_kF_k^T+Q_k)^{-1})\sigma_{\max}(F_k)} \quad (44)$$

As,

$$\sigma_{\max}\left(F_k^T\breve{\beta}_k(F_kP_kF_k^T+Q_k)^{-1}\breve{\beta}_kF_k\right) \leq \left(\sigma_{\max}(\breve{\beta}_k)\right)^2\sigma_{\max}(F_k^T)\sigma_{\max}((F_kP_kF_k^T+Q_k)^{-1}) \times \sigma_{\max}(F_k) \quad (45)$$

We have then :

$$\begin{aligned}&\sigma_{\max}\left(F_k^T\breve{\beta}_k(F_kP_kF_k^T+Q_k)^{-1}\breve{\beta}_kF_k\right)\\ &\leq \left(\sigma_{\max}(\breve{\beta}_k)\right)^2\sigma_{\max}(F_k^T)\sigma_{\max}((F_kP_kF_k^T+Q_k)^{-1}) \times \\ &\quad \sigma_{\max}(F_k)\\ &\leq (1-\zeta)\sigma_{\min}(P_k^{-1}) \end{aligned} \quad (46)$$

which induce that (38) is satisfied.
On the other hand, substituting (43) into (39), we have :

$$\lambda^{-1}(C_xP_{k+1/k}^xC^{x^T})^{-1}(C_xP_{k+1/k}^xC^{x^T}) \leq I_q \quad (47)$$

which is satisfied for all $\lambda \geq 1$.
Therefore, (45) and (47) induce that that V_k is a decreasing Lyapunov function.

Finally, as P_k is a bounded matrix, it follows from (36) that :

$$0 \leq \mu \begin{pmatrix}\tilde{x}_k\\ \tilde{\theta}_k\end{pmatrix}^T \begin{pmatrix}\tilde{x}_k\\ \tilde{\theta}_k\end{pmatrix} \leq V_k \leq (1-\zeta)^kV_0 \quad (48)$$

which implies :

$$\begin{aligned}0 \leq \mu \lim_{k\to\infty}\left(\begin{pmatrix}\tilde{x}_k\\ \tilde{\theta}_k\end{pmatrix}^T \begin{pmatrix}\tilde{x}_k\\ \tilde{\theta}_k\end{pmatrix}\right) &\leq \lim_{k\to\infty}(V_k)\\ &\leq V_0\lim_{k\to\infty}(1-\zeta)^k = 0\end{aligned} \quad (49)$$

with $\mu I_{n+p} \leq P_k^{-1}$.
Therefore :

$$\lim_{k\to\infty}\begin{pmatrix}x_k-\hat{x}_k\\ \theta-\hat{\theta}_k\end{pmatrix} = 0 \quad (50)$$

Remark 1 **:** The sufficient conditions (42) represents, in a way, the attraction region in the state and parameters spaces. Indeed, in the body of the paper we introduced a linearisation technique in the form of (30) to quantify the distance between the actual and estimated trajectories and in the same time we deduce a sufficient condition (42) on the norm of β_k so that we obtain a decreasing Lyapunov function, i.e. asymptotic convergence of the observer. Contrary to the standard EKO where β_k is assumed to be very close to the identity matrix (that is very small initialization errors), the proposed approach ensures asymptotic convergence for all matrix β_k (which represent the linearisation errors) bounded by Π_k^β. On the other hand, as long as $\begin{pmatrix}\hat{x}_k\\ \hat{\theta}_k\end{pmatrix}$ goes to $\begin{pmatrix}x_k\\ \theta_k\end{pmatrix}$, the matrix β_k converges to the identity and the approximation (31) usually used in the literature becomes valid. The stability analysis is then similar to the linear case.

Design of the arbitrary matrices Q_k^x and Q_k^θ, in order to satisfy the condition (42), depends on the evaluation of the upper bound $\bar{\beta}_k$ on the norm of β_k in particular at the initialization of the algorithm. In practice, from our knowledge of the system to synchronize like *Lorentz* system, *Rossler* system or *Chua* circuit, most often we have an idea on the bounds of the state and parameter vectors to be estimated and by the use of the non linearities $f(.)$ we may evaluate realistic values of $\bar{\beta}_k$. In addition, we should recall that the proposed approach in only local but improve the convergence significantly.

Remark 2 **:** Hereafter, by a quantitative analysis, the aim is to design the weighting matrices Q_k^x and Q_k^θ so that the bound Π_k^β is as large as possible in order to satisfy condition (42).
Before we introduce a simple design of Q_k^x and Q_k^θ, we notice that the upper bound Π_k^β is large when Q_k (consequently Q_k^x and Q_k^θ) is large. Indeed, since the matrices P_k and F_k depend on Q_{k-1} computed at time instant $k-1$, Q_k is a free weighting matrix to enlarge the above upper bound in the sense that we have $\left(\frac{(1-\zeta)\sigma_{\min}(F_kP_kF_k^T+Q_k^1)}{\sigma_{\max}(F_k^T)\sigma_{\max}(P_k)\sigma_{\max}(F_k)}\right)^{1/2} \geq \left(\frac{(1-\zeta)\sigma_{\min}(F_kP_kF_k^T+Q_k^2)}{\sigma_{\max}(F_k^T)\sigma_{\max}(P_k)\sigma_{\max}(F_k)}\right)^{1/2}$ for $Q_k^1 \geq Q_k^2 > 0$. However, we should have in mind that matrices P_{k+1} and $P_{k+1/k}$ have to remain bounded and, in order to avoid numerical instabilities, Q_k can not be chosen extremely large.

Remark 3 **:** In the following we propose a simple and efficient design, which is not unique, for Q_k^x and Q_k^θ :

$$Q_k^x = \zeta^xe_k^Te_kI_n + \varepsilon^xI_n \quad (51)$$
$$Q_k^\theta = \zeta^\theta e_k^Te_kI_p + \varepsilon^\theta I_p \quad (52)$$

where $e_k = y_k - C^x\hat{x}_{k/k-1}$ and the positive scalars ζ^x and ζ^θ have to be chosen sufficiently large especially in case of bad initial conditions and high non-linearities of $f(.)$, ε^x and ε^θ are small enough scalars introduced to avoid numerical instabilities.

The original idea of designing Q_k^x and Q_k^θ is, in fact, the presence of the output error e_k which controls automatically width of the upper bound on β_k. Indeed, in case of high non-linearities and/or arbitrary large initialization errors, the terms $\zeta^xe_k^Te_k$ and $\zeta^\theta e_k^Te_k$ (and consequently the bound) become large.

Remark 4 **:** Considering the weighting matrix (43), we remark that setting high values of λ leads to a very slow convergence rate or even a divergence of the algorithm (the gains K_{k+1}^x and K_{k+1}^θ quickly go to zero) so much so that the choice of λ must be a judicious compromise between stability and rate of convergence of the proposed observer.

Since the proposed approach is only local but may be applied to a very large class of non-linear systems, it is very difficult to give an idea, a priori, on the values of the parameters ζ^x, ζ^θ and λ which depend on many factors such as initializations or degree of non linearity of the system. In practice, suitable values of these parameters are determined, by an iterative approach. From our experience (we have tried this approach on several chaotic systems and on a large variety of physical processes under severe conditions) ζ^x, ζ^θ and λ belong in general to the intervals $\left[10^0 \quad 10^{20}\right]$, $\left[10^0 \quad 10^{20}\right]$ and $[1 \quad 20]$ respectively.

Remark 5 **:** Using the special structures of $F_k = \begin{pmatrix} F_k^x & F_k^\theta \\ 0 & I_q \end{pmatrix}$ and $C = \begin{pmatrix} C_x & 0 \end{pmatrix}$, the observability matrix $\mathcal{O}_{[k-N+1,k]}$ may be partitioned as follows :

$$\mathcal{O}_{[k-N+1,k]} = \left[\mathcal{O}^x_{[k-N+1,k]} \quad \mathcal{O}^\theta_{[k-N+1,k]}\right]$$

with

$$\mathcal{O}^x_{[k-N+1,k]} = \begin{pmatrix} C^x \\ C^x F^x_{k-N+1} \\ \vdots \\ C^x F^x_{k-1} \dots F^x_{k-N+1} \end{pmatrix}$$

$$\mathcal{O}^\theta_{[k-N+1,k]} = \begin{pmatrix} 0 \\ C^x F^\theta_{k-N+1} \\ \vdots \\ C^x\big(F^x_{k-1} \dots F^\theta_{k-N+1} + F^x_{k-1} \dots F^\theta_{k-N+2} + \dots + F^\theta_{k-1}\big) \end{pmatrix}$$

As we can expect, if the original system (1a)-(1b) isn't in some sense "observable" or "persistently excited" that is if $\operatorname{rang}(\mathcal{O}^x_{[k-N+1,k]}) < n$ or $\operatorname{rang}(\mathcal{O}^\theta_{[k-N+1,k]}) < p$, then the observability condition (40) is not verified. In particular, when a state has null initial condition and is not reachable, then all parameters multiplying only this state variable couldn't be identified because they are not persistently excited i.e. the corresponding columns in F_k^θ are zero and then we have rank $\operatorname{rang}(\mathcal{O}^\theta_{[k-N+1,k]}) < p$.

Remark 6 **:** In order to reduce the computational requirements and then the numerical instabilities, in particular, for large scale systems the proposed two stage algorithm may be implemented, in real time applications, on a separate processor system.

4 Numerical examples

Example 1 : *Synchronization of Chua's cirucit*

Consider the example of chaotic synchronization where both transmitter and receiver systems are implemented as *Chua* circuits with unknown constant parameters [10]. The transmitter system, so-called double scroll chatic attractor (see Figure 1), is described by the following set of nonlinear differential equations :

$$\begin{aligned} \dot{x}_1(t) &= \alpha(-x_1(t) + x_2(t) - g(x_1(t))) \\ \dot{x}_2(t) &= x_1(t) - x_2(t) + x_3(t) \\ \dot{x}_3(t) &= -\gamma x_2(t) \end{aligned}$$

where $g(x_1) = m_1 x_1 + \dfrac{m_0 - m_1}{2}(|x_1 + 1| - |x_1 - 1|)$ with $m_0 = -8/7$ and $m_1 = -5/7$. The transmitted driven signal is $y(t) = x_1(t)$. We suppose that the parameters α and λ are unknown but constants (i.e. $\theta = \begin{pmatrix}\alpha & \gamma\end{pmatrix}^T$, $\dot{\theta} = 0$).

Using Euler discretization method, the discrete-time model of *Chua*'s circuit is :

$$\begin{aligned} x_{1k+1} &= (1 - \alpha T)x_{1k} + \alpha T(x_{2k} - g(x_{1k})) \\ x_{2k+1} &= T x_{1k} + (1 - T)x_{2k} + T x_{3k} \\ x_{3k+1} &= x_{3k} - \gamma T x_{2k} \end{aligned}$$

where T is the sampling period and the driven signal is $y_k = x_{1k}$.

The initial conditions for the transmitter were taken as $x_0 = \begin{pmatrix}0.1 & 0.1 & 0.1\end{pmatrix}^T$ while the true values of the supposed unknown parameters are : $\alpha = 15.6$ and $\gamma = 27$. The sampling period is $T = 0.01$ second.

For the receiver, the initial conditions are very bad for the states as well as for the unknown parameters : $\hat{x}_0 = \begin{pmatrix}1 & 1 & 1\end{pmatrix}^T$ and $\hat{\theta}_0 = \begin{pmatrix}10 & 20\end{pmatrix}^T$ with $P_0^x = 10^{10} I_3$ and $P_0^\theta = 10^{10} I_2$. In order to ensure convergence of the algorithm, the weighting matrices Q_k^x, Q_k^θ and R_{k+1} are chosen as in (51), (52) and (43) with $\zeta^x = \zeta^\theta = 10^{10}$, $\lambda = 1$ and $\varepsilon^x = \varepsilon^\theta = 10^{-3}$.

The results of the simulations are reported in Figure 2 and 3. A comparison has been realized with the classical extended Kalman filter when the weighting matrices Q_k^x, Q_k^θ and R_{k+1} are chosen as $Q_k^x = Q^x = 10^{-3} I_3$, $Q_k^\theta = Q^\theta = 10^{-3} I_2$ and $R_{k+1} = R = 10^{-3}$. The plots clearly show high performances of the proposed adaptive observer and the important role of the instrumental matrices Q_k^x, Q_k^θ and R_{k+1} to ensure strong convergence even with very bad initial conditions.

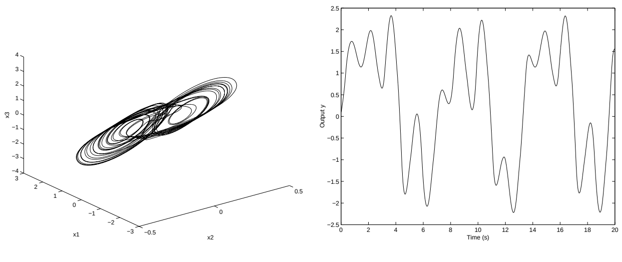

Figure 1: The double-scroll attractor and the output of Chua's circuit

Example 2 : *Secure communications using Chua's cirucit*

In the current example, we are going to present the potential applications of synchronized *Chua*'s systems in secure transmissions. In this case, the driven signal y_k contains a binary

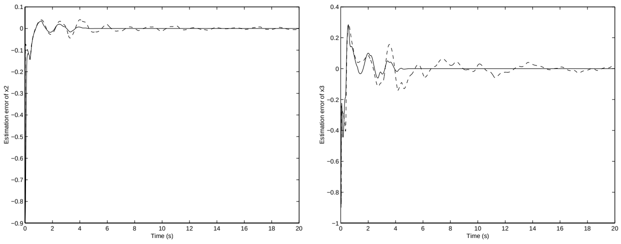

Figure 2: Convergence behavior of $\|\tilde{x}_{2k}\|$ and $\|\tilde{x}_{3k}\|$. Proposed observer (–) and standard EKO (- -)

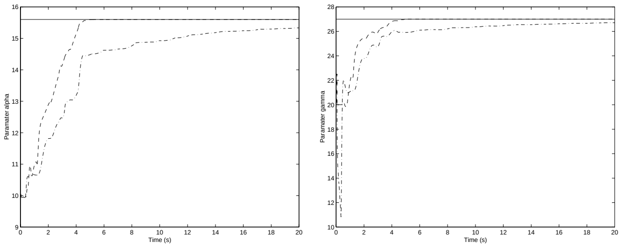

Figure 3: Estimation of parameters α and γ. Proposed observer (- -) and standard EKO (-.)

encoding message s_k that has been used to modulate one or more parameters of the chaotic transmitter.

Then we use the proposed decentralized algorithm to estimate the modulated parameters in order to recover the information signal s_k. Here we suppose that the message modulate the parameters α of the transmitter with the following modulation rule :

$$\alpha_k = \alpha_0 + \alpha_1 s_k$$

where $s_k = 0.5(1 - \text{sgn}\left(\sin\left(\frac{2\pi kT}{T0}\right)\right))$ with $\alpha_0 = 15.6$, $\alpha_1 = 0.5$ and the period of s_k is $T_0 = 20$ s.

The initial conditions for the transmitter are the same as in the first example, we suppose that only the parameter α is unknown.

The initial conditions of the receiver are : $\hat{x}_0 = \begin{pmatrix}1 & 1 & 1\end{pmatrix}^T$ and $\hat{\alpha}_0 = 16.5$ with $P_0^x = 10^{20} I_3$ and $P_0^\theta = 10^{20}$. The weighting matrices Q_k^x, Q_k^θ and R_{k+1} are chosen as in (51), (52) and (43) with $\zeta^x = 1$, $\zeta^\theta = 10^{15}$, $\lambda = 15$ and $\varepsilon^x = \varepsilon^\theta = 10^{-3}$.

Figure 4 shows the secure communication via binary parameter modulation. We easily verify that the proposed algorithm recovers the information message s_k at the end of the communication link without any information on the dynamics of s_k.

5 Conclusion

In this contribution a simple decoupled adaptive observer for chaos synchronization has been derived. Sufficient conditions for local asymptotic stability were established. Convergence and strong tracking ability were illustrated by chaotic synchronization using the classical *Chua*'s circuit.

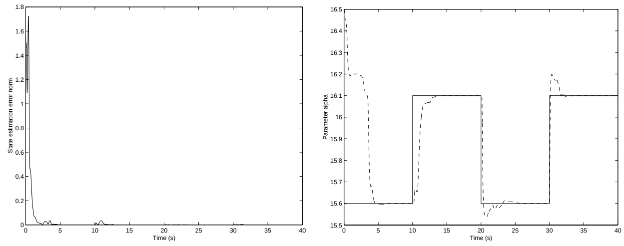

Figure 4: Convergence behavior of $\|\tilde{x}_k\|$ and estimation of the parameter α with the proposed observer

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