On the Principle of Linearized Stability for Nonlinear Hyperbolic Systems

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Abstract. The principle of linearized stability is proved to be valid for a class of hyperbolic nonlinear systems. One example in fluid mechanics is worked out to show how the principle is applied for determining the local stability of its stationary solution. The example is concerned with the irrigation canal system governed by the Saint Venant equation. The principle of linearized stability should be useful in the construction of stabilizing feedback laws for networks of irrigation canals.

1 Introduction and main results

Consider the following linear system of partial differential equations with two independent variables (x,t) :

$$\Sigma_{\text{autonom}} \ : \ \begin{cases} \dot{\psi}(x,t) = A(x)\psi'(x,t), \\ \psi^1(0,t) = D_0\psi^2(0,t), \\ \psi^2(\ell,t) = D_\ell\psi^1(\ell,t), \\ \psi(x,0) = \psi^0(x), \end{cases} \tag{1.1}$$

where $(x,t) \in (0,\ell) \times \mathbb{R}^+$, ℓ is a positive constant, $\psi(x,t) = [\psi^1(x,t)^\tau, \psi^2(x,t)^\tau]^\tau$, $\psi^i(x,t)$, $i = 1,2$, is a vector function for $(x,t) \in (0,\ell)\times\mathbb{R}^+$ such that $\psi^1(x,t) \in \mathbb{R}^p$, $\psi^2(x,t) \in \mathbb{R}^q$, and $p+q = n$, where τ denotes the transpose of a matrix or a vector, the dot and the prime denote derivatives respect to time variable t and space variable x, respectively. Moreover the matrix $A(x)$ is diagonal: $A(x) = \text{diag}(A_1(x), A_2(x))$, where

$$A_1(x) = \text{diag}(\lambda_1(x), \cdots, \lambda_p(x)),$$

$$A_2(x) = \text{diag}(\lambda_{p+1}(x), \cdots, \lambda_{p+q}(x)),$$

and D_0, D_ℓ are real constant matrices of appropriate dimension. We assume that the following hypothesis is satisfied for (1.1):

(H1) $A(\cdot) \in C^1([0,\ell]; \mathbb{R}^{n\times n})$ such that $A_1(x) < 0$ and $A_2(x) > 0$ on $[0,\ell]$.

We consider the nonlinear (autonomous) system governed by semilinear or quasilinear hyperbolic systems of PDE as follows:

$$\Sigma_n \ : \ \begin{cases} \dot{\varphi}(x,t) = A(x,\varphi(x,t))\varphi'(x,t) + F(\varphi(x,t)), \\ \varphi^1(0,t) = D_0\varphi^2(0,t) + F_0(\varphi^2(0,t)), \\ \varphi^2(\ell,t) = D_\ell\ \varphi^1(\ell,t) + F_\ell(\varphi^1(\ell,t)), \\ \varphi(x,0) = \varphi^0(x), \end{cases} \tag{1.2}$$

where $A(x,y) = \text{diag}(\lambda_1(x,y), \cdots, \lambda_n(x,y))$, is an $n\times n$ matrix for $x \in [0,\ell], y \in \mathbb{R}^n$, the equation and the related matrices have the same structure as that we have described for the corresponding linear system (1.1). Moreover, the following condition is satisfied for the nonlinearity.

(H2) Each of the nonlinear functions $F : \mathbb{R}^n \to \mathbb{R}^n$, $F_0 : \mathbb{R}^q \to \mathbb{R}^p$ and $F_\ell : \mathbb{R}^p \to \mathbb{R}^q$ is of class C^3 in some neighborhood of the origin. They satisfy the condition that $F(0) = 0$, $F'(0) = 0$, $F_0(0) = 0$, $F_0'(0) = 0$, $F_\ell(0) = 0$ and $F_\ell'(0) = 0$, where the prime denotes the Jacobian matrix with respect to suitable variables.

The nonlinear system is called semilinear if $A(x,\varphi(x,t)) = A(x,0)$, i.e., $A(x,\varphi(x,t))$ does not depend on the unknown function $\varphi(x,t)$. It is called quasilinear if $A(x,\varphi(x,t)) \neq A(x,0)$, i.e., $A(x,\varphi(x,t))$ depends on the unknown function. In any case we suppose that $A(x) = A(x,0)$ satisfy the hypothesis (H1). Our main result is the following.

Theorem 1.1. *If the null solution is an equilibrium point exponentially stable for the linearized system (1.1), then it is locally exponentially stable for the semilinear system (1.2): there exist some constants ε, M, $\omega > 0$ such that for any initial data φ^0 satisfying the C^1 compatibility condition and $|\varphi^0|_1 < \varepsilon$,*

$$|\varphi(\cdot,t)|_1 \leq Me^{-\omega t}|\varphi^0|_1.$$

Theorem 1.2. *If the linearized system (1.1) with $A(x) = A(0)$ satisfies the condition:*

$$\begin{cases} A_1 + D_\ell^\tau A_2 D_\ell > 0 \\ -A_2 - D_0^\tau A_I D_0 > 0, \end{cases} \tag{1.3}$$

then the null solution is an unstable equilibrium point for (1.1) as well as for the quasilinear system (1.2).

Remark 1.3. The result of Theorem 1.1 is the best that we can expect for the stable case of the semilinear system (1.2). Our proof is constructed by using the semigroup system theory (see [6], [7]) combined with the PDE theory [5]. To determine the local stability of the semilinear system (1.2) it is sufficient to study that of its linearized one.

Remark 1.4. The local existence and the uniqueness of C^1 classical solutions is guaranteed by the Li-Yu theorem (see [5]). Theorem 1.2 is proved as in the proof of Theorem 2 using $V(R)$ as a Lyapunov function: $V(R) = \int_0^\ell R^\tau(x) e^{-\theta x A(0)} R(x)dx$, $\theta > 0$.

Remark 1.5. When we take $p = q$ and D_0 and D_ℓ as diagonal matrices (called diagonal case), the condition (1.3) is equivalent to that $D_0^2 D_\ell^2 > I$ (modulo a diagonal transformation). The condition is only sufficient but not necessary. Indeed, if $D_0^2 D_\ell^2$ has an element greater than one, then the null solution is an unstable equilibrium point for the quasilinear system. On the other hand we can prove that if $D_0^2 D_\ell^2 < I$, then the null solution is locally exponentially stable for the quasilinear system. As shown in the proof of Lemma 1 of [11] we obtain the best result that we can expect for the diagonal case. However, the situation is more complicated for general quasilinear systems.

For the quasilinear case we consider the non autonomous linear system as follows:

$$\Sigma\text{evo}: \quad \begin{cases} \dot\phi(x,t) = A(x, \xi(x,t))\phi'(x,t), \\ \phi^1(0,t) = D_0 \phi^2(0,t), \\ \phi^2(\ell,t) = D_\ell\, \phi^1(\ell,t), \\ \phi(x,0) = \phi^0(x), \end{cases} \tag{1.4}$$

where $\xi \in C^1([0,\ell] \times \mathbb{R}^+)$ is a known function. We say that the non autonomous system is exponentially stable if its evolution operator $U(t,s)$ satisfies the condition : $|U(t,s)|_{\mathcal{L}(X)} \le Me^{-\alpha(t-s)}$ for some $M > 0$ and $\alpha > 0$.

We have the following sufficient condition for the null solution to be exponentially stable for the quasilinear system (1.2).

Theorem 1.6. *If there exists some constant $\epsilon > 0$ such that the evolution system (1.4) is exponentially stable whenever $\xi \in C^1([0,\ell] \times [0,\infty))$ satisfying $|\xi(\cdot,t)|_1 \le \epsilon$, then the null solution is a locally exponentially stable equilibrium point for the quasilinear system (1.2).*

2 Application example

To show how the principle of linearized stability is useful in applications, we consider the irrigation canal system governed by the Saint Venant equation as formulated by Coron, d'Andréa-Novel and Bastin [2]:

$$\frac{\partial Y}{\partial t} + V\frac{\partial Y}{\partial x} + Y\frac{\partial V}{\partial x} = 0, \quad \frac{\partial V}{\partial t} + g\frac{\partial Y}{\partial x} + V\frac{\partial V}{\partial x} = 0, \tag{2.1}$$

with the boundary conditions:

$$\begin{aligned} V(0,t)Y(0,t) &= \sqrt{u_a[y_a - Y(0,t)]}, \\ V(\ell,t)Y(\ell,t) &= \sqrt{u_b[Y(\ell,t) - y_b]}, \end{aligned} \tag{2.2}$$

where ℓ is the reach's length of a canal, x is the space variable in $[0,\ell]$ and t is time, $V(x,t)$ is the water velocity at (x,t), $Y(x,t)$ is the water level and g is the gravitation constant. We assume the following conditions satisfied for the constants:

$$\text{(H)} \quad \begin{cases} u_a,\ u_b,\ y_a,\ y_b > 0, \quad y_a > y_b, \\ g(u_a y_a + u_b y_b)^3 > u_a u_b (u_a + u_b)^2 (y_a - y_b). \end{cases}$$

Physically, u_a and u_b represent the upstream gate opening and the down stream opening, respectively. They would be taken as control variables in the feedback control design (*see* [2]). The constants y_a and y_b represent the upstream water level and the down stream water level, respectively, outside the reach. Under the condition of (H) the system (2.1) and (2.2) admits a unique stationary solution (y_e, v_e) which is given by

$$y_e = (u_a y_a + u_b y_b)(u_a + u_b)^{-1},$$

$$v_e = (u_a y_a + u_b y_b)^{-1} \sqrt{u_a u_b (u_a + u_b)(y_a - y_b)}.$$

The third condition in (H) is equivalent to $g y_e > v_e^2$. It is meant by the latter that the nonlinear system (2.2) is hyperbolic and has two characteristic curves of opposite directions (positive and negative). Since the boundary conditions (2.2) are prescribed on the two boundary points, respectively, from Li and Yu [5] the assumption (H) guarantees the local existence of a unique classical solution with any initial data in some neighborood of (y_e, v_e).

To apply the principle of linearized stability we first write the PDE governing the variation of (Y,V) relative to (y_e, v_e). Let us set $\tilde y = Y - y_e$ and $\tilde v = V - v_e$. Then the nonlinear system (2.2) takes the following equivalent form:

$$\partial_t \begin{bmatrix} \tilde y \\ \tilde v \end{bmatrix} + \begin{bmatrix} \tilde v + v_e & \tilde y + y_e \\ g & \tilde v + v_e \end{bmatrix} \partial_x \begin{bmatrix} \tilde y \\ \tilde v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.3}$$

$$[\tilde v + v_e]\,[\tilde y + y_e](0,t) = \sqrt{u_a(y_a - \tilde y(0,t) - y_e)}, \tag{2.4}$$

$$[\tilde v + v_e]\,[\tilde y + y_e](\ell,t) = \sqrt{u_b(\tilde y(\ell,t) + y_e - y_b)}. \tag{2.5}$$

Hence the irrigation system is governed by the quasilinear PDE. With the notation of Section 1 we have

$$\widetilde{A}(x, (\tilde y, \tilde v)) = -\begin{bmatrix} \tilde v + v_e & \tilde y + y_e \\ g & \tilde v + v_e \end{bmatrix}. \tag{2.6}$$

The two eigenvalues of $\widetilde{A}(x,(\tilde{y},\tilde{v}))$ are given by

$$\begin{aligned} \lambda_1(x,(\tilde{y},\tilde{v})) &= -(\tilde{v}+v_e) - \sqrt{g(\tilde{y}+y_e)}, \\ \lambda_1(x,(\tilde{y},\tilde{v})) &= -(\tilde{v}+v_e) + \sqrt{g(\tilde{y}+y_e)}. \end{aligned} \tag{2.7}$$

We develop each nonlinear term in (2.3)-(2.3) in terms of limited Taylor series around $(0,0)$. By removing all the terms of order equal or superior to 2 after the development we obtain easily the linearized system around $(0,0)$ as follows:

$$\partial_t \begin{bmatrix} y \\ v \end{bmatrix} = \widetilde{A}(x,0)\partial_x \begin{bmatrix} y \\ v \end{bmatrix}, \tag{2.8}$$

$$2v_e y_e^2\ v(0,t) + (u_a + 2v_e^2 y_e)\ y(0,t) = 0, \tag{2.9}$$

$$2v_e y_e^2\ v(\ell,t) + (2v_e^2 y_e - u_b)\ y(\ell,t) = 0. \tag{2.10}$$

However the matrix $\widetilde{A}(x,0)$ is not diagonal as required in our formulation (1.1). To diagonalize it we take the following linear regular transformation on the unknown functions (y,v) such that

$$y(x,t) = \eta_1(x,t) + \eta_2(x,t),$$

$$v(x,t) = \sqrt{\frac{g}{y_e}}\left(\eta_1(x,t) - \eta_2(x,t)\right).$$

The the linear system on (η_1,η_2) takes the forme (1.1), i.e.,

$$\partial_t \begin{bmatrix} y \\ v \end{bmatrix} = A(x,0)\partial_x \begin{bmatrix} y \\ v \end{bmatrix}, \tag{2.11}$$

$$\eta_1(0,t) = \left[\frac{2v_e y_e^{3/2} g^{1/2} - u_a - 2v_e^2 y_e}{u_a + 2v_e^2 y_e + 2v_e y_e^{3/2} g^{1/2}}\right]\eta_2(0,t), \tag{2.12}$$

$$\eta_2(\ell,t) = \left[\frac{2v_e y_e^{3/2} g^{1/2} + 2v_e^2 y_e - u_b}{u_b - 2v_e^2 y_e + 2v_e y_e^{3/2} g^{1/2}}\right]\eta_1(\ell,t), \tag{2.13}$$

where

$$A(x,(\tilde{y},\tilde{v})) = \begin{bmatrix} \lambda_1(x,(\tilde{y},\tilde{v})) & 0 \\ 0 & \lambda_2(x,(\tilde{y},\tilde{v})) \end{bmatrix}. \tag{2.14}$$

From Greenberg and Li [3] and Xu and Feng [11] we get that the system (2.11)-(2.13) is exponentially stable if and only if the following property holds

$$\left|\frac{2v_e y_e^{3/2} g^{1/2} - u_a - 2v_e^2 y_e}{u_a + 2v_e^2 y_e + 2v_e y_e^{3/2} g^{1/2}}\right| \left|\frac{2v_e y_e^{3/2} g^{1/2} + 2v_e^2 y_e - u_b}{u_b - 2v_e^2 y_e + 2v_e y_e^{3/2} g^{1/2}}\right| < 1. \tag{2.15}$$

Moreover, according to Li [4] there exists an $\epsilon > 0$ such that for $(\tilde{y},\tilde{v}) \in C^1([0,\ell]\times\mathbb{R}^+)$ with $|(\tilde{y}(\cdot,t),\tilde{v}(\cdot,t))|_1 \le \epsilon$, the condition (2.15) implies also exponential stability of the time-varying system (2.11)- (2.13) obtained by replacing $A(x,0)$ by $A(x,(\tilde{y},\tilde{v}))$. Applying our Theorem 1.6 we get that the equilibrium point (y_e,v_e) is locally exponentially stable for the irrigation system (2.1)-(2.2) if and only if the property (2.15) holds.

To prove our Theorems 1.1-1.6 we need the existence and uniqueness theorems in Li and Yu [5], the semigroup and evolution system theory in Pazy [6] and the notion of well-posed control system in Weiss [8]. The validity of our proof is not limited to the specific case (1.2) and can be generalized to a larger class of quasilinear hyperbolic systems as studied in Li and Yu [5] and d'Andréa-Novel et al. [1]. The paper [9] with our complete proof is already available and will be submitted later for publication.

References

[1] B. d'Andréa-Novel, G. Bastin, J. de Halleux, J.M. Coron, G. Sallet et C.Z. Xu, Commande en dimension infinie, *Automatique pour la gestion des ressources en eau*, Editeurs: D. Georges et X. Litrico, pp.187-237, 2002.

[2] J.M. Coron, B. d'Andréa-Novel and G. Bastin, A Lyapunov approach to control irrigation canals modeled by Saint Venant equations, *European Control Conference ECC'99*, Karlsrube, September, 1999.

[3] J.M. Greenberg and T.T. Li, The effect of boundary damping for the quasilinear wave equation, *Journal of Differential Equations* **52**, 66-75, 1984.

[4] T.T. Li, *Global classical solutions for quasilinear hyperbolic systems*, Series Editors: P.G. Ciarlet and J.L. Lions, Masson, 1994.

[5] T.T. Li and W. Yu, *Boundary Value Problems for Quasilinear Hyperbolic Systems*, Duke University Mathematics, Series V, 1985.

[6] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[7] E.D. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, Texts in Applied Mathematics **6**, Springer-Verlag, New York, 1990.

[8] G. Weiss, Admissibility of unbounded control operators, *SIAM J. Control and Optimization* **27**, pp.527-545, 1989.

[9] C.Z. Xu, Ph. Anin and D.X. Feng, On the principle of linearized stability for nonlinear hyperbolic systems, Rapport interne du LAGEP, 17 pages, June, 2003.

[10] C.Z. Xu and D.X. Feng, Symmetric hyperbolic systems and applications to exponential stability of heat exchangers and irrigation canals, *Proceedings of Mathematical Theory of Network and Systems*, Perpignan, 2000.

[11] C.Z. Xu and D.X. Feng, Linearization Method to Stability Analysis, *C.R. Acad. Sci. Paris, t.332, Série I*, pp.809-814, 2001.