OPTIMAL TRANSIENT RESPONSE SHAPING FOR THE DISCRETE-TIME SERVOMECHANISM PROBLEM

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Abstract

The problem of constructing a controller that results in a time response which is smooth, has a desired speed of response, and has little cross-channel interaction, is considered in this paper for the discrete-time servomechanism problem. Subject to fundamental limitations on achievable performance, we satisfy this objective by introducing a new cheap-control quadratic performance index that has the desired transient response characteristics embedded within it. Two examples are used to demonstrate that minimizing the performance index indeed results in a controller which has good transient performance and, in addition, reasonable control effort.

1 Introduction

When designing a controller, control engineers have to consider a number of different issues, including time-response characteristics, robustness properties, controller complexity, and control signal magnitude, for example. This paper focuses on the *time-response performance aspects* of controller design related to the robust servomechanism problem (RSP) [2, 5]. Specifically, the goal is to design a controller, subject to the fundamental limitations of achievable performance [1], which solves the RSP and which, for the nominal plant, results in a "good" transient response in the sense that the closed-loop system response is "smooth", the speed of response is at a desirable level, and there are "small" interactions between the output channels. It is shown in this paper that "good" transient behaviour is indeed attainable if one designs a controller that minimizes a performance index of the form

$$\begin{aligned} J_\theta &:= \sum_{k=1}^{\infty} \left\{ (\hat{z}^\theta_{[k]})'(\hat{z}^\theta_{[k]}) + \varepsilon \cdot \phi(u_{[k]})'\phi(u_{[k]}) \right\} \\ \text{with } \hat{z}^\theta_{[k]} &:= (I-\theta/h)\, e_{[k-p]} + (\theta/h) e_{[k-p+1]}, \end{aligned}$$

where $e_{[k]}$ is the output tracking error, h is the designer-specified sampling period, θ is a designer-specified diagonal matrix whose diagonal elements satisfy $\theta_i > h$, ε is a designer-specified positive constant, $u_{[k]}$ is the control signal, and $\phi(u_{[k]})$ is a special polynomial function of $u_{[k]}$ and its previous values which, like integer p, depends on the class of the disturbance/tracking signals. Note that this performance index differs from the more standard index

$$J = \sum_{k=1}^{\infty} \left\{ (e_{[k-p]})'(e_{[k-p]}) + \varepsilon \cdot \phi(u_{[k]})'\phi(u_{[k]}) \right\},$$

which has been used elsewhere in the literature (e.g., see [2, 5, 1]), although both indices are of the "cheap-control" type [4, 7]. The motivation for the structure of the term $\hat{z}^\theta_{[k]}$ in the proposed index is that, in the ideal case when $\hat{z}^\theta_{[k]} = 0$, we have $(I-\theta/h)e_{[k-p]} + (\theta/h)e_{[k-p+1]} = 0$, or, equivalently, $e_{[k]} = [I - h\theta^{-1}]^k e_{[0]}$, i.e., the error response in the system has a smooth, non-interacting, exponentially-decaying behaviour whose response time can be adjusted by choice of the matrix θ. Note that the ideal response, $e_{[k]} = [I - h\theta^{-1}]^k e_{[0]}$, can be obtained by sampling $e(t) = \exp(-\theta^{-1}t)e(0)$ at $t = kh$ and using the approximation $\exp(-h\theta^{-1}) \approx I - h\theta^{-1}$; hence, the i^{th} diagonal element of θ essentially determines the desired time constant of the i^{th} output channel.

2 Preliminary Results

2.1 Plant, Disturbance, and Reference Signal Models

The discrete-time plant to be controlled is assumed to be described by the linear shift-invariant model

$$x_{[k+1]} = Ax_{[k]} + Bu_{[k]} + E\omega_{[k]} \tag{1}$$
$$y_{[k]} = Cx_{[k]} + Du_{[k]} + F\omega_{[k]} \tag{2}$$
$$e_{[k]} = y_{[k]} - y_{\text{ref}[k]}, \tag{3}$$

where $x_{[k]} \in \mathbb{R}^n$ is the state, $u_{[k]} \in \mathbb{R}^m$ is the control input, $y_{[k]} \in \mathbb{R}^r$ is the (measurable) output to be controlled, $\omega_{[k]} \in \mathbb{R}^{m_1}$ is the vector of (unmeasurable) disturbances acting on the system, $y_{\text{ref}[k]} \in \mathbb{R}^r$ is the tracking reference signal, and $e_{[k]} \in \mathbb{R}^r$ is the tracking error.

It is assumed that the disturbances $\omega_{[k]}$ are generated from an unforced system with the structure

$$x_{\text{dist}[k+1]} = \mathcal{A}_1 x_{\text{dist}[k]}, \tag{4}$$
$$\omega_{[k]} = \mathcal{C}_1 x_{\text{dist}[k]} \tag{5}$$

where $x_{\text{dist}[k]} \in \mathbb{R}^{n_1}$. Similarly, we assume that the tracking reference signal $y_{\text{ref}[k]}$ arises from a system with the structure

$$x_{\text{ref}[k+1]} = \mathcal{A}_2 x_{\text{ref}[k]}, \tag{6}$$
$$\rho_{[k]} = \mathcal{C}_2 x_{\text{ref}[k]}, \tag{7}$$
$$y_{\text{ref}[k]} = G\rho_{[k]}, \tag{8}$$

where $x_{\text{ref}[k]} \in \mathbb{R}^{n_2}$ and $\rho_{[k]} \in \mathbb{R}^q$. For nontriviality, we assume that $\text{sp}(\mathcal{A}_1) \subset \mathbb{D}^c$, $\text{sp}(\mathcal{A}_2) \subset \mathbb{D}^c$, where $\text{sp}(\cdot)$ denotes the eigenvalues of $(\cdot)$ and $\mathbb{D}^c$ denotes the closed region outside of the unit circle. Finally, we assume that, without loss of generality, $(\mathcal{C}_1, \mathcal{A}_1)$ and $(\mathcal{C}_2, \mathcal{A}_2)$ are observable, rank $\begin{bmatrix} E \\ F \end{bmatrix}$ = rank $\mathcal{C}_1$ = m_1, and rank G = rank $\mathcal{C}_2$ = q. This class of signals is quite broad and includes classes of signals which commonly occur in application problems, e.g., constants, polynomials, and sinusoidals, and polynomial-sinusoidals.

Two definitions will be useful in the development to follow. Given (4)–(8), let

$$S(\lambda) := \lambda^p + \delta_p \lambda^{p-1} + \ldots + \delta_2 \lambda + \delta_1 \tag{9}$$

be the least common multiple of the minimal polynomial of $\mathcal{A}_1$ and the minimal polynomial of $\mathcal{A}_2$, and let $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ be the zeros of $S(\lambda)$ (multiplicities included); call

$$\Lambda := \{\lambda_1, \lambda_2, \ldots, \lambda_p\} \tag{10}$$

the *disturbance/tracking poles* of (4)–(8), and call $S(\lambda)$ the *disturbance/tracking polynomial* of (4)–(8). As an example, consider a SISO system subject to constant tracking reference signals and constant disturbance signals: in this case we have $\Lambda = \{0\}$. On the other hand, for a SISO system subject to ramp tracking reference signals and constant disturbance signals, we have $\Lambda = \{0 \;\; 0\}$.

2.2 The Robust Servomechanism Problem

The *robust servomechanism problem* (RSP) for (1)–(3), as described in [2], involves finding a linear, shift-invariant controller which has inputs $y_{[k]}, y_{\text{ref}[k]}$ and output $u_{[k]}$ so that:

(a) the resulting closed-loop system is asymptotically stable;

(b) asymptotic tracking/regulation occurs for the disturbances and reference signal, i.e.,

$$\lim_{k \to \infty} e_{[k]} = 0, \;\; \forall x_{[0]} \in \mathbb{R}^n, \;\; \forall x_{\text{dist}[0]} \in \mathbb{R}^{n_1}, \;\; \forall x_{\text{ref}[0]} \in \mathbb{R}^{n_2}$$

and for all controller initial conditions; and

(c) condition (b) holds for arbitrary perturbations in the plant model (1)–(3) (e.g., plant parameters or plant dynamics, including changes in model order) which do not cause the resulting perturbed closed-loop system to become unstable.

2.3 Solution to the Robust Servomechanism Problem

The following existence result is well established:

Theorem 2.1 [2]: There exists a solution to the robust servomechanism problem with plant (1)–(3) if and only if the following conditions are all satisfied:

(a) (C, A, B) is stabilizable and detectable,

(b) $m \geq r$, and

(c) the transmission zeros of (C, A, B, D) exclude the disturbance/tracking poles $\lambda_i, \; i = 1, 2, \ldots, p$. □

To describe the structure of a controller that solves the RSP, it is helpful to introduce the notion of a *servo-compensator*. Towards this end, define the matrix $\mathcal{A} \in \mathbb{R}^{p \times p}$, which depends on the disturbance/tracking polynomial (9), and define vector $\beta \in \mathbb{R}^p$ as

$$\mathcal{A} := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\delta_1 & -\delta_2 & -\delta_3 & \cdots & -\delta_p \end{bmatrix} \quad \text{and} \quad \beta := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Also define the matrices

$$\begin{aligned} \mathcal{A}^* &:= \text{block diag}\,\underbrace{(\mathcal{A}, \mathcal{A}, \ldots, \mathcal{A})}_{r} \\ \mathcal{B}^* &:= \text{block diag}\,\underbrace{(\beta, \beta, \ldots, \beta)}_{r}. \end{aligned}$$

Finally, define $\gamma := [1 \;\; 0 \;\; 0 \;\; \cdots \;\; 0] \in \mathbb{R}^{1 \times p}$ and introduce $\mathcal{C} \in \mathbb{R}^{r \times rp}$ as

$$\mathcal{C} := \text{block diag}\,\underbrace{(\gamma, \gamma, \ldots, \gamma)}_{r}.$$

Then, a *servo-compensator* for plant (1)–(3) with the class of disturbance/tracking signals (4)–(8) is a dynamic system with input $e_{[k]} \in \mathbb{R}^r$, output $z_{[k]} \in \mathbb{R}^r$, and state $\eta_{[k]} \in \mathbb{R}^{rp}$ which satisfies

$$\eta_{[k+1]} = \mathcal{A}^* \eta_{[k]} + \mathcal{B}^* e_{[k]} \tag{11}$$

$$z_{[k]} = \mathcal{C} \eta_{[k]}. \tag{12}$$

In the frequency domain, the i^{th} component of the servo-compensator output is obtained by filtering the i^{th} component of the error signal, where the filter has transfer function $1/S(\lambda)$, with $S(\lambda)$ defined as in (9). For example, in the scalar case with $\omega_{[k]}$ as an unknown constant, we have $S(\lambda) = \lambda - 1$ and the output is $z_{[k]} = z_{[0]} + \sum_{i=0}^{k} e_{[i]}$, i.e., the (discrete-time) integral of the error.

Let us define the *augmented plant* to be the system formed by cascading the servo-compensator in (11)–(12) with the plant in (1)–(3). A realization for the augmented plant, where the measurable outputs (denoted $y_{\mathrm{meas}[k]}$) are the output, is as follows:

$$\begin{aligned}\begin{bmatrix} x_{[k+1]} \\ \eta_{[k+1]} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ \mathcal{B}^*C & \mathcal{A}^* \end{bmatrix}\begin{bmatrix} x_{[k]} \\ \eta_{[k]} \end{bmatrix} + \begin{bmatrix} B \\ \mathcal{B}^*D \end{bmatrix} u_{[k]} + \\ &\qquad \begin{bmatrix} E & 0 \\ \mathcal{B}^*F & -\mathcal{B}^* \end{bmatrix}\begin{bmatrix} \omega_{[k]} \\ y_{\mathrm{ref}[k]} \end{bmatrix} \qquad (13)\\ y_{\mathrm{meas}[k]} &= \begin{bmatrix} C & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{[k]} \\ \eta_{[k]} \end{bmatrix} + \begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix} u_{[k]} + \\ &\qquad \begin{bmatrix} F & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \omega_{[k]} \\ y_{\mathrm{ref}[k]} \end{bmatrix}. \qquad (14)\end{aligned}$$

It is shown in [2] that, if the three conditions of Theorem 2.1 hold, then there exists a compensator that stabilizes the augmented plant in (13)–(14); moreover, any compensator that stabilizes the augmented plant solves the RSP. In the next section we show how to design a compensator that, in addition to stabilizing the augmented plant and thereby solving the RSP, satisfies the objective of having a smooth time response whose speed can be set to any desired value, with little cross-channel interaction.

3 Main Results

3.1 A Useful Realization of the Augmented Plant

Our goal of achieving a "good" transient response will be satisfied if we can force each component of the tracking error, $e_{[k]} = y_{[k]} - y_{\mathrm{ref}[k]}$, to decay exponentially to zero at any prespecified rate. With this in mind, we introduce the matrix

$$\theta := \mathrm{diag}(\theta_1, \theta_2, ..., \theta_r), \qquad (15)$$

where $\theta_i \geq h$ determines the desired time constant for the i^{th} error channel. Assuming $p \geq 2$ (the special case $p = 1$ is handled later), we also define $\gamma_i \in \mathbb{R}^{1\times p}$, $i = 1, 2, \ldots, r$ as

$$\gamma_i := \begin{bmatrix} \left(1 - \frac{\theta_i}{h}\right) & \frac{\theta_i}{h} & 0 & \ldots & 0 \end{bmatrix}$$

and define

$$\mathcal{C}_\theta := \text{block diag}(\gamma_1, \gamma_2, \ldots, \gamma_r). \qquad (16)$$

We now introduce a modified servo-compensator (11)–(12) which has output $z^\theta_{[k]} := \mathcal{C}_\theta \cdot \eta_{[k]}$. This output, like $z_{[k]}$, is obtained by filtering $e_{[k]}$ by $1/S(\lambda)$, but now the desired time constants are also incorporated in such a way that making $z^\theta_{[k]}$ "small" will result in a "good" transient response. The augmented plant with the modified output is

$$\begin{aligned}\begin{bmatrix} x_{[k+1]} \\ \eta_{[k+1]} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ \mathcal{B}^*C & \mathcal{A}^* \end{bmatrix}\begin{bmatrix} x_{[k]} \\ \eta_{[k]} \end{bmatrix} + \begin{bmatrix} B \\ \mathcal{B}^*D \end{bmatrix} u_{[k]} + \\ &\qquad \begin{bmatrix} E & 0 \\ \mathcal{B}^*F & -\mathcal{B}^* \end{bmatrix}\begin{bmatrix} \omega_{[k]} \\ y_{\mathrm{ref}[k]} \end{bmatrix} \qquad (17)\\ z^\theta_{[k]} &= \begin{bmatrix} 0 & \mathcal{C}_\theta \end{bmatrix}\begin{bmatrix} x_{[k]} \\ \eta_{[k]} \end{bmatrix}. \qquad (18)\end{aligned}$$

Roughly speaking, the objective now is to minimize the size of $z^\theta_{[k]}$ for the system (17)–(18). Unfortunately, the presense of the external disturbance and reference signal in (17) means that standard optimization methods, such as LQR methods, cannot be directly applied. To write the equations in a form for which standard optimization methods can be applied, we introduce modified versions of $x_{[k]}$, $\eta_{[k]}$, and $u_{[k]}$ as follows:

$$\hat{x}_{[k]} := x_{[k]} + \delta_p x_{[k-1]} + \cdots + \delta_2 x_{[k-(p-1)]} + \delta_1 x_{[k-p]}, \qquad (19)$$

$$\hat{\eta}_{[k]} := \begin{bmatrix} \hat{e}_{1,[k]} \\ \hat{e}_{2,[k]} \\ \vdots \\ \hat{e}_{r,[k]} \end{bmatrix}, \text{ where } \hat{e}_{i,[k]} := \begin{bmatrix} e_{i,[k-p]} \\ e_{i,[k-(p-1)]} \\ \vdots \\ e_{i,[k-1]} \end{bmatrix}, \qquad (20)$$

$$\hat{u}_{[k]} := u_{[k]} + \delta_p u_{[k-1]} + \ldots + \delta_2 u_{[k-(p-1)]} + \delta_1 u_{[k-p]}. \qquad (21)$$

Note that these new variables are obtained by filtering the old ones with $S(\lambda)$.

In these new coordinates, the signals $\omega_{[k]}$ and $y_{\mathrm{ref}[k]}$ no longer appear directly as external inputs. Indeed, it can be verified that system (17)–(18) is equivalent to

$$\begin{bmatrix} \hat{x}_{[k+1]} \\ \hat{\eta}_{[k+1]} \end{bmatrix} = \begin{bmatrix} A & 0 \\ \mathcal{B}^*C & \mathcal{A}^* \end{bmatrix}\begin{bmatrix} \hat{x}_{[k]} \\ \hat{\eta}_{[k]} \end{bmatrix} + \begin{bmatrix} B \\ \mathcal{B}^*D \end{bmatrix} \hat{u}_{[k]} \qquad (22)$$

$$\hat{z}^\theta_{[k]} = \begin{bmatrix} 0 & \mathcal{C}_\theta \end{bmatrix}\begin{bmatrix} \hat{x}_{[k]} \\ \hat{\eta}_{[k]} \end{bmatrix}, \qquad (23)$$

where $\hat{z}^\theta_{[k]} := \mathcal{C}_\theta \hat{\eta}_{[k]}$ equals $(I - \theta/h)\, e_{[k-p]} + (\theta/h) e_{[k-(p-1)]}$.

In the special case when $p = 1$, we have $S(\lambda) = \lambda + \delta_1$. We also have $\mathcal{A}^* = -\delta_1 I_r$ and $\mathcal{B}^* = I_r$ for the servo-compensator. We still can introduce a matrix θ as in (15), and again introduce the new coordinates (19)–(21), which in this case simplify to $\hat{x}_{[k]} = x_{[k]} + \delta_1 x_{[k-1]}$, $\hat{\eta}_{[k]} = e_{[k-1]}$, and $\hat{u}_{[k]} = u_{[k]} + \delta_1 u_{[k-1]}$. In terms of these variables, system (17)–(18) can be rewritten as

$$\begin{bmatrix} \hat{x}_{[k+1]} \\ \hat{\eta}_{[k+1]} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & -\delta_1 I \end{bmatrix}\begin{bmatrix} \hat{x}_{[k]} \\ \hat{\eta}_{[k]} \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} \hat{u}_{[k]} \qquad (24)$$

$$\hat{z}^\theta_{[k]} = \begin{bmatrix} \frac{\theta}{h}C & I - \frac{\theta}{h}(1+\delta_1) \end{bmatrix}\begin{bmatrix} \hat{x}_{[k]} \\ \hat{\eta}_{[k]} \end{bmatrix} + \frac{\theta}{h} D \hat{u}_{[k]}, \qquad (25)$$

where the output, $\hat{z}^\theta_{[k]}$, now equals $(I - \theta/h)\, e_{[k-1]} + (\theta/h) e_{[k]}$.

From the structure of (22)–(25), the following result can be obtained:

Theorem 3.1: Given (1)–(3) and the class of disturbance/reference signals (4)–(8):

(a) The augmented plant representations (22)–(25) are stabilizable and detectable, and possess the same fixed modes [3] as (1)–(3), iff the existence conditions of Theorem 2.1 hold.

(b) The augmented plant representations (22)–(25) are minimum phase iff the plant (1)–(3) is minimum phase. □

It follows from this theorem that, if the plant (1)–(3) satisfies the RSP existence conditions, then there exists a compensator that stabilizes the augmented plant in the new coordinates (22)–(25); such a compensator solves the RSP.

3.2 Proposed Performance Index and Optimal Controller

As indicated in Section 1, we now propose the performance index

$$J_\theta := \sum_{k=1}^{\infty} \left\{ (\hat{z}^\theta_{[k]})'(\hat{z}^\theta_{[k]}) + \varepsilon \cdot (\hat{u}_{[k]})'(\hat{u}_{[k]}) \right\}, \qquad (26)$$

where θ is a given matrix of the form (15), ε is a given positive scalar, $\hat{u}_{[k]}$ is defined in (21), and $\hat{z}^\theta_{[k]}$ equals $(I-\theta/h)e_{[k-p]} + (\theta/h)e_{[k-(p-1)]}$. Recall that the motivation for using such a performance index is that we desire to make $\hat{z}^\theta_{[k]}$ small, which, in turn, will result in a "good" transient response.

Assuming that the conditions of Theorem 2.1 hold, we know from Theorem 3.1 and standard LQR theory that there exists an optimal stabilizing controller that minimizes (26) subject to (22)–(25) and which solves the RSP for (1)–(3). The optimal controller is a state-feedback controller of the form

$$\hat{u}_{[k]} = \begin{bmatrix} K_0 & K_1 \end{bmatrix} \begin{bmatrix} \hat{x}_{[k]} \\ \hat{\eta}_{[k]} \end{bmatrix} = K_0\hat{x}_{[k]} + K_1\hat{\eta}_{[k]},$$

where K_0 and K_1 are the controller gains computed from the LQR Riccati equation. It follows from the definitions of $\hat{x}_{[k]}$, $\hat{\eta}_{[k]}$ and $\hat{u}_{[k]}$ that an equivalent representation of this controller is

$$u_{[k]} = K_0 x_{[k]} + K_1 \eta_{[k]}.$$

Combining this with the servo-compensator state equation, (11), we obtain the following final control structure:

$$\eta_{[k+1]} = \mathcal{A}^*\eta_{[k]} + \mathcal{B}^* e_{[k]} \qquad (27)$$

$$u_{[k]} = K_0 x_{[k]} + K_1 \eta_{[k]}. \qquad (28)$$

Note that, in this optimization problem, the usual limitations on performance apply if the plant is non-minimum-phase (see [1]). In addition, we note that standard observer construction can be used to estimate the state if it is not measurable (see [3]).

4 Examples

To illustrate the type of performance achievable using the proposed control scheme, we now consider two examples.

The first example, taken from [6], is a non-minimum-phase single-input single-output flexible crane system, pictured in Figure 1(a). The input is the force applied to the cart, and the output is the horizontal position of the mass. For certain masses, spring constants, etc., the basic equations of motion, linearized about the "down" equilibrium point and sampled with sampling period $h = 0.1$ seconds, fit a model of the form (1)–(3) where

$$A = \begin{bmatrix} 1 & 0.1 & 0.04882 & 0.001628 & 0.0006816 & 0.0001857 \\ 0 & 1 & 0.9748 & 0.04882 & 0.08267 & 0.0004959 \\ 0 & 0 & 0.9902 & 0.09967 & 0 & 0 \\ 0 & 0 & -0.1950 & 0.9902 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7555 & 0.008402 \\ 0 & 0 & 0 & 0 & -40.509 & 0.7471 \end{bmatrix},$$

$$B = \begin{bmatrix} 5.0959\cdot 10^{-6} \\ 1.0169\cdot 10^{-4} \\ -4.9919\cdot 10^{-7} \\ -9.9674\cdot 10^{-6} \\ 5.0705\cdot 10^{-7} \\ 8.4020\cdot 10^{-5} \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D=0, \quad E=0, \quad F=0.$$

Initially consider the servomechanism problem for *constant* tracking/disturbance signals, i.e., $\Lambda = \{0\}$ in (10). The solid curves in Figure 2 are simulation results for a unit step reference signal when the control weighting in (26) is $\varepsilon = 10^{-5}$ and when the desired transient shape has parameter $\theta = 3$, corresponding to an ideal error transient that is a decaying exponential with a time constant of 3 seconds (i.e., the desired settling time is 12 seconds or 120 samples). For comparison, Figure 2 also shows in dashed curves the response of the "standard" controller corresponding to $\theta = 0$, while still using $\varepsilon = 10^{-5}$. In both cases, state-feedback is used to implement the controller. Notice that the proposed control scheme gives a much smoother response than the "standard" controller, without using extra control effort. Also note that the settling time is, indeed, close to the desired 120 samples.

Next, still using the flexible crane system, we consider the servomechism problem where *ramp* reference signals are possible, i.e., $\Lambda = \{0\ 0\}$ in (10). Figure 3 shows simulation results for the reference signal $y_{\mathrm{ref}[k]} = 50 + 0.1k$, i.e., the discretized version of the signal $50+t$. The parameter values in the simulation are $\varepsilon = 10^{-7}$ and $\theta = 20$ (for the proposed controller simulation, shown using solid curves) and $\varepsilon = 10^{-7}$ and $\theta = 0$ (for the standard controller simulation, shown using dashed curves). The second frame in the figure, showing the tracking error, clearly illustrates that the transient performance is smoother when using the proposed controller.

The second example we consider is a minimum-phase multivariable mass-spring-damper system, shown in Figure 1(b), where the system inputs are forces and the output are positions. We assume that the masses are $M = 10$ and $m_1 = m_2 = 1$, the spring constant is $K = 1$, the damper coefficient is $B = 1$, and the sampling period is $h = 0.1$. The resulting model has parameters

$$A = \left[\begin{matrix} 9.990\cdot 10^{-1} & 9.987\cdot 10^{-2} & 4.975\cdot 10^{-4} \\ -1.984\cdot 10^{-2} & 9.970\cdot 10^{-1} & 9.920\cdot 10^{-3} \\ 4.975\cdot 10^{-3} & 6.636\cdot 10^{-4} & 9.950\cdot 10^{-1} \\ 9.920\cdot 10^{-2} & 1.489\cdot 10^{-2} & -9.927\cdot 10^{-2} \\ 4.975\cdot 10^{-3} & 6.636\cdot 10^{-4} & 2.070\cdot 10^{-6} \\ 9.920\cdot 10^{-2} & 1.489\cdot 10^{-2} & 6.610\cdot 10^{-5} \end{matrix}\right.$$

$$\left.\begin{matrix} 6.636\cdot 10^{-5} & 4.975\cdot 10^{-4} & 6.636\cdot 10^{-5} \\ 1.489\cdot 10^{-3} & 9.920\cdot 10^{-3} & 1.489\cdot 10^{-3} \\ 9.934\cdot 10^{-2} & 2.070\cdot 10^{-6} & 2.568\cdot 10^{-7} \\ 9.851\cdot 10^{-1} & 6.610\cdot 10^{-5} & 8.680\cdot 10^{-6} \\ 2.568\cdot 10^{-7} & 9.950\cdot 10^{-1} & 9.934\cdot 10^{-2} \\ 8.680\cdot 10^{-6} & -9.927\cdot 10^{-2} & 9.851\cdot 10^{-1} \end{matrix}\right],$$

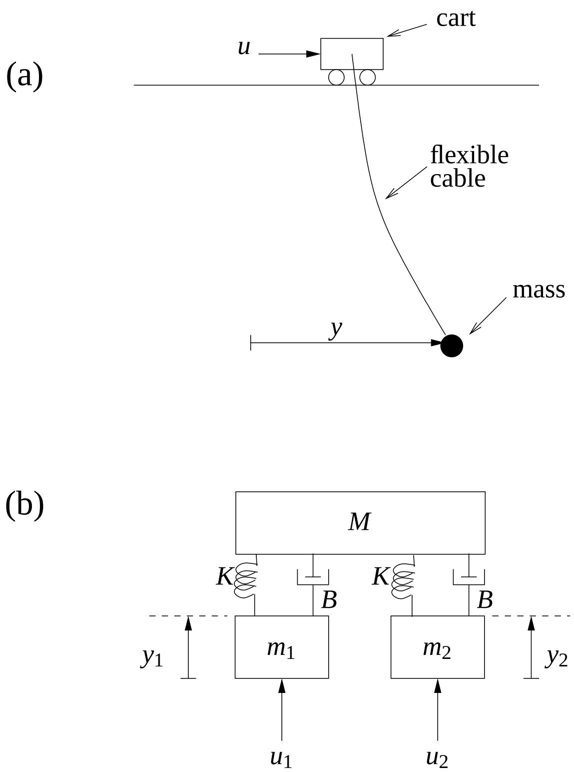

Figure 1: *(a) Flexible crane system; (b) Multivariable mass-spring-damper system.*

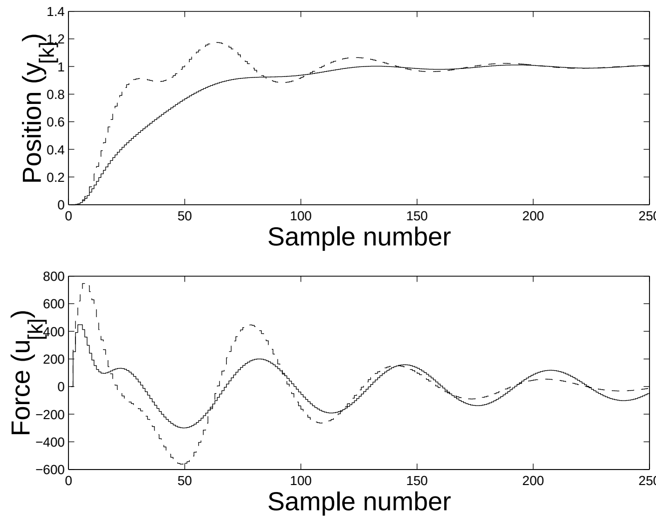

Figure 2: *Step response simulation results for the flexible crane system. The solid curves are for the proposed controller, and the dashed for the "standard" controller.*

$$B = \begin{bmatrix} 2.076 \cdot 10^{-6} & 2.076 \cdot 10^{-6} \\ 6.636 \cdot 10^{-5} & 6.636 \cdot 10^{-5} \\ 4.979 \cdot 10^{-3} & 5.943 \cdot 10^{-9} \\ 9.934 \cdot 10^{-2} & 2.568 \cdot 10^{-7} \\ 5.943 \cdot 10^{-9} & 4.979 \cdot 10^{-3} \\ 2.568 \cdot 10^{-7} & 9.934 \cdot 10^{-2} \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \ D = 0, \ E = 0, \ F = 0.$$

Consider the servomechanism problem for *constant* tracking/disturbance signals. Figures 4 and 5 show simulation results, using $\varepsilon = 100$ and $\theta = 4$ for the proposed controller (solid), and using $\varepsilon = 100$ and $\theta = 0$ for the "standard" controller (dashed). In Figure 4, a step reference signal is applied to the first channel of $y_{\text{ref}[k]}$; in Figure 5, a step is applied to the second channel of $y_{\text{ref}[k]}$. Notice that the proposed controller gives a response that, relative to the "standard" controller, is smooth and has little cross-channel interaction. Moreover, the controller gives excellent tracking performance for both channels.

5 Conclusions

This paper has proposed a new type of cheap-control quadratic performance index for the discrete-time servomechanism problem. Minimizing this new index results in a nominal transient response that has a desired speed of response, is "smooth", and

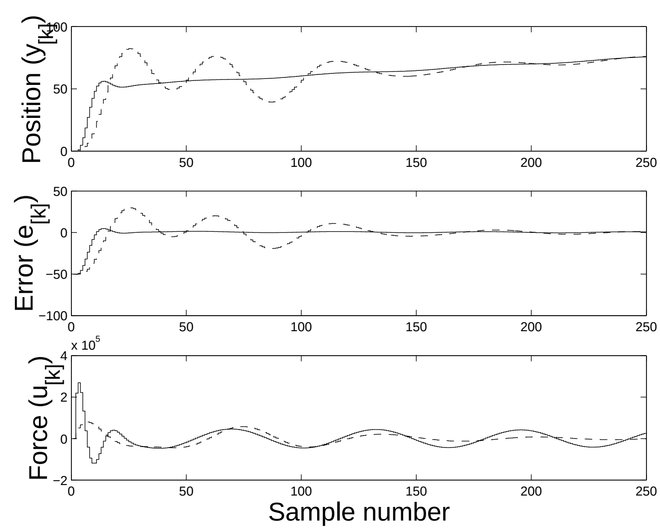

Figure 3: *More simulation results for the flexible crane system. The reference signal is $50 + 0.1k$. The solid curves are for the proposed controller, and the dashed for the "standard" controller.*

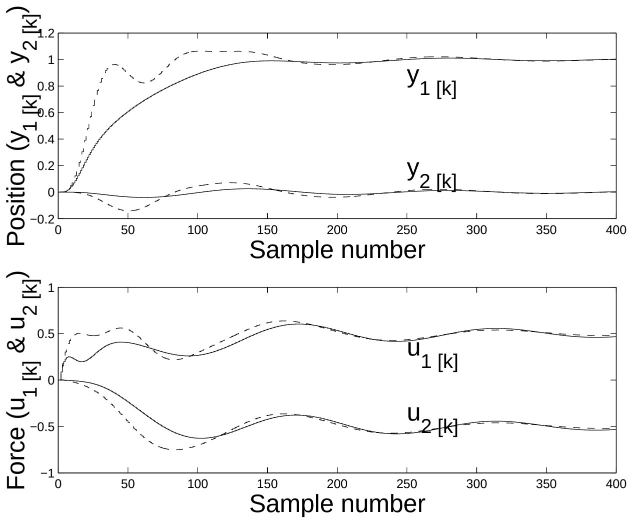

Figure 4: *Simulation results for the multivariable mass-spring-damper system. The reference signal has a unit step on the first channel. The solid curves are for the proposed controller, and the dashed for the "standard" controller.*

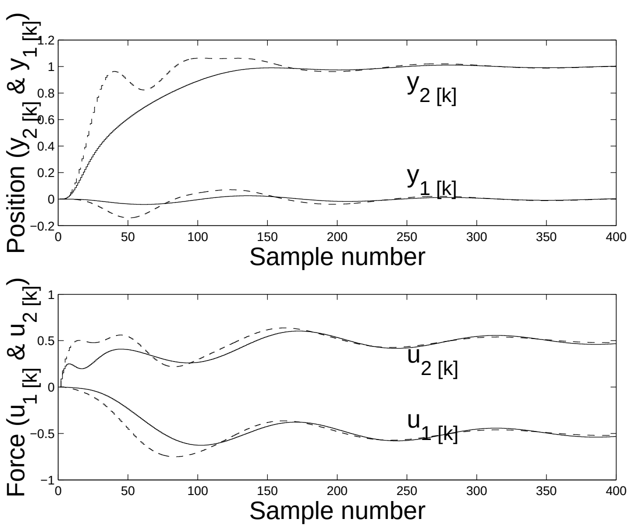

Figure 5: *More simulation results for the multivariable mass-spring-damper system. The reference signal has a unit step on the second channel. The solid curves are for the proposed controller, and the dashed for the "standard" controller.*

is "almost free" of interaction between output channels. It is interesting to note that, relative to the "standard" controller, a significant improvement in transient performance can be attained without using significantly more control effort. Understanding and quantifying this effect is a topic of future research.

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