ON ROBUST STABILITY OF UNCERTAIN LINEAR NEUTRAL SYSTEMS WITH TIME-VARYING DISCRETE DELAY

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Abstract

This paper investigates the robust stability of uncertain linear neutral systems with time-varying discrete delay. A delay-dependent stability criterion is obtained and formulated in the form of a linear matrix inequality (LMI). Two numerical examples are given to indicate significant improvements over some existing results.

1 Introduction

The problem of stability of neutral systems has received considerable attention in the last two decades, see for example, [3]. The direct Lyapunov method is a powerful tool for studying stability of the systems. Some early results are based on a rather simple form of Lyapunov-Krasovskii functional, with stability criteria independent of time-delay [11]. A model transformation technique is often used to transform the pointwise delay system to a distributed delay system, and delay-dependent stability criteria are obtained [4, 8, 9]. These results are usually less conservative than the delay-independent stability ones. Some of these results can be improved by applying tighter bounding of the cross term introduced in Park [10]. Furthermore, time-varying discrete delays are not considered in the references mentioned above.

In this paper, the result in [10] will be extended to uncertain linear neutral systems with time-varying discrete delay. The uncertainty under consideration will be norm-bounded one. A delay-dependent stability criterion will be given to reduce the conservatism of the existing ones.

2. Problem statement

Consider the following linear neutral system with norm-bounded uncertainty

$$\dot{x}(t) - C\dot{x}(t-\tau) = \left(A + LF(t)E_a\right)x(t) + \left(B + LF(t)E_b\right)x(t-h(t)) \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ are constant matrices, and $F(t) \in \mathbb{R}^{p\times q}$ is an unknown real and possibly time-varying matrix with Lebesgue measurable elements satisfying

$$F^T(t)F(t) \le I \qquad (2)$$

and L, E_a, E_b, and E_d are known real constant matrices which characterize how the uncertainty enters the nominal matrices *A* and *B*. The delay $\tau \ge 0$ is a constant neutral delay and the discrete delay $h(t)$ is a time-varying function that satisfies

$$0 \le h(t) \le h_M\,,\ \dot{h}(t) \le h_d \qquad (3)$$

where h_M , and h_d are constants, and $0 \le h_d < 1$.

The initial condition of system (1) is given by

$$x(t_0+\theta)=\varphi(\theta)\,,\ \forall\,\theta\in[-\max\{\tau,h_M\},\,0] \qquad (4)$$

where $\varphi(\cdot)$ is a continuous vector-valued initial function.

The purpose of this paper is to formulate a practically computable criterion to check the stability of system described by (1)~(4).

3. Main result

System (1) can be written as

$$\dot{x}(t)-C\dot{x}(t-\tau)=Ax(t)+Bx(t-h(t))+Lu \qquad (5a)$$

$$y=E_a x(t)+E_b x(t-h(t)) \qquad (5b)$$

with the constraint

$$u=F(t)y \qquad (6)$$

We further rewrite (5)~(6) as

$$\dot{x}(t)-C\dot{x}(t-\tau)=\left(A+B\right)x(t)-B\int_{t-h(t)}^{t}\dot{x}(\xi)d\xi+Lu \quad (7)$$

$$u^T u\le\left(E_a x(t)+E_b x(t-h(t))\right)^T\left(E_a x(t)+E_b x(t-h(t))\right) \quad (8)$$

Define the operator $\mathcal{D}$: $\mathcal{C}\left([-\max\{\tau,h_M\}],\mathbb{R}^n\right)\to\mathbb{R}^n$ as

$$\mathcal{D}x_t=x(t)-Cx(t-\tau)$$

Throughout this paper, we assume that

A1. All the eigenvalues of matrix C are inside the unit circle.

We now state and establish the following result.

Proposition 1. Under A1, the system described by (1) to (4) is asymptotically stable if there exist real matrix X, symmetric positive definite matrices P, R, S, W, Y and a scalar $\varepsilon\ge 0$ such that the LMI (9), as shown at the bottom of the last page of the paper, holds, where

$$(1,1)\overset{\Delta}{=}\left(A+B\right)^T P+P\left(A+B\right)+R+S+X^T B+B^T X$$

$$(1,3)\overset{\Delta}{=}-\left(A+B\right)^T PC-B^T XC$$

Proof. Choose the Lyapunov-Krasovskii functional candidate for system (7) as $V=V_1+V_2+V_3+V_4+V_5$, where

$$V_1=(\mathcal{D}x_t)^T P(\mathcal{D}x_t)$$

$$V_2=\frac{1}{1-h_d}\int_{t-h(t)}^{t}\left(h(t)-t+\xi\right)\dot{x}^T(\xi)B^T QB\dot{x}(\xi)d\xi$$

$$V_3=\int_{t-h(t)}^{t}x^T(\xi)Rx(\xi)d\xi$$

$$V_4=\int_{t-\tau}^{t}x^T(\xi)Sx(\xi)d\xi$$

$$V_5=\int_{t-\tau}^{t}\dot{x}^T(\xi)W\dot{x}(\xi)d\xi$$

where symmetric positive definite matrices P, R, W, $Y(=h_M Q)$ are solutions of (9).

The derivative of V along the trajectory of system (7) is given by $\dot{V}=\dot{V}_1+\dot{V}_2+\dot{V}_3+\dot{V}_4+\dot{V}_5$.

$$\dot{V}_1=2(\mathcal{D}x_t)^T P\left(A+B\right)x(t)$$

$$-2(\mathcal{D}x_t)^T PB\int_{t-h(t)}^{t}\dot{x}(\xi)d\xi+2(\mathcal{D}x_t)^T PLu$$

Define $a(\xi)=B\dot{x}(\xi)$, $b(\xi)=P\left(\mathcal{D}x_t\right)$ and use Lemma 1 in Park [10] to obtain

$$-2(\mathcal{D}x_t)^T PB\int_{t-h(t)}^{t}\dot{x}(\xi)d\xi$$

$$\le h_M\left(\mathcal{D}x_t\right)^T P(M^T Q+I)Q^{-1}(QM+I)P\left(\mathcal{D}x_t\right)$$

$$+2\left(\mathcal{D}x_t\right)^T PM^T QB\int_{t-h(t)}^{t}\dot{x}(\xi)d\xi$$

$$+\int_{t-h(t)}^{t}\dot{x}^T(\xi)B^T QB\dot{x}(\xi)d\xi$$

Let $X=QMP$ and $Y=h_M Q$, then

$$\dot{V}_1\le x^T(t)\Big(P(A+B)+(A+B)^T P$$

$$+h_M^2\left(X^T+P\right)Y^{-1}\left(X+P\right)+X^T B+BX\Big)x(t)$$

$$-2x^T(t)X^T Bx(t-h(t))$$

$$-2x^T(t)\Big(\left(A+B\right)^T PC + B^T XC$$

$$+h_M^2\left(X^T+P\right)Y^{-1}\left(X+P\right)C\Big)x(t-\tau)$$

$$+2x^T(t)PLu\ +2x^T(t-h(t))B^T XCx(t-\tau)$$

$$+h_M^2 x^T(t-\tau)C^T\left(X^T+P\right)Y^{-1}\left(X+P\right)Cx(t-\tau)$$

$$-2x^T(t-\tau)C^T PLu + \int_{t-h(t)}^{t}\dot{x}^T(\xi)B^T QB\dot{x}(\xi)d\xi$$

Noting that (3), one can easily compute $\dot{V}_2$, $\dot{V}_3$, $\dot{V}_4$ and $\dot{V}_5$ as

$$\dot{V}_2 \le \frac{1}{1-h_d}\dot{x}^T(t)B^T YB\dot{x}(t) - \int_{t-h(t)}^{t}\dot{x}^T(\xi)B^T QB\dot{x}(\xi)d\xi$$

$$\dot{V}_3 \le x^T(t)Rx(t) - (1-h_d)x^T(t-h(t))Rx(t-h(t))$$

$$\dot{V}_4 = x^T(t)Sx(t) - x^T(t-\tau)Sx(t-\tau)$$

$$\dot{V}_5 = \dot{x}^T(t)W\dot{x}(t) - \dot{x}^T(t-\tau)W\dot{x}(t-\tau)$$

Then we have

$$\dot{V} \le x^T(t)\Big(P(A+B)+(A+B)^T P + R + S$$

$$+h_M^2\left(X^T+P\right)Y^{-1}\left(X+P\right)+X^T B + BX\Big)x(t)$$

$$-2x^T(t)X^T Bx(t-h(t))$$

$$-2x^T(t)\Big(\left(A+B\right)^T PC + B^T X$$

$$+h_M^2\left(X^T+P\right)Y^{-1}\left(X+P\right)C\Big)x(t-\tau)$$

$$+2x^T(t)PLu\ -(1-h_d)x^T(t-h(t))Rx(t-h(t))$$

$$+2x^T(t-h(t))B^T XCx(t-\tau)$$

$$-x^T(t-\tau)\Big(S-h_M^2 C^T\left(X^T+P\right)Y^{-1}\left(X+P\right)C\Big)x(t-\tau)$$

$$-2x^T(t-\tau)C^T PLu\ -\dot{x}^T(t-\tau)W\dot{x}(t-\tau)$$

$$+\dot{x}^T(t)\left(W+\frac{1}{1-h_d}B^T YB\right)\dot{x}(t)$$

Noting that $\dot{x}(t) = Ax(t) + Bx(t-h(t)) + C\dot{x}(t-\tau) + Lu$, we further have

$$\dot{V} \le q^T(t)\Pi q(t)$$

where

$$q(t) = \left(x^T(t)\ \ x^T(t-h(t))\ \ x^T(t-\tau)\ \ \dot{x}^T(t-\tau)\ \ u^T\right)^T$$

and

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & \Pi_{15} \\ \Pi_{12}^T & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} \\ \Pi_{13}^T & \Pi_{23}^T & \Pi_{33} & \Pi_{34} & \Pi_{35} \\ \Pi_{14}^T & \Pi_{24}^T & \Pi_{34}^T & \Pi_{44} & \Pi_{45} \\ \Pi_{15}^T & \Pi_{25}^T & \Pi_{35}^T & \Pi_{45}^T & \Pi_{55} \end{pmatrix}$$

with

$$\Pi_{11} = P(A+B)+(A+B)^T P + R + S$$

$$+h_M^2\left(X^T+P\right)Y^{-1}\left(X+P\right)+X^T B$$

$$+BX + A^T\left(W+\frac{1}{1-h_d}B^T YB\right)A$$

$$\Pi_{12} = -X^T B + A^T\left(W+\frac{1}{1-h_d}B^T YB\right)B$$

$$\Pi_{13} = -\left(A+B\right)^T PC - B^T X - h_M^2\left(X^T+P\right)Y^{-1}\left(X+P\right)C$$

$$\Pi_{14} = A^T\left(W+\frac{1}{1-h_d}B^T YB\right)C$$

$$\Pi_{15} = PL + A^T\left(W+\frac{1}{1-h_d}B^T YB\right)L$$

$$\Pi_{22} = -(1-h_d)R + B^T\left(W+\frac{1}{1-h_d}B^T YB\right)B$$

$$\Pi_{23} = B^T XC$$

$$\Pi_{24} = B^T\left(W+\frac{1}{1-h_d}B^T YB\right)C$$

$$\Pi_{25} = B^T\left(W+\frac{1}{1-h_d}B^T YB\right)L$$

$$\Pi_{33} = -S + h_M^2 C^T\left(X^T+P\right)Y^{-1}\left(X+P\right)C$$

$$\Pi_{34}=0$$

$$\Pi_{35}=-C^T PL$$

$$\Pi_{44}=-W+C^T\left(W+\frac{1}{1-h_d}B^T YB\right)C$$

$$\Pi_{45}=C^T\left(W+\frac{1}{1-h_d}B^T YB\right)L$$

$$\Pi_{55}=L^T\left(W+\frac{1}{1-h_d}B^T YB\right)L$$

A sufficient condition for asymptotic stability of system (1) is that the operator $\mathcal{D}$ is stable and there exist real matrix X, symmetric positive definite matrices P, R, S, W and Y such that

$$\dot{V}(t)\le q^T(t)\Pi q(t)<0 \qquad (10)$$

for all $q(t)\neq 0$ satisfying (8). Using the S-procedure [1] we see that this condition is implied by the existence of a nonnegative scalar $\varepsilon\ge 0$ such that

$$q^T(t)\Pi q(t)+\varepsilon\left(E_a x(t)+E_b x(t-h(t))\right)^T \times\left(E_a x(t)+E_b x(t-h(t))\right)-\varepsilon u^T u<0 \qquad (11)$$

for all $q(t)\neq 0$. Thus, if there exist real matrix X, symmetric positive definite matrices P, R, S, W and Y and a scalar $\varepsilon\ge 0$ such that LMI (9) is satisfied, then (11) holds. Note that assumption A1 guarantees that the operator $\mathcal{D}$ is stable. Therefore, system (1)~(4) is asymptotically stable according to Theorem 8.1 (pp. 292-293, in [3]). Q. E. D.

Remark 1. For the case that $h(t)=h=const$ and $\tau=h$, system (1) becomes

$$\dot{x}(t)-C\dot{x}(t-h)=\left(A+LF(t)E_a\right)x(t)+\left(B+LF(t)E_b\right)x(t-h) \qquad (12)$$

By Proposition 1, we conclude that system (12), (2), (4) is asymptotically stable if there exist real matrix X, symmetric positive definite matrices P, R, W, Y and a scalar $\varepsilon\ge 0$ such that the LMI (13), as shown at the bottom of the last page of the paper, holds, where

$$(1,1)\triangleq(A+B)^T P+P(A+B)+R+X^T B+B^T X$$

$$(1,2)\triangleq-(A+B)^T PC-B^T XC-X^T B$$

$$(2,2)=-R+B^T XC+C^T X^T B$$

Furthermore, if $C=0$ and $L=0$, the result in [10] is recovered.

Remark 2. The experience by author using MATLAB LMI Toolbox shows that direct coding (9) or (13) is not computationally efficient because the high dimensional linear matrix inequality (9) or (13) is computationally costly with current algorithm. To improve the efficiency, (9) or (13) can be broken into two lower dimensional linear matrix inequalities. The idea regarding how to decompose these LMIs can be found in [5].

4. Examples

Example 1. Consider the following uncertain neutral system with time-varying discrete delay

$$\dot{x}(t)-\begin{pmatrix}c&0\\0&c\end{pmatrix}\dot{x}(t-\tau)=\begin{pmatrix}-2+\delta_1\cos(t)&0\\0&-1+\delta_2\sin(t)\end{pmatrix}x(t)+\begin{pmatrix}-1+\gamma_1\cos(t)&0\\-1&-1+\gamma_2\sin(t)\end{pmatrix}x(t-h(t)) \qquad (14)$$

where $0\le|c|<1$ and δ_1, δ_2, γ_1 and γ_2 are unknown parameters satisfying

$$|\delta_1|\le 1.6,\ |\delta_2|\le 0.05,\ |\gamma_1|\le 0.1,\ |\gamma_2|\le 0.3$$

For $c=0$, system (14) reduces to the system studied in [6]. Applying the criteria in [6] and this paper, the following table gives the maximum value of h_M for stability of system (14) for different h_d. It is clear to see that the results in this paper are much less conservative than those in [6].

h_d	0.0	0.1	0.2	0.3	0.4
[6]	0.24	0.23	0.22	0.21	0.20
This paper	1.03	0.92	0.82	0.71	0.61
h_d	0.5	0.6	0.7	0.8	0.9
[6]	0.18	0.16	0.14	0.11	0.06
This paper	0.50	0.40	0.29	0.18	0.08

For $h_d = 0.1$, the maximum value of h_M is listed in the following table for various parameter c. As $|c|$ increases, h_M decreases.

$\|c\|$	0.0	0.1	0.2	0.3
h_M	0.92	0.73	0.55	0.41
$\|c\|$	0.4	0.5	0.6	0.7
h_M	0.29	0.19	0.11	0.04

For $c = 0.1$, we obtain the maximum value of h_M in the following table. One can see that as h_d increases, h_M decreases.

h_d	0.0	0.1	0.2	0.3	0.4
h_M	0.80	0.73	0.65	0.57	0.49
h_d	0.5	0.6	0.7	0.8	0.9
h_M	0.41	0.33	0.24	0.16	0.07

Example 2. Consider system (1) with

$$A = \begin{pmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{pmatrix},\ B = \begin{pmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{pmatrix},$$

$$C = \begin{pmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{pmatrix},\ E_a = E_b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$L = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},\ \alpha \geq 0$$

For $\alpha = 0$, $h(t) = h$ (const) and $\tau = h$, the system under consideration reduces to the system discussed in [7]. Using the criterion in this paper, the maximum value of h_M for the nominal system to be asymptotically stable is $h_M = 1.6014$. By the criteria in [7], [8] and [2], the nominal system is asymptotically stable for any h satisfying $h \leq 0.3$, $h \leq 0.71$, and $h \leq 0.74$, respectively. This example shows that the stability criterion in this paper gives a much less conservative result than these in [7], [8] and [2].

For $\alpha = 0.2$, the following table gives different h_M for different h_d. It is clear that as h_d increases, the corresponding h_M decreases.

h_d	0.0	0.1	0.2	0.3	0.4
h_M	1.08	0.94	0.82	0.70	0.58
h_d	0.5	0.6	0.7	0.8	0.9
h_M	0.47	0.37	0.27	0.17	0.07

For $h_d = 0.1$, the effect of the parameter α on the maximum time-delay for stability h_M is also studied. The following table illustrates the numerical results for different α. One can see that as $\alpha \to 0$, the stability limit for delay approaches the uncertainty-free case. As α increases, h_M decreases.

h_d	0.0	0.1	0.2	0.3	0.4
h_M	1.31	1.11	0.94	0.80	0.67
h_d	0.5	0.6	0.7	0.8	0.9
h_M	0.55	0.44	0.33	0.22	0.07

5. Conclusion

A delay-dependent stability criterion for neutral systems with time-varying discrete delay has been obtained. The criterion has been expressed in the form of a linear matrix inequality (LMI). Numerical examples have shown that the results derived by criterion in this paper are much less conservative than some existing ones in the literature.

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$$\begin{pmatrix} (1,1) & -X^T B & (1,3) & 0 & PL & A^T B^T Y & A^T W & h_M(X^T+P) & \varepsilon E_a^T \\ -B^T X & -(1-h_d)R & B^T XC & 0 & 0 & B^T B^T Y & B^T W & 0 & \varepsilon E_b^T \\ (1,3)^T & C^T X^T B & -S & 0 & -C^T PL & 0 & 0 & -h_M C^T(X^T+P) & 0 \\ 0 & 0 & 0 & -W & 0 & C^T B^T Y & C^T W & 0 & 0 \\ L^T P & 0 & -L^T PC & 0 & -\varepsilon I & L^T B^T Y & L^T W & 0 & 0 \\ YBA & YBB & 0 & YBC & YBL & -(1-h_d)Y & 0 & 0 & 0 \\ WA & WB & 0 & WC & WL & 0 & -W & 0 & 0 \\ h_M(X+P) & 0 & -h_M(X+P)C & 0 & 0 & 0 & 0 & -Y & 0 \\ \varepsilon E_a & \varepsilon E_b & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon I \end{pmatrix} < 0 \quad (9)$$

$$\begin{pmatrix} (1,1) & (1,2) & 0 & PL & A^T B^T Y & A^T W & h_M(X^T+P) & \varepsilon E_a^T \\ (1,2)^T & (2,2) & 0 & -C^T PL & B^T B^T Y & B^T W & -h_M C^T(X^T+P) & \varepsilon E_b^T \\ 0 & 0 & -W & 0 & C^T B^T Y & C^T W & 0 & 0 \\ LP & -LPC & 0 & -\varepsilon I & L^T B^T Y & L^T W & 0 & 0 \\ YBA & YBB & YBC & 0 & -Y & 0 & 0 & 0 \\ WA & WB & WC & 0 & 0 & -W & 0 & 0 \\ h_M(X+P) & -h_M(X+P)C & 0 & 0 & 0 & 0 & -Y & 0 \\ \varepsilon E_a & \varepsilon E_b & 0 & 0 & 0 & 0 & 0 & -\varepsilon I \end{pmatrix} < 0 \quad (13)$$