OBSERVER-BASED SWITCHED CONTROL DESIGN FOR DISCRETE-TIME SWITCHED SYSTEMS

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Abstract

In this paper, we consider switched linear discrete-time systems. We propose a method to design an observer-based switched control which guarantees that the switched system is asymptotically stable. The main result consists in proving a separation principle for linear discrete-time switched systems. Hence, the design of the switched state feedback control and the switched observer can be carried out independently. Such a design is formulated in terms of $\mathcal{L}$inear $\mathcal{M}$atrix $\mathcal{I}$nequalities ($\mathcal{LMI}$).

1 Introduction

In recent years, the study of switched systems has received a growing attention in control theory and practice. By switched systems we mean a class of hybrid dynamical systems consisting of a family of continuous (or discrete) time subsystems and a rule that governs the switching between them. A survey of basic problems in stability and design of switched systems is given in [1] where some contributions are summarized. Most of these contributions deal with stability analysis or design of state feedback based control laws (see [2], [3], and references therein).

In this paper, the problem of switched output feedback control design is addressed for switched discrete time systems. We propose a method to design an observer-based switched control with guarantee of asymptotic stability of the closed loop switched system. The main result consists in proving a separation principle for linear discrete-time switched systems. Hence, one may perform a separate design of the switched state feedback control and the switched observer. The advantage of the proposed control in addition to be an output feedback based control is that the observer allows a direct access to all the components of the state vector. This may be useful for people interested by fault detection problems in the switched systems framework.

The paper is organized as follows. The next section gives the problem formulation. In section 3, the design of a switched state feedback and a switched observer are considered separately. In Section 4 a separation principle is proved. This shows that the observer-based switched control, obtained by using simultaneously the switched state feedback and the switched observer of section 3, stabilizes the closed switched system. We end the paper by an illustrative example and a conclusion.

Notations: We use standard notations throughout the paper. M^T is the transpose of the matrix M. $M > \mathbf{0}$ ($M < \mathbf{0}$) means that M is positive definite (negative definite). $\mathbf{0}$ and $\mathbf{I}$ denote the null matrix and the identity matrix with appropriate dimensions. X is the state space $X \subset \mathbb{R}^n$. $B(0, R)$ denotes the ball with center 0 and radius R and :

$\|x\|$ denotes the euclidian norm.
$\|A\| = \sup \frac{\|Ax\|}{\|x\|}$ hence $\|AB\| < \|A\| \, \|B\|$

$$\prod_{k=1}^{p} A_{i_k} = A_{i_p} A_{i_{p-1}} \cdots A_{i_1}.$$

2 Problem formulation

Consider the switched system defined by:

$$x_{k+1} = A_\alpha x_k + B_\alpha u_k \quad (1)$$
$$y_k = C_\alpha x_k \quad (2)$$

where $x_k \in \mathbb{R}^v$ is the state, $u_k \in \mathbb{R}^q$ is the control input and $y_k \in \mathbb{R}^t$ is the output vector. $\{(A_i, B_i, C_i) : i \in \mathcal{E}\}$ are a family of matrices of appropriate dimensions parameterized by an index set $\mathcal{E} = \{1, 2, ..., N\}$ and $\alpha : X \times \mathbb{N} \to \mathcal{E}$ is a switching signal ($i = \alpha(x_k, k)$). The switching sequence may also be generated by any strategy or supervisor. We assume that the switching signal is unknown a priori but real time available.

The problem addressed in this paper concerns the design of an observer-based switched control law of the following form :

$$\begin{aligned} \hat{x}_{k+1} &= A_\alpha \hat{x}_k + B_\alpha u_k + L_\alpha (y_k - \hat{y}_k) && (3) \\ \hat{y}_k &= C_\alpha \hat{x}_k \\ u_k &= K_\alpha \hat{x}_k, && (4) \end{aligned}$$

such that the corresponding closed loop switched system

$$\left(\begin{array}{c} x_{k+1} \\ \epsilon_{k+1} \end{array} \right) = \left[\begin{array}{cc} \tilde{A}_\alpha & \tilde{B}_\alpha \\ \mathbf{0} & \hat{A}_\alpha \end{array} \right] \left(\begin{array}{c} x_k \\ \epsilon_k \end{array} \right) \quad (5)$$

where $\epsilon_k = x_k - \hat{x}_k$ denotes the observation error and

$$\begin{array}{rcl} \tilde{A}_\alpha & = & A_\alpha + B_\alpha K_\alpha \\ \hat{A}_\alpha & = & A_\alpha - L_\alpha C_\alpha \\ \tilde{B}_\alpha & = & -B_\alpha K_\alpha \end{array} \quad (6)$$

is asymptotically stable. Such a switched control law is more realistic than the classical switched state feedback which requires the availability of all the state vector components.

3 Separate design of the control and the observer

3.1 Switched state feedback design

The classical switched state feedback design reduces to the computation of

$$u_k = K_\alpha x_k, \quad (7)$$

ensuring stability of the corresponding closed loop switched system

$$x_{k+1} = \underbrace{(A_\alpha + B_\alpha K_\alpha)}_{\tilde{A}_\alpha} x_k \quad (8)$$

under arbitrary switching signal. A solution has been proposed to this problem in [4]. It is based on the use of switched quadratic Lyapunov functions and stability conditions from [5].

Theorem 1 *If there exist symmetric matrices S_i, matrices G_i, and R_i, $\forall i \in \mathcal{E}$, such that*

$$\left(\begin{array}{cc} G_i + G_i^T - S_i & (A_i G_i + B_i R_i)^T \\ A_i G_i + B_i R_i & S_j \end{array} \right) > \mathbf{0} \quad \forall (i,j) \in \mathcal{E} \times \mathcal{E} \quad (9)$$

then the state feedback control given by (7) with

$$K_i = R_i G_i^{-1} \quad \forall i \in \mathcal{E} \quad (10)$$

stabilizes asymptotically the system (8).

Proof: see Theorem 2 in [4] □

Remark 1 : As shown in [4] for analysis and state feedback design, the condition given in the previous Theorem is equivalent to the following one :

$$\left(\begin{array}{cc} S_i & (A_i S_i + B_i R_i)^T \\ A_i S_i + B_i R_i & S_j \end{array} \right) > \mathbf{0} \quad \forall (i,j) \in \mathcal{E} \times \mathcal{E} \quad (11)$$

with

$$K_i = R_i S_i^{-1} \quad \forall i \in \mathcal{E} \quad (12)$$

The main difference is the introduction of an additional variable G_i. The introduction of an additional variable G has been first used in [6] for uncertain linear time invariant discrete time systems. Unlike [6] where this variable cannot be dropped since it plays a key role for stability analysis and control design, in the case of switched systems the additional variables G_i can be omitted in analysis and state feedback design and are useful for constrained control design problems only. This has been proved in [4]. However, we prefer to formulate the results on the basis of condition (9) to allow a direct extension of the results proposed in this paper to other constrained control laws. For example, the case of decentralized control design can be immediately solved using condition (9) with an appropriate partitioning of the matrices G_i. This prevents conservatism since it allows to keep the structure of the Lyapunov matrices S_i unchanged.

Remark 2 : A switched state feedback computed as indicated in Theorem 1 ensures that the closed loop switched system (8) is asymptotically stable under arbitrary switching signal α. This is equivalent to

$$\forall x_0 \in \mathbb{R}^n, \\ \forall s \in \{(i_0, i_1, i_2, \ldots, i_k, \ldots) : \forall k \geq 0,\ i_k \in \{1, 2, ..., N\}\},$$

$$\lim_{p \to \infty} \prod_{k=0}^{p} \tilde{A}_{i_k} x_0 = 0$$

$\Leftrightarrow$

$$\forall x_0 \in B(0,R), \exists n_{\tilde{A}}, \forall p \geq 0, \\ \forall s_{n_{\tilde{A}}+p} \in \{(i_0, i_1, i_2, \ldots, i_{n_{\tilde{A}}+p}) : \\ \forall k = 0, 1, ..., n_{\tilde{A}} + p, i_k \in \{1, 2, ..., N\}\}$$

$$x_{n_{\tilde{A}}+p} = \prod_{k=0}^{n_{\tilde{A}}+p} \tilde{A}_{i_k} x_0 \in B(0, R/2)$$

$\Leftrightarrow$

$$\forall \sigma > 1, \exists n_{\tilde{A}}(\sigma), \forall p \geq 0, \\ \forall s_{n_{\tilde{A}}+p} \in \{(i_0, i_1, i_2, \ldots, i_{n_{\tilde{A}}+p}) : \forall k = 0, 1, ..., n_{\tilde{A}} + p, \\ i_k \in \{1, 2, ..., N\}\},$$

$$\left\| \prod_{k=0}^{n_{\tilde{A}}+p} \tilde{A}_{i_k} \right\| < \frac{1}{\sigma}$$

3.2 Switched observer design

Consider the switched system defined by (1). The design of a switched observer :

$$\begin{array}{rcl} \hat{x}_{k+1} & = & A_\alpha \hat{x}_k + B_\alpha u_k + L_\alpha (y_k - \hat{y}_k) \\ \hat{y}_k & = & C_\alpha \hat{x} \end{array} \quad (13)$$

for this system consists in computing the gain matrices L_i, $i \in \mathcal{E}$ such that the observation error between the state x_k of the switched system (1) and the state $\hat{x}_k$ of the observer (13) is asymptotically stable. The convergence to the origin of the observation error has to be independent of the initial conditions x_0 and $\hat{x}_0$, the input u_k and the switching signal α.

Define the observation error by $\epsilon_k = x_k - \hat{x}_k$, the error dynamic is given by:

$$\epsilon_{k+1} = \underbrace{(A_\alpha - L_\alpha C_\alpha)}_{\hat{A}_\alpha} \epsilon_k. \tag{14}$$

The following Theorem, gives sufficient conditions to build such a switched observer.

Theorem 2 *If there exist symmetric matrices S_i, matrices F_i and G_i solutions of:*

$$\begin{bmatrix} G_i + G_i^T - S_i & G_i^T A_i - F_i^T C_i \\ A_i^T G_i - C_i^T F_i & S_j \end{bmatrix} > 0, \ \forall (i,j) \in \mathcal{E} \times \mathcal{E}, \tag{15}$$

then a switched observer (13) *exists and the resulting gains L_i are given by*

$$L_i = G_i^{-T} F_i^T \quad \forall i \in \mathcal{E}$$

.

Proof: See theorem 4 in [7]. □

Remark 3 : The same comments in Remark 1 hold with (11) replaced by

$$\begin{bmatrix} S_i & S_i^T A_i - F_i^T C_i \\ A_i^T S_i - C_i^T F_i & S_j \end{bmatrix} > 0, \ \forall (i,j) \in \mathcal{E} \times \mathcal{E}, \tag{16}$$

and (12) by

$$L_i = S_i^{-T} F_i^T \quad \forall i \in \mathcal{E}$$

.

Remark 4 : A switched observer computed as indicated in Theorem 2 ensures that the error dynamic (14) is asymptotically stable under arbitrary switching signal α. The properties stated in Remark 1 are verified with $n_{\tilde{A}}$ and $\tilde{A}$ replaced by $n_{\hat{A}}$ and $\hat{A}$ respectively.

4 A separation principle for discrete-time switched systems

In this section, we show that the switched output feedback control obtained by combining the switched state feedback and the switched observer computed independently in the previous section guarantees that the closed loop switched system (5) is asymptotically stable.

Theorem 3 *Assume that the matrix gains K_i and L_i, $\forall i \in \mathcal{E}$, have been computed as indicated in Theorems 1 and 2. Then the observer-based switched control (3)-(4) stabilizes asymptotically the closed switched system (5).*

Proof : Assume that the matrix gains K_i and L_i, $\forall i \in \mathcal{E}$, have been computed as indicated in Theorems 1 and 2. The closed loop system resulting from the combination of the switched state feedback and the switched observer is given by (5). As the error ϵ_k is asymptotically stable, it remains to show that

$$x_{k+1} = \tilde{A}_\alpha x_k + \tilde{B}_\alpha \epsilon_k \tag{17}$$

is also asymptotically stable. The later equation writes :

$$\begin{aligned} x_1 &= \tilde{A}_{i_0} x_0 + \tilde{B}_{i_0} e_0 \qquad (18) \\ x_2 &= \tilde{A}_{i_1} \tilde{A}_{i_0} x_0 + \tilde{A}_{i_1} \tilde{B}_{i_0} e_0 + \tilde{B}_{i_1} \hat{A}_{i_0} e_0 \\ &\cdots \\ &\cdots \\ x_{p+1} &= \prod_{k=0}^{p} \tilde{A}_{i_k} x_0 + \Bigg[\prod_{k=1}^{p} \tilde{A}_{i_k} \tilde{B}_{i_0} + \prod_{k=2}^{p} \tilde{A}_{i_k} \tilde{B}_{i1} \hat{A}_{i_0} \\ &+ \cdots + \prod_{k=j}^{p} \tilde{A}_{i_k} \tilde{B}_{i_{j-1}} \prod_{k=0}^{j-2} \hat{A}_{i_k} + \cdots \\ &+ \tilde{A}_{i_p} \tilde{B}_{i_{p-1}} \prod_{k=0}^{p-2} \hat{A}_{i_k} + \tilde{B}_{i_p} \prod_{k=0}^{p-1} \hat{A}_{i_k} \Bigg] e_0 \end{aligned}$$

According to Remarks 3 and 4, let

$$m = \max\left(n_{\tilde{A}}, n_{\hat{A}}\right)$$

we have,

$$\forall s_n \in \{(i_0, i_1, i_2, \ldots, i_n) : \forall k = 0, 1, ..., n, \ i_k \in \{1, 2, ..., N\}\},$$

$$\left\| \prod_{k=0}^{n} \tilde{A}_{i_k} \right\| < \left(\frac{1}{2}\right)^l \sigma^r$$

where $n = lm + r$ with $r < m$ and $\sigma = \max_{i \in \{1,2,\ldots,N\}} \left\| \tilde{A}_i \right\|$

Hence,

$$\left\| \prod_{k=j}^{pm} \tilde{A}_{i_k} \tilde{B}_{i_{j-1}} \prod_{k=0}^{j-2} \hat{A}_{i_k} \right\| \leq \left\| \prod_{k=j}^{pm} \tilde{A}_{i_k} \right\| \left\| \tilde{B}_{i_{j-1}} \right\| \left\| \prod_{k=0}^{j-2} \hat{A}_{i_k} \right\| \tag{19}$$

$$\left\| \prod_{k=j}^{pm} \tilde{A}_{i_k} \tilde{B}_{i_{j-1}} \prod_{k=0}^{j-2} \hat{A}_{i_k} \right\| \leq \hat{\sigma} \beta \hat{\gamma} \frac{1}{2^{p-1}} \tag{20}$$

with

$$\begin{aligned} \sigma &= \max_{i \in \{1,2,\ldots,N\}} \left\| \tilde{A}_i \right\| \\ \beta &= \max_{i \in \{1,2,\ldots,N\}} \left\| \tilde{B}_i \right\| \\ \gamma &= \max_{i \in \{1,2,\ldots,N\}} \left\| \hat{A}_i \right\| \\ \hat{\sigma} &= \max(\sigma, \sigma^m) \\ \hat{\gamma} &= \max(\gamma, \gamma^m) \end{aligned} \tag{21}$$

As :

$$
\begin{aligned}
\|x_{pm+1}\| \quad \leq \quad & \left\|\prod_{k=0}^{pm}\tilde{A}_{i_k}x_0\right\| + \left[\left\|\prod_{k=1}^{pm}\tilde{A}_{i_k}\tilde{B}_{i_0}\right\|\right. \\
& + \left\|\prod_{k=2}^{pm}\tilde{A}_{i_k}\tilde{B}_{i1}\hat{A}_{i_0}\right\| + \cdots \\
& + \left\|\prod_{k=j}^{pm}\tilde{A}_{i_k}\tilde{B}_{i_{j-1}}\prod_{k=0}^{j-2}\hat{A}_{i_k}\right\| + \cdots \\
& + \left\|\tilde{A}_{i_{pm}}\tilde{B}_{i_{pm-1}}\prod_{k=0}^{pm-2}\hat{A}_{i_k}\right\| \\
& + \left.\left\|\tilde{B}_{i_{pm}}\prod_{k=0}^{pm-1}\hat{A}_{i_k}\right\|\right]\|e_0\|
\end{aligned}
$$

we have :

$$
\|x_{pm+1}\| \leq \left\|\prod_{k=0}^{pm}\tilde{A}_{i_k}x_0\right\| + \hat{\sigma}\beta\hat{\gamma}\sum_{k=1}^{pm}\frac{1}{2^{p-1}}\|e_0\| \tag{22}
$$

$$
\|x_{pm+1}\| \leq \left(\frac{1}{2}\right)^p\|x_0\| + \hat{\sigma}\beta\hat{\gamma}\frac{pm}{2^{p-1}}\|e_0\| \tag{23}
$$

$$
\|x_{pm+1}\| \to 0 \text{ if } p \to \infty \qquad \square
$$

It is well known that the separation principle does not hold in general and one has to check carefully the validity of this principle in other cases than the classical linear time invariant case. Hence, it is not obvious that the switched system (17) is asymptotically stable even if the error ϵ_k converges asymptotically to 0. Theorem 3 gives a rigorous justification for a separate design of the switched control and the switched observer.

5 Illustrative example

Consider a switched system given by (1) where $\{A_i : i \in \mathcal{E}\}$, $\{B_i : i \in \mathcal{E}\}$ and $\{C_i : i \in \mathcal{E}\}$ are a family of matrices parameterized by an index set $\mathcal{E} = \{1, 2\}$ and

$$
A_i = \begin{bmatrix} 0 & 0.89 & 0.5 \\ h_i & 0.89 & 0 \\ -0.1 & 0 & 0.9 \end{bmatrix}
$$

with $i = 1, 2$ and $h_1 = -a\lambda_1$, $h_2 = \lambda_2$

$$
B_1 = [0\ 0\ 1]^T, \quad B_2 = [0\ -6(\lambda_1+\lambda_2)\ 0]^T
$$

$$
C_1 = [-1\ \ 1\ \ -2] \quad \text{and} \quad C_2 = [-2\ \ 0.35\ \ 1]
$$

If the switching signal is characterized by

$$
\alpha = \begin{cases} 1 & \text{if } x_k^1 < 6, \quad \text{with} \quad x_k = [x_k^1\ \ x_k^2\ \ x_k^3]^T \\ 2 & \text{otherwise} \end{cases} \tag{24}
$$

and the parameters $\lambda_1 = 1.12$, $\lambda_2 = 2$ then the open loop switched system may exhibit a chaotic motion as shown in figure 1 by using an open loop control $u_k = 0$ if $x_k^1 < 6$ and $u_k = 1$ otherwise.

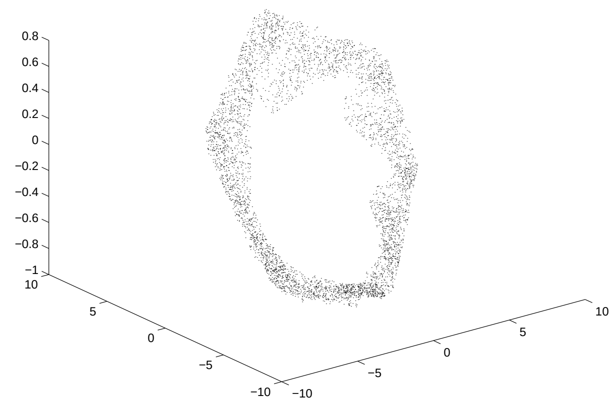

Figure 1: Chaotic attractor generated by the switched system in the three dimensional state space

First, we design a switched state feedback using Theorem 1. The corresponding LMIs are found to be feasible and the obtained matrix gains are :

$$
K_1 = \begin{bmatrix}1.3724 & -0.7652 & -0.7618\end{bmatrix}
$$

$$
K_2 = \begin{bmatrix}0.1021 & 0.0360 & 0.0358\end{bmatrix}
$$

We design separately a switched observer using Theorem 2. The corresponding LMIs are found to be feasible and the obtained matrix gains are :

$$
L_1 = \begin{bmatrix} -0.3450 \\ 0.5650 \\ -0.1756 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -0.3033 \\ -1.3599 \\ -0.0463 \end{bmatrix}
$$

The observer-based switched control obtained by combining the previous results leads to a closed behavior corresponding to the one depicted in figure 2.

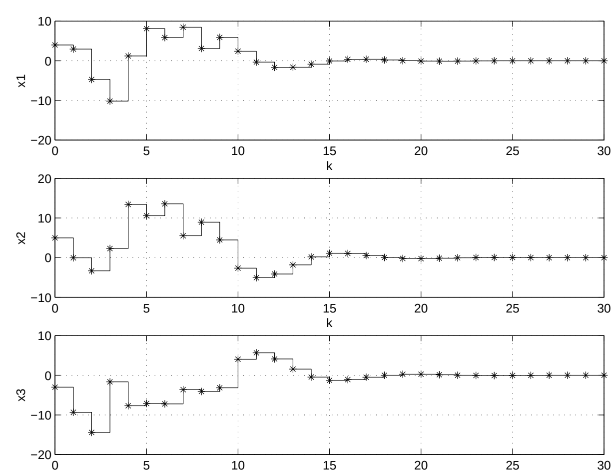

Figure 2: The state components $x_k = [x_k^1\ \ x_k^2\ \ x_k^3]^T$ of the closed loop switched system

The error behavior is shown in figure 3. The simulation is performed for the same switching rule as in (24).

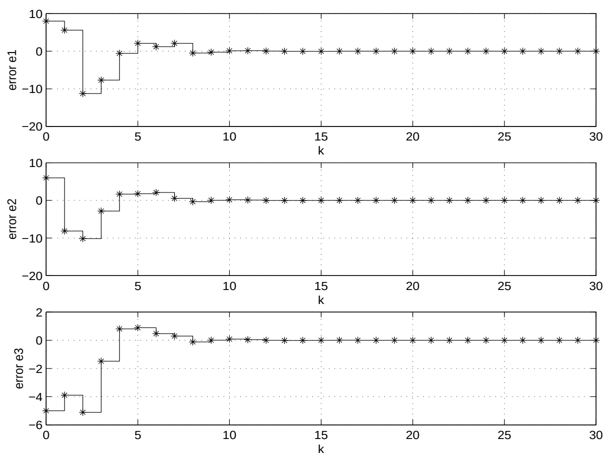

Figure 3: The error behavior $\epsilon_k = x_k - \hat{x}_k$

The designed observer-based control guarantees that the closed loop system in inconditionally stable that is under arbitrary switching signal α. A simulation has been performed with $\alpha = 1$ for even samples k and $\alpha = 2$ for odd samples k and the results are depicted in figures 4 and 5.

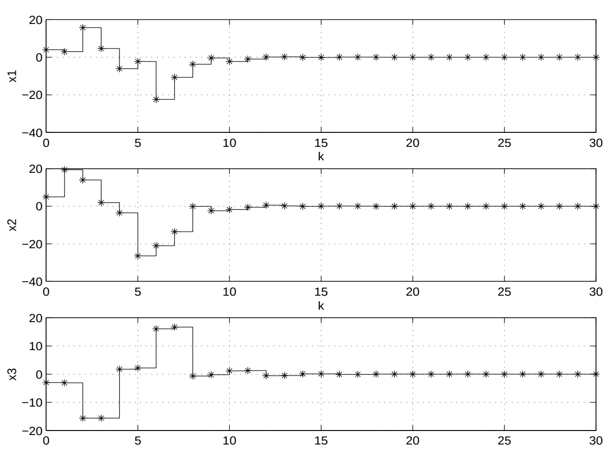

Figure 4: The state components $x_k = [x_k^1 \quad x_k^2 \quad x_k^3]^T$ of the closed loop switched system

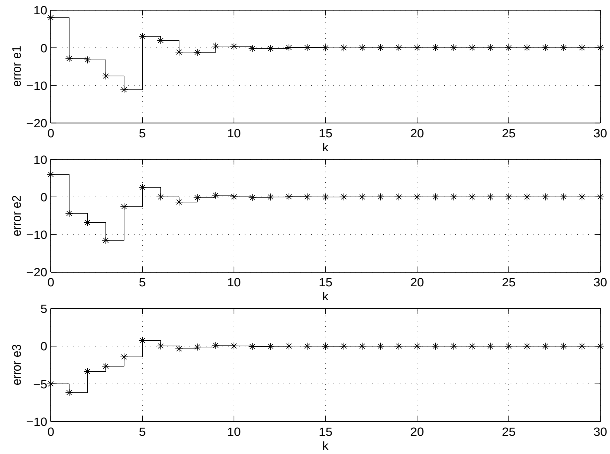

Figure 5: The error behavior $\epsilon_k = x_k - \hat{x}_k$

6 Conclusion

In this paper, the design of a stabilizing switched output feedback control is addressed. A separation principle for linear discrete-time switched systems is proved. It allows to perform the control and the observer designs independently. The design reduces to check the feasibility of two sets of LMIs. This reduces to solve a convex optimization problem for which many tools are available (LMI toolbox by MATALB, Sedumi solver...). Obviously, the proposed results are less conservative than the ones based on single quadratic Lyapunov functions $V = x_k^T P x_k$ with P a constant matrix. Finally, these results are also applicable to the classical discrete time varying case. This case corresponds to a fixed switching sequence and can be solved by checking only the LMIs corresponding to the allowable sequence.

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