IMPULSIVE BEHAVIORS OF DISCRETE AND CONTINUOUS TIME VARYING SYSTEMS: A UNIFIED APPROACH

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Abstract

A unified theory of continuous time impulsive behaviors (consisting of linear combinations of the Dirac distribution and its derivatives, generated by "inconsistent initial conditions") and of discrete time impulsive behaviors (consisting of backward solutions with finite support, generated by "inconsistent final consitions") is developed first in an analytic setting, then in an algebraic one; the latter is valid for possibly time varying systems. The effectiveness of the theory is illustrated through several examples.

1 Introduction

Impulsive behaviors of continuous time-invariant (CTI) linear systems are due to "inconsistent initial conditions" following an abrupt change in the system. Such a change is called a "temporal interconnection" [2]; typical temporal interconnections are: switches in electrical circuits, mechanical linkages made active or inactive, valves switched in hydraulic systems, etc. [8]. A system resulting from a temporal interconnection is a temporal system [2]. Impulsive behaviors of CTI temporal systems have been extensively studied for more than 20 years: in [25] and [26], which are among the first relevant contributions, systems are considered in descriptor form or as defined by a "polynomial matrix description" (a more detailed theory can be found in [24]); systems described by a "behavioral representation" (in Willems' spirit [27]), are considered in [15], [14]. According to Fliess [6], a linear system can also be defined as a finitely generated module over a differential polynomial ring, and this setting is "dual" (in the categorical point of view widely used by Oberst [22]) of the behavioral one. Such a duality has also been used in [2] to extend the theory of impulsive behaviors to continuous time-varying (CTV) temporal systems. Assuming that the system is defined by an equation of the form

$$B(\partial)\, w = 0 \tag{1}$$

where $B(\partial)$ is a polynomial matrix and ∂ is the distributional derivative (referred to as the "continuous-time derivative" in the sequel), the impulsive behavior is determined by the structure of the zero at infinity of $B(\partial)$ [24], [15], or equivalently by the structure of the module of *uncontrollable poles at infinity* (also called *input-decoupling zeros at infinity* when the input variables are specified) of the temporal system defined by the matrix $B(\partial)$ [2], [4].

Discrete time-invariant (DTI) linear systems in descriptor form have been studied in [19], [20], [17], [18], and it has been shown that they may have forward *and backward* solutions, determined by boundary conditions consisting in a combination of initial *and final* conditions. The existence of backward solutions is linked to the Kronecker canonical form of the matrix pencil associated with the descriptor system, and more specifically to its infinite elementary divisors. This connection is also made in [1], [12], [13] in the case of discrete-time auto-regressive representations, not necessarily in descriptor form. Such a representation can be put in the form (1) with $\partial := q - 1$, where q is the usual shift forward operator (this ∂ is referred to as the "discrete-time derivative" in the sequel). For a discrete-time system, a "temporal interconnection" is an interconnection made *up to a final time* (as opposed to the case of continuous-time systems, where it is made *from an initial time*).

Without loss of generality, this initial or final time is assumed to be zero in the sequel. System (1) can be viewed as resulting from the interconnection of the subsystem

$$B(\partial)\, \hat{w} = e \tag{2}$$

with the trivial subsystem $e = 0$ (where $\hat{w}$ denotes the variable w when leaving aside the interconnection); assuming that this interconnection is *temporal*, i.e. $e(t) = 0$ for $t \in T$, where $T = [0, +\infty[$ in the continuous-time case and $T =]-\infty, 0] \cap \mathbf{Z}$ ($\mathbf{Z}$ denoting the set of integers) in the discrete-time one, system (2) is said to be a *temporal system*.

The aim of this paper, which elaborates upon some remarks already made in [2], is to develop a unified theory of continuous time and discrete time behaviors. An analytic approach is first developed in Section 2 for time-invariant systems. A more general algebraic approach is then developed in Section 3 for possibly time-varying systems (detailed proofs are omitted, due to the lack of place, and can be found in [3]).

2 Impulsive behaviors of time-invariant systems: a unified analytic approach

Impulsive behaviors of CTI systems can be studied using the "causal Laplace transform" (proposed in [2] to clarify the classic "L_- Laplace transform" and briefly recalled in Section 2.1). Impulsive behaviors of DTI systems can be studied using the "anticausal Z-transform", developed in Section 2.2. As shown below, these transforms have similar properties, giving rise to a "unified analytic approach" for impulsive behaviors of CTI and DTI systems.

2.1 A brief review of the causal Laplace transform

In the continuous-time case, interconnecting two systems from time 0 consists in multiplying a function (such as the function e in Section 1) by $1-\Upsilon$, where Υ is the Heaviside function. Let W be the subspace of $\Re^{\Re}$ consisting of all functions f such that: f has a left bounded support, is infinitely differentiable and is dominated by the function $t \to e^{\alpha t}$ for some real α in a neighborhood of $t=+\infty$. Set

$$\Delta = \oplus_{n\geq 0} \Re \delta^{(n)} \tag{3}$$

where δ is the Dirac distribution. The $\Re[\partial]$-module generated by ΥW is (as $\Re$-vector space): $Q = \Upsilon W \oplus \Delta$. Set $S := (1-\Upsilon) W \oplus Q$. For any signal w belonging to S, let w_+ be its causal component (i.e. its component in Q). The causal Laplace transform $L_c(w)$ of w is defined as [2]: $L_c(w) = L(w_+)$, where $L(w_+)$ is the two-sided Laplace transform of w_+. The causal Laplace transform is not injective on S, since $\ker L_c = (1-\Upsilon) W$. For any $w \in S$,

$$L_c(\partial^n w) = s^n L_c(w) - \left(\sum_{i=1}^{n} s^{n-i} \partial_0^{i-1}\right) w \tag{4}$$

for every $n \geq 0$, where ∂_0^n is the operator defined on S as: $\partial_0^n w := \partial^n w(0^-)$.

2.2 Anticausal Z-transform

We now consider the discrete-time case. Let Υ be the sequence defined by $\Upsilon(t) = 1$ for $t > 0$ and 0 otherwise (sequences are considered and noted as functions defined on $\mathbf{Z}$). Let $S = Q = W$ be the subspace of $\Re^{\mathbf{Z}}$ consisting of sequences with left bounded support, and let $\Delta := (1-\Upsilon) W$.[1] The identity (3) still holds with $\delta := \partial \Upsilon$; δ is the sequence such that $\delta(t) = 1$ for $t = 0$ and 0 otherwise.

Definition 1: Let $w \in S$, and let w_- be its anticausal component, *i.e.* $w_- = (1-\Upsilon) w$. The anticausal Z-transform of w, noted $Z_a(w)$, is: $Z_a(w) = Z(w_-)$, where $Z(w_-)$ is the two-sided Z-transform of w_-.

The Z_a-transform inherits all classic properties (not detailed here) of the usual Z-transform; it is not injective, since $\ker Z_a = \Upsilon W$.

Let $\partial_0^n, n \geq 0$, be the operator defined on S as follows: for any $w \in S$, $\partial_0^0 w = -w(1)$ and $\partial_0^n w = \partial_0^0(\partial^n w), n \geq 0$. The following theorem is easily proved by induction:

Theorem 2: Set $s := z-1$; for every $n \geq 0$,

$$Z_a(\partial^n w) = s^n Z_a(w) - \left(\sum_{i=1}^{n} s^{n-i} \partial_0^{i-1}\right) w. \tag{5}$$

2.3 Impulsive behaviors of CTI and DTI temporal systems

Setting $\pounds = L_c$ in the CTI case and $\pounds = Z_a$ in the DTI one, (4) and (5) reduce to one formula. In both cases, the following result is clear:

Lemma 3: For every $n \geq 0$, $\partial_0^n \delta = 0$.

Therefore using (4) and (5) one obtains (with $\delta^{(n)} := \partial^n \delta$):

Proposition 4: For every $n \geq 0$, $\pounds^{-1}(s^n) = \delta^{(n)} + \ker \pounds$.

Note that:

$$\Delta \cong_{\Re} \frac{Q}{\Upsilon W} := \bar{\Delta}. \tag{6}$$

From the above, and according to the characterizations given in [24], [15] for the impulsive behavior $\mathcal{B}_\infty$ in the CTI case, one and deduce the following result[2], as stated in [2]:

Proposition 5: The structure of the impulsive behavior $\mathcal{B}_\infty$ is determined (in the CTI case and in the DTI one as well) by the structure of the zero at infinity of $B(\partial)$.

This result is specified below (see Proposition 7 and Theorem 14). According to [1], in the DTI case, $\mathcal{B}_\infty$ is the "backward subspace" corresponding to a maximal forward/backward decomposition of the whole space of solutions [17].

3 Impulsive behaviors of time-varying systems: a unified algebraic approach

3.1 Rings, modules and cogenerators

Let $\mathbf{K}$ be a commutative field; equipped with an α-derivation γ (a notion which is clarified below in the two cases we are concerned with), it becomes a "differential field"; α and γ are endomorphisms of $\mathbf{K}$. Then, $\mathbf{R} := \mathbf{K}[\partial; \alpha, \gamma]$ denotes the ring of "differential polynomials" of the form $\sum_{i\in I} a_i \partial^i$, where $a_i \in \mathbf{K}$ and I is a finite set of non-negative integers, equipped with the "commutation rule": $\partial a = a^\alpha \partial + a^\gamma$ [5]. The field of constants of $\mathbf{K}$ (consisting of all elements $a \in \mathbf{K}$ such that $a^\gamma = 0$ and $a^\alpha = a$) is denoted by $\mathbf{k}$. Two cases are considered in what follows:

- In the continuous-time case, ∂ is the continuous-time derivative, $\gamma = \partial$ and α is the identity. The above commutation rule is the usual Leibniz rule.
- In the discrete-time case, ∂ is the discrete-time derivative, $\gamma = \partial$ and α is the shift forward operator q [9].

[1] In the CTI case as in the DTV one (and *a fortiori* for time varying systems studied in Section 3.5), S may be too small for System (1) to have all its solutions in that space; however, as far as the impulsive behavior is concerned, S is large enough.

[2] An equivalent result has been recently established for the DTI case in [12], [13].

The commutation rule, in this case, is explained by the following identity: $a(t+1)w(t+1) - a(t)w(t) = a(t+1)(w(t+1) - w(t)) + (a(t+1) - a(t))w(t)$.

In both cases (assuming, in the latter, that α is an automorphism of $\mathbf{K}$), $\mathbf{R}$ is a principal ideal domain [5]. When no confusion is possible, ∂w and a^{γ} are respectively noted $\dot{w}$ and $\dot{a}$. Strictly speaking, there is a difference between ∂ and γ: the former is an operator on signals, the latter on coefficients belonging to $\mathbf{K}$.

Set $\sigma = \partial^{-1}$, $\beta = \alpha^{-1}$ and let $\mathbf{S} := \mathbf{K}[[\sigma;\beta,\gamma]]$ be the ring of formal power series in σ, equipped with the commutation rule $\sigma a = a^{\beta}\sigma - \sigma a^{\beta\gamma}\sigma$ [5] deduced from the commutation rule of $\mathbf{R}$; this ring is a principal ideal domain and is local with maximal ideal (σ). The skew field of Laurent series in σ (equipped with the same commutation rule as $\mathbf{S}$) is denoted by $\mathbf{L} := \mathbf{K}((\sigma;\beta,\gamma))$; it is obtained by *localizing and completing at infinity* the ring $\mathbf{R}$.

Let $_{\mathbf{S}}$**Mod** (resp. $_{\mathbf{S}}$**Modf**) be the category of left $\mathbf{S}$-modules (resp. of finitely generated left $\mathbf{S}$-modules), and let $M^+ \in_{\mathbf{S}}$**Modf**. The module M^+ can be decomposed as a direct sum: $M^+ = \mathcal{T}^+ \oplus \Phi^+$, where $\mathcal{T}^+ = \mathcal{T}(M^+)$ is the torsion submodule of M^+ and where $\Phi^+ \cong_{\mathbf{S}} M^+/\mathcal{T}^+$ is free. The ascending chain of invariant factors of $\mathcal{T}^+$, if any, is of the form $(\sigma^{\mu_r}) \subset ... \subset (\sigma^{\mu_1})$, where $1 \leq \mu_1 \leq ... \leq \mu_r$, and

$$\mathcal{T}^+ \cong_{\mathbf{S}} \bigoplus_{i=1}^{r} \frac{\mathbf{S}}{(\sigma^{\mu_i})}. \tag{7}$$

Set $C_\mu = \frac{\mathbf{S}}{(\sigma^{\mu})}$, and, for $\mu \geq 1$, let $\tilde{\delta}^{(\mu-1)}$ be the canonical image of $1 \in \mathbf{S}$ in C_μ. Consider the following $\mathbf{S}$-module morphism from C_μ to $C_{\mu+1}$: $\mathbf{s} + (\sigma^{\mu}) \to \sigma\,\mathbf{s} + (\sigma^{\mu+1})$. It is a monomorphism through which C_μ is embedded in $C_{\mu+1}$ and $\tilde{\delta}^{(\mu-1)}$ is identified with $\sigma\tilde{\delta}^{(\mu)}$, hence

$$C_\mu = \oplus_{i=1}^{\mu} \left\langle \tilde{\delta}^{(i-1)} \right\rangle \tag{8}$$

where $\left\langle \tilde{\delta}^{(i)} \right\rangle := \mathbf{K}\,\tilde{\delta}^{(i)}$. Set

$$\tilde{\Delta} := \text{direct.}\lim C_\mu = \oplus_{\mu \geq 0} \left\langle \tilde{\delta}^{(\mu)} \right\rangle. \tag{9}$$

The $\mathbf{S}$-module $\tilde{\Delta}$ is called the *canonical cogenerator* of $_{\mathbf{S}}$**Mod**. It is the injective hull of C_1 (thus it is an injective cogenerator of $_{\mathbf{S}}$**Mod**). This construction of $\tilde{\Delta}$ is classic and due to Matlis in the commutative case (see [16]). As $\mathbf{S}$ is complete (for the (σ)-adic topology), the endomorphism ring $\mathbf{E}$ of $\tilde{\Delta}$ is isomorphic to $\mathbf{S}$.

Let $M^+ \in_{\mathbf{S}}$**Modf**; this module is *presented* by a matrix $B^+(\sigma)$ over $\mathbf{S}$ (which may be assumed to be right regular without loss of generality, with k columns). This means that there exist generators $w_1^+, ..., w_k^+$ of M^+ (written $M^+ = [w^+]_{\mathbf{S}}$, where $w^+ := \left[w_1^+, ..., w_k^+\right]^T$) such that $B(\sigma)\,w^+ = 0$. Here, $M^+ = \operatorname{coker} \bullet B^+(\sigma)$, where $\bullet B^+(\sigma)$ is the $\mathbf{S}$-morphism $\mathbf{S}^q \ni e \to eB^+(\sigma) \in \mathbf{S}^k$ (the elements of the free modules $\mathbf{S}^q$ and $\mathbf{S}^k$ are represented by row matrices in the canonical bases, and $\bullet B^+(\sigma)$ operates on the left).

For any $M^+ \in_{\mathbf{S}}$**Modf**, set $(M^+)^* = \operatorname{Hom}_{\mathbf{S}}\left(M^+, \tilde{\Delta}\right)$; $(M^+)^*$ in an $\mathbf{E}$-module, isomorphic to the set of elements $\tilde{w} \in \tilde{\Delta}^k$ such that $B(\sigma)\,\tilde{w} = 0$, i.e. to $\ker B(\sigma) \bullet$, where $B(\sigma) \bullet$ right-operates on the elements of $\tilde{\Delta}^k$ (represented by column matrices). Similarly, set $(M^+)^{**} = \operatorname{Hom}_{\mathbf{E}}\left((M^+)^*, \tilde{\Delta}\right)$. The module $\tilde{\Delta}$ defines a "Morita duality" from $\mathbf{S}$ to $\mathbf{E}$, thus the $\mathbf{S}$-morphism $M \to M^{**}$ is an automorphism of $_{\mathbf{S}}$**Modf** ([16], § 19E).

3.2 A key isomorphism

It is assumed in this § and in the next one that $\mathbf{K} = \Re$.

Consider first the continuous-time case. The operator ∂ is an automorphism of the $\Re$-vector space S, and $\sigma = \partial^{-1}$ is the operator defined on S by: $(\sigma w)(t) = \int_{-\infty}^{t} w(\tau)\,d\tau$ (this operator is called the "continuous-time integrator" in the sequel). The space S is an $\mathbf{S}$-module, Q is an $\mathbf{S}$-submodule of S and ΥW is an $\mathbf{S}$-submodule of Q. The $\mathbf{R}$-module Δ is not an $\mathbf{S}$-module, but $\bar{\Delta}$, defined by (6), is an $\mathbf{S}$-module. Note that

$$\Delta \cong_{\Re} \frac{S}{(1-\Upsilon)\,W \oplus \Upsilon W} \cong_{\Re} \bar{\Delta}. \tag{10}$$

Let τ be the $\Re$-isomorphism (6), or equivalently (10) :

$$\tau : \Delta \longrightarrow \bar{\Delta} \tag{11}$$

One has $\sigma\delta = \Upsilon$; setting $\bar{\delta} = \tau(\delta)$, $\sigma\bar{\delta} = 0$, thus $\bar{\delta}$ and $\tilde{\delta}$ can be identified, as well as the $\mathbf{S}$-modules $\tilde{\Delta}$ and $\bar{\Delta}$.

In the discrete-time case, the operator ∂ is still an automorphism of the $\Re$-vector space S, and $\sigma = \partial^{-1}$ is the operator defined on S by: $(\sigma w)(t) := \sum_{j=-\infty}^{t-1} w(j)$ (this operator is called the "discrete-time integrator" in the sequel). The isomorphism (6), (11) still holds and the $\mathbf{S}$-modules $\tilde{\Delta}$ and $\bar{\Delta}$ can still be identified. Obviously, the discrete-time case is completely analogous to the continuous-time one.

3.3 Impulsive behavior in the time-invariant case

Consider the temporal system (2) where $g \in S^q$ and $\hat{w} \in S^k$. The matrix $B(\partial)$ is right regular. Let $\tilde{\phi}$ be the canonical epimorphism $S \to \tilde{\Delta}$. We are interested in the impulsive solutions, i.e. in those solutions whose components are linear combinations of elements of Δ and which are generated by a temporal interconnection. Due to isomorphism (11), these solutions can be found by *localizing and completing at infinity* the ring $\mathbf{R}$ (i.e. by embedding $\mathbf{R}$ in $\mathbf{L}$), and by replacing the elements $\hat{w}_i$ $(1 \leq i \leq k)$ and g_j $(1 \leq j \leq q)$ by their canonical images $\tilde{w}_i = \tilde{\phi}\hat{w}_i$ and $\tilde{g}_j = \tilde{\phi}g_j$ in $\tilde{\Delta}$. Let us detail this.

As is well known, there exist unimodular matrices $U(\sigma) \in$

$\mathbf{S}^{q\times q}$ and $V(\sigma) \in \mathbf{S}^{k\times k}$ such that

$$U(\sigma)\, B(\partial)\, V^{-1}(\sigma) = \left[\begin{array}{cc} diag\{\sigma^{\nu_i}\} & 0 \end{array}\right], \qquad (12)$$

$(\nu_1 \leq ... \leq \nu_q)$, which is the Smith-MacMillan form at infinity of $B(\partial)$ [11]. Define the finite sequences $(\bar{\pi}_i)_{1\leq i\leq q}$ and $(\bar{\mu}_i)_{1\leq i\leq q}$ as: $\bar{\pi}_i = \max(0, -\nu_i)$ and $\bar{\mu}_i = \max(0, \nu_i)$. Among the elements $\bar{\pi}_i$ (resp. $\bar{\mu}_i$), those which are nonzero (if any) are called the *structural indexes* of the *pole at infinity* (resp. of the *zero at infinity*) of the matrix $B(\partial)$ [4]; they are put in decreasing (resp. increasing) order and denoted by π_i $(1 \leq i \leq \rho)$ (resp. μ_i $(1 \leq i \leq r)$). Set $\tilde{v} = V(\sigma)\,\tilde{w}$ and $\tilde{h} = U^{-1}(\sigma)\,\tilde{g}$; from (2),

$$\left.\begin{array}{l} \tilde{v}_i = \sigma^{\bar{\pi}_i}\tilde{h}_i \\ \sigma^{\bar{\mu}_i}\tilde{v}_i = \tilde{h}_i \end{array}\right\} \quad 1 \leq i \leq q.$$

As for every $i \in \{1, ..., q\}$, $\tilde{g}_j = 0$, one has $\tilde{h}_i = 0$; therefore,

$$\left[\begin{array}{cc} \Sigma(\sigma) & 0_{q\times(k-q)} \end{array}\right] \tilde{v} = 0 \qquad (13)$$

where $\Sigma(\sigma) = \mathrm{diag}\,\{\sigma^{\bar{\mu}_i}\}_{1\leq i\leq q}$. Obviously:

Proposition 6: The set of solutions of (13) is the $\mathbf{E}$-module $\mathcal{A}_\infty \times \tilde{\Delta}^{k-q}$, where $\mathcal{A}_\infty = \ker \Sigma(\sigma) \bullet$.

The $\mathbf{E}$-submodule $\mathcal{A}_\infty$ of $\mathcal{A}_\infty \times \tilde{\Delta}^{k-q}$ consists of those solutions which are generated by the temporal interconnection. Therefore, one obtains the following result, which is a reformulation of Proposition 5 (but is obtained by a rationale where tedious calculations are avoided):

Proposition 7: i) The algebraic impulsive behavior of the temporal system (2) is $\mathcal{A}_\infty$; ii) the analytic impulsive behavior of the temporal system (2) is the $\Re$-vector space $\mathcal{B}_\infty \subset \Delta^q$ consisting of the elements $v = [v_1, ..., v_q]^T$ such that $v_i = \tau^{-1}(\tilde{v}_i)$ and $[\tilde{v}_1, ..., \tilde{v}_q]^T \in \mathcal{A}_\infty$.

3.4 Impulsive behavior of systems with time-varying coefficients in a field

The Smith-MacMillan form at infinity has been defined in [4] for a matrix over $\mathbf{R}$ when the coefficient field $\mathbf{K}$ is not necessarily a field of constants (see also [21]); this situation is considered here. Let $B(\partial) \in \mathbf{R}^{q\times k}$ be a right regular matrix, let $\hat{\Sigma}(\sigma) = \left[\begin{array}{cc} \Sigma(\sigma) & 0_{q\times(k-q)} \end{array}\right]$ be the matrix defined from its Smith-MacMillan form at infinity as in § 3.3, and set $M^+ = \mathrm{coker} \bullet \hat{\Sigma}(\sigma)$; thus $M^+ = \mathcal{T}^+ \oplus \Phi^+$, where $\Phi^+ \cong_{\mathbf{S}} \mathbf{S}^{k-q}$ and where $\mathcal{T}^+ = \mathcal{T}(M^+)$ has the structure (7).

Proposition 8: Let $A^{-1}(\sigma)\, B^+(\sigma)$ be any left coprime factorization of $B(\partial)$ over $\mathbf{S}$; then, $M^+ \cong_{\mathbf{S}} \mathrm{coker} \bullet B^+(\sigma)$.

Definition 9: The $\mathbf{S}$-module $M^+ = \mathrm{coker} \bullet B^+(\sigma)$ (uniquely defined from $B(\partial)$, up to isomorphism) is called the *impulsive system* defined by $B(\partial)$. The $\mathbf{E}$-module $\mathcal{A}_\infty = (\mathcal{T}^+)^*$ is called the *algebraic impulsive behavior* defined by $B(\partial)$.

Proposition 10: i) For any $\mu \geq 0$, $(C_\mu)^* \cong_{\mathbf{K}} C_\mu$, given by (8). ii) Assuming that $\mathcal{T}^+$ has the decomposition (7), $\mathcal{A}_\infty \cong_{\mathbf{K}} \prod_{i=1}^{r} (C_{\mu_i})^*$.

Let us take the following general definition:

Definition 11: i) Generally speaking, an *algebraic impulsive behavior* is an $\mathbf{E}$-submodule of $\tilde{\Delta}^k$, for some natural integer k. ii) A *subbehavior* of an algebraic impulsive behavior a_∞ is an $\mathbf{E}$-submodule of a_∞.

Algebraic impulsive behaviors enjoy the following properties:

Proposition 12: i) Every algebraic impulsive behavior is of the form $\ker C^+(\sigma) \bullet$, where $C^+(\sigma)$ is a right regular matrix over $\mathbf{S}$. ii) Let $\mathrm{a}_\infty \subset \tilde{\Delta}^k$ be an algebraic impulsive behavior, and let $L(\sigma)$ be a matrix over $\mathbf{S}$ with k columns; then, $L(\sigma)\,\mathrm{a}_\infty$ is a subbehavior of a_∞.

Let us define an *order relation* on algebraic behaviors:

Definition 13: Let a_∞^1 and a_∞^2 be two algebraic behaviors; $\mathrm{a}_\infty^2 \leq \mathrm{a}_\infty^1$ means that there exists a right regular matrix $L(\sigma)$ over $\mathbf{S}$ such that $\mathrm{a}_\infty^2 = L(\sigma)\,\mathrm{a}_\infty^1$ (where $L(\sigma)$ has the suitable number of columns for this expression to make sense).

Note that by Proposition 12, ii), $\mathrm{a}_\infty^2 \leq \mathrm{a}_\infty^1$ implies that a_∞^2 is a subbehavior of a_∞^1, but there are subbehaviors of a_∞^1 which are not of this form (for example, if $\mathrm{a}_\infty^1 = \tilde{\Delta}^k$, every set of the form $L(\sigma)\,\mathrm{a}_\infty^1$ is equal to $\tilde{\Delta}^l$ for some integer l, $0 \leq l \leq k$).

The algebraic impulsive behavior $\mathcal{A}_\infty = (\mathcal{T}^+)^*$ is directly characterized from $(M^+)^*$ (where $M^+ = \mathrm{coker} \bullet B^+(\sigma)$) by the following property, which also characterizes $\mathcal{T}^+$ (and therefore gives all the desirable information).

Theorem 14: i) Let $\mathcal{L}$ be the set of all algebraic impulsive behaviors a_∞ such that $\mathrm{a}_\infty \leq (M^+)^*$ and for which there exists a natural integer μ such that $\sigma^\mu \mathrm{a}_\infty = 0$. An element of $\mathcal{L}$ is maximal if, and only if it is $\mathbf{E}$-isomorphic to $\mathcal{A}_\infty$. ii) $\mathcal{T}^+ = (\mathcal{A}_\infty)^*$.

Let us briefly indicate on which properties are based the above results: Proposition 8 is proved by an elementary calculation. Proposition 10, i) is easily proved by induction, and ii) results from a basic property of the functor Hom. Proposition 12, i) is a consequence of Matlis' theory, and ii) is true because $\tilde{\Delta}$ is an injective $\mathbf{S}$-module [22]. Theorem 14, i) is a consequence of Proposition 10, ii) and of the fact that the $\mathbf{S}$-module $\tilde{\Delta}$ is faithful; ii) is true because $\tilde{\Delta}$ is a cogenerator [22], [16].

3.5 Analytic impulsive behavior in the time-varying case

In § 3.4, where $\mathbf{K}$ is a *field* containing nonconstant elements, an algebraic impulsive behavior has been defined and studied, but not an analytic one. The difficulty is that the spaces W, Q, Δ, etc., defined in Sect. 2 no longer have a natural structure of $\mathbf{R}$-module. The isomorphism (11) does no longer make sense.

Therefore, it is now assumed that $\mathbf{K}$ is a differential *domain*, e.g. $\mathbf{K} = \Re[t]$ equipped with the α-derivation γ. The operators ∂ and σ are those defined in Sect. 1 and § 3.2. The rings $\mathbf{R} = \mathbf{K}[\partial; \alpha, \gamma]$ and $\mathbf{S} = \mathbf{K}[[\sigma; \beta, \gamma]]$ (where $\beta = \alpha^{-1}$) are Noetherian domains; they can still be embedded in the domain $\mathbf{L} = \mathbf{K}((\sigma; \beta, \gamma))$ (which is no longer a field). The ideal (σ) is two-sided. The units of $\mathbf{S}$ are the power series whose constant term is a unit of $\mathbf{K}$, i.e. a real number. The spaces W, Q, Δ, as introduced in Sect. 2, are again $\mathbf{R}$-modules, Q, S and ΥW are again $\mathbf{S}$-modules, and the isomorphism (11) holds. The difficulty is that $\mathbf{R}$ and $\mathbf{S}$ are no longer principal ideal domains. Therefore, to obtain relevant results, a regularity assumption must be made.

Definition 15: The temporal system (2) is said to be *impulsively regular*, if its matrix of definition $B(\partial)$ has a Smith-MacMillan form at infinity, i.e. if there exist unimodular matrices $U(\sigma)$ and $V(\sigma)$ over $\mathbf{S}$ such that (12) holds.

Assuming that the temporal system (2) is impulsively regular, Propositions 6 and 7 are still valid. As in § 3.4, set $M^+ = \text{coker}\,\bullet\hat{\Sigma}(\sigma)$, so that $M^+ = \mathcal{T}^+ \oplus \Phi^+$, where $\Phi^+ \cong_{\mathbf{S}} \mathbf{S}^{k-q}$ and where $\mathcal{T}^+ = \mathcal{T}(M^+)$ has the structure (7). To recover Proposition 8, the following notion must be introduced:

Definition 16: Let $A(\sigma)$ and $B^+(\sigma)$ be two matrices over $\mathbf{S}$, having the same number of rows, and assume that $V(\sigma) := \left[\begin{array}{cc} A(\sigma) & B^+(\sigma) \end{array}\right]$ is right regular. The pair $(A(\sigma), B^+(\sigma))$ is said to be *completely left coprime* if $V(\sigma)$ is *completable*, which means that there exists a matrix $W(\sigma)$ such that $\left[\begin{array}{cc} V^T(\sigma) & W^T(\sigma) \end{array}\right]$ is unimodular [5].

Proposition 8 can now be extended as follows:

Proposition 17: i) Assuming that the temporal system (2) is impulsively regular, $B(\partial)$ has a completely left coprime factorization (CLCF) over $\mathbf{S}$. ii) Let $A^{-1}(\sigma)B^+(\sigma)$ be any CLCF of $B(\partial)$ over $\mathbf{S}$; then, $M^+ \cong_{\mathbf{S}} \text{coker}\,\bullet B^+(\sigma)$.

Let ${}_{\mathbf{S}}$**Modstruc** be the be the category of all finitely generated $\mathbf{S}$-modules of the form $\Phi \oplus \mathcal{T}^+$, where Φ is free and $\mathcal{T}^+$ is torsion with the structure (7). One can show that $\tilde{\Delta}$ is an injective cogenerator *for the category* ${}_{\mathbf{S}}$**Modstruc**. Therefore, the following results are still valid: Proposition 10, Proposition 12, ii), and Theorem 14. The analytic impulsive behavior $\mathcal{B}_\infty$ can be calculated using Proposition 7, ii).

Consider the following example (CTV or DTV):

$$B(\partial) = \left[\begin{array}{cccc} -1 & \partial^2 + t & 0 & 0 \\ 0 & 0 & \partial^2 & -1 \\ 0 & 1 & -1 & 0 \end{array}\right] \tag{14}$$

and write $w = \left[\begin{array}{cccc} u_1 & y_1 & u_2 & y_2 \end{array}\right]^T$. The corresponding temporal system can be viewed as the series temporal interconnection of System 1, with input u_1, output y_1 and equation $\ddot{y}_1 + t y_1 - u_1 = 0$, with System 2 with input u_2, output y_2 and equation $\ddot{u}_2 - y_2 = 0$; the interconnection equation is $u_2 = y_1$. In the interconnected temporal system with input u_1 and output y_2, the two derivatives are "hidden". It is easy to show that this temporal system is impulsively regular; in addition, one has the following CLCF: $B(\partial) = A^{-1}(\sigma)B^+(\sigma)$ with $A(\sigma) = diag(\sigma^2, \sigma^2, 1)$ and

$$B^+(\sigma) = \left[\begin{array}{cccc} -\sigma^2 & 1+\sigma^2 t & 0 & 0 \\ 0 & 0 & 1 & -\sigma^2 \\ 0 & 1 & -1 & 0 \end{array}\right].$$

The matrix $B^+(\sigma)$ is equivalent over $\mathbf{S}$ to $\left[\begin{array}{cc} \Sigma & 0 \end{array}\right]$ with $\Sigma = diag(1, 1, \sigma^2)$, thus $M^+ \cong_{\mathbf{S}} \mathbf{S} \oplus \frac{\mathbf{S}}{(\sigma^2)}$. The impulsive system M^+ is defined by the following equations:

$$(1+\sigma^2 t)\, y_1^+ - \sigma^2 u_1^+ = 0; \quad u_2^+ - \sigma^2 y_2^+ = 0; \quad y_1^+ = u_2^+$$

and $\mathcal{T}^+ = [v^+]_{\mathbf{S}}$ where

$$\sigma^2(v^+) = 0, \quad v^+ = t\, y_1^+ + y_2^+ - u_1^+. \tag{15}$$

The space $\mathcal{B}_\infty$ can also be analytically calculated. One obtains: for $t \geq 0$ in the CTV case[3], and for $t \leq 0$ in the DTV one,

$$t\, y_1(t) + y_2(t) - u_1(t) = \alpha_1 \delta(t) + \alpha_0 \dot{\delta}(t) := v(t) \tag{16}$$

where $\alpha_0 := \partial_0^0 (y_1 - u_2)$ and $\alpha_1 := \partial_0^1 (y_1 - u_2)$, which is consistent with Proposition 7, ii): $\mathcal{B}_\infty$ is the subspace of Δ spanned by v as (α_0, α_1) spans $\Re \times \Re$. This example illustrates the fact that, for impulsively regular time-varying temporal systems as for time-invariant ones, impulsive motions occur due to "inconsistent initial conditions" in the continuous-time case and to "inconsistent final conditions" in the discrete-time one (with respect to the interconnection equation).

An impulsive singularity arises if a system coefficient annihilates a part of an impulsive motion when vanishing (e.g., in the continuous-time case, an impulsive motion proportional to δ is annihilated by a coefficient a such that $a(0) = 0$). A "regularization procedure" can then be used. In the CTV case, one has $t^k \delta^{(n)} = (-1)^k \delta^{(n-k)}$, thus when operating on Δ (resp. $\tilde{\Delta}$), the operators $t^k \partial^n$ and $(-1)^k \partial^{n-k}$ (resp. $t^k \sigma^n$ and $(-1)^k \sigma^{n+k}$) are equivalent, written

$$t^k \partial^n \Delta \simeq (-1)^k \partial^{n-k}, \; t^k \sigma^n \tilde{\Delta} \simeq (-1)^k \sigma^{n+k} \tag{17}$$

Consider the following example, which is impulsively singular:

$$B(\partial) = \left[\begin{array}{ccc} -t\partial^4 & \partial & -t \\ 1 & 0 & 0 \end{array}\right] \tag{18}$$

The variable w_1 is discontinuous at $t = 0$ due to the second row, and its 4st order derivative in the first row generates elements of Δ. One can use (17). For the "regularized temporal system", $M^+ \cong_{\mathbf{S}} \mathbf{S} \oplus \mathcal{T}^+$ and $\mathcal{T}^+ = [v^+]_{\mathbf{S}}$ where $\sigma^2(v^+) = 0$, $v^+ = w_2^+ - \sigma t w_3^+$. The space $\mathcal{B}_\infty$ can be analytically calculated (with the help of Theorem 14); one obtains the following explicit expression: for $t \geq 0$, $w_1(t) = 0$ and $w_2(t) - \int_{0^-}^t \tau\, w_3(\tau)\, d\tau = f_i(t) + f_s(t)$ with $f_i = -3 w_1(0^-)\,\dot{\delta} - 2\dot{w}_1(0^-)\,\delta$ and $f_s = (w_2(0^-) - \ddot{w}_1(0^-))\,\Upsilon$. The space $\mathcal{B}_\infty \subset \Delta$ is spanned by f_i. This is consistent with the expression of $\mathcal{T}^+$ and Proposition 7.

Remark 18: In accordance with Theorem 14 and Proposition 7, "impulsive motions" such as the above f_i are generated by $\mathbf{S}$-linear (not, in general, $\Re$-linear) combinations of w; to the best of our knowledge, this point was not clear in the literature (even in the time-invariant case).

[3] With the slight abuse of notation since the signals involved here are distributions; but as they belong to the signal space S, this notation can be justified.

4 Concluding remarks

The impulsive behavior of an "impulsively regular" (or of a "regularized") temporal system can be explicitly calculated using the theory developed in this paper. The algebraic approach developed in Section 3 is the main contribution. It has several advantages: on one hand, it points out the *structure* of the impulsive behavior; on the other hand, it can be easily computerized, making it possible to calculate the impulsive behavior of a possibly *time-varying large-scale system.*

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