

# ESTIMATING THE LYAPUNOV EXPONENTS OF CHAOTIC TIME SERIES: A MODEL BASED METHOD

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## Abstract

In this paper, the problem of Lyapunov Exponents (LEs) computation from chaotic time series based on Jacobian approach by using polynomial modelling is considered. The embedding dimension which is an important reconstruction parameter, is interpreted as the most suitable order of model. Based on a global polynomial model fitting to the given data, a novel criterion for selecting the suitable embedding dimension is presented. By considering this dimension as the model order, by evaluating the prediction error of different models, the best nonlinearity degree of polynomial model is estimated. This selected structure is used in each point of the reconstructed state space to model the system dynamics locally and calculate the Jacobian matrices which are used in QR factorization method in the LEs estimation. This procedure is also applied to multivariate time series to include information from other time series and resolve probable shortcoming of the univariate case. Finally, simulation results are presented for some well-known chaotic systems to show the effectiveness of the proposed methodology.

## 1 Introduction

The pattern in output measurements reflects different behaviours, which can be classified as periodic cycles or stable equilibrium, non-linear chaotic, and pure random processes. Obviously, different types of models and analysis

are needed for each case such that improper selection of required tool may lead to lose the deterministic structure. However, uncovering this deterministic structure is important because it leads to more realistic and better modelling [12].

Deterministic chaos appears in variety of fields of science like engineering, biomedical and life sciences, social sciences, and physical sciences. Therefore, recognizing the chaotic behaviour of dynamical systems when only output data are available, is an important field of research. To achieve this, there are some quantitative measures included fractal dimension, entropy and Lyapunov exponents (LEs).

The LEs are conceptually the most basic and useful dynamical diagnostic for deterministic chaotic systems. The calculation of LEs for systems whose dynamical equations are known is straightforward. However, these methods cannot be applied directly to a set of measurement data. A system with one or more positive LEs is defined to be chaotic. These exponents provide not only a qualitative characterization of dynamical behaviour but also the exponent itself determines the measure of predictability. Hence, the estimation of the LEs as the useful dynamical classifier for deterministic chaotic systems is an important issue in nonlinear time series analysis. Two general approaches for computing the LEs from output time series are *geometrical* and *Jacobian* approaches. In geometrical approach, LEs are calculated based on the long term evolution of an infinitesimal sphere of initial conditions [19,9]. On the other hand, in the Jacobian approach, local Jacobian matrices are estimated and the long term product of matrices is computed. This is presented in [16] and [7] and its idea has been extended in several

references, e.g. [5], and [15]. The important step in latter approach is to estimate the Jacobian matrices. Since the LEs are derived from the eigenvalues of the Jacobians, any small error in the computation of Jacobians can cause major error in the LE computation. Some general perturbation results and error analysis in QR algorithms for computing LEs can be found in [6].

In the Jacobian approach, the Jacobians are usually found by *locally linear mapping* the neighborhoods near the reference trajectory to neighborhoods at a subsequent time. In [16] and [7], the linearized flow map from the neighbor data set into  $m$  step ahead of this set is considered as an approximation for the tangent map. In [5], it is shown that using the local neighborhood-to-neighborhood mappings with higher order Taylor series, can lead to superior results. In addition, in [3] it is suggested that the estimation of Jacobian matrices is best achieved in the case of noisy data by least squares polynomial fitting. An adaptive method for the computation of the Jacobian matrices has been presented in [11]. Four other methods for estimating the Jacobian has been referred to in [13], including the local thin-plate splines, radial basis functions, projection pursuit and neural nets. In [15], the Jacobians are estimated over boxes of the state space to speed up the algorithm of LEs computation.

In this paper, the problem of LEs computation from chaotic time series based on Jacobian approach by using polynomial models is considered. Different polynomial models lead to different Jacobians which result in various LEs. Moreover, large value of orders obtains the spurious LEs which confuses the selection of true LEs. Therefore, in proposed method two problems including the embedding dimension estimation and also a transparent decision criterion for choosing the proper structure of model are considered. In the first part of this paper, it is discussed that, the order of model plays the role of embedding dimension in the state space reconstruction. Therefore, by using global polynomial modeling of underlying dynamical system and considering the first step ahead prediction error, a criterion for estimating the suitable model order is presented. Then, the best degree of nonlinearity of polynomial model is estimated by evaluating the prediction errors of different models. The selected structure is used for local estimating of Jacobians which are used in QR factorization to calculate the LEs. This procedure is also applied to multivariate time series and it is shown that it can resolve probable shortcoming of the univariate case.

The background materials are given in Section 2. In Section 3.1, by considering the global polynomial models, a model based method for estimating the minimum embedding dimension and determining the suitable structure of the model are presented. The estimation of LEs based on local polynomial models is presented in Section 3.2. Finally, simulation results are provided to show the effectiveness of the proposed methodology for the well-known chaotic dynamical systems in section 4. The superiority of using multiple time series in this procedure is shown by applying the procedure to the given data from Rössler system.

## 2 Some preliminaries

In order to use the Jacobian approach to calculate the LEs, related mathematical background is explained in this section. Moreover, the general form of polynomial models which used in the next section is presented.

### 2.1. Computation of LEs by Jacobian approach

Consider the discrete dynamical system described in the following form:

$$\underline{x}_{k+1} = F(\underline{x}_k) \quad k = 0, 1, \dots \quad (1)$$

where  $\underline{x}_k$  is the state vector in the  $R^m$  space and  $F(\cdot)$  is a continuously differentiable nonlinear function. The linearized system for a small range around the operational trajectory in the phase space can be written as:

$$\underline{\delta x}_{k+1} = J_k \cdot \underline{\delta x}_k, \text{ where } J_k = \left. \frac{\partial F}{\partial \underline{x}} \right|_{\underline{x}_k} \in R^{m \times m} \quad k = 0, 1, \dots \quad (2)$$

where  $J_k$  is the Jacobian matrix in point  $k$ . The LE is defined as follows [6]:

*Definition 1-* Let  $Y^k = J_{k-1} \cdot J_{k-2} \cdots J_0$ , then the following symmetric positive definite matrix exists:

$$\Lambda = \lim_{k \rightarrow \infty} \left( (Y^k)^T \cdot Y^k \right)^{\frac{1}{2k}} \quad (3)$$

and the logarithms of its eigenvalues are called the *Lyapunov Exponents*.

However, the computation of the LEs by using definition 1 has some problems. The first problem is that for large value of  $k$ , the fundamental solution  $Y^k$  may go to very large values and actually, the calculation of  $\Lambda$  is not possible.

Further, the computation of  $Y^k$  should be such that the linear independency of the columns is maintained. Otherwise, this computation leads only to the largest LEs. To deal with these problems, the QR factorisation algorithm is used for approximation of LEs [7, 5, 15, 6]. The steps involved in this method can be summarized as follows:

- 1) Given orthogonal  $Q_0$  such that  $Q_0^T \cdot Q_0 = I$ .
- 2) Solve  $Z_{k+1} = J_k \cdot Q_k$ ,  $k = 0, 1, \dots$  and obtain the decomposition:  $Z_{k+1} = Q_{k+1} \cdot R_{k+1}$  where  $Q_{k+1}$  is an orthogonal matrix and  $R_{k+1}$  is upper triangular matrix with positive diagonal elements.
- 3) The LEs can be calculated as follows:

$$\begin{aligned} \lambda_j &= \lim_{k \rightarrow \infty} \frac{1}{k} \log \left( (R_k)_{jj} \cdots (R_1)_{jj} \right) \\ &= \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k \log \left( (R_i)_{jj} \right) \quad j = 1, \dots, m \end{aligned} \quad (4)$$

## 2.2. Polynomial Model

The dynamical behavior of system is considered with the following nonlinear difference equation:

$$y(k+1) = f(\underline{x}_k) \quad (5)$$

where  $f(\cdot)$  is a continuously differentiable function and  $\underline{x}_k$  is state vector. It is supposed that the output data of the dynamical system is available as the following univariate time series:

$$y(t+t_s), y(t+2t_s), \dots, y(t+Nt_s) \quad (6)$$

where  $t_s$  is the sampling time and N is total number of measurements.

In many practical situations the structure of the underlying dynamical system which generates the data is unknown. Depending on the objectives, there are different theories, such as the functional theory, which are suitable for special analysis of nonlinear systems. In this paper, an arbitrary degree polynomial as follows is selected to fit the output data:

$$\begin{aligned} y(k+1) = & \theta_0 + \sum_{i=0}^{d-1} \theta_{1i} y(k-i) + \sum_{i=0}^{d-1} \sum_{j=i}^{d-1} \theta_{2ij} y(k-i)y(k-j) + \\ & \dots + \sum_{i=0}^{d-1} \sum_{j=i}^{d-1} \dots \sum_{v=u}^{d-1} \theta_{nijp} y(k-i)y(k-j) \dots y(k-u)y(k-v) \end{aligned} \quad (7)$$

where  $d$  is the order of model and  $n$  is degree of nonlinearity. For the model order  $d$  and degree of nonlinearity  $n$  the number of parameters in vector  $\Theta$  that should be estimated to identify the underlying model is:

$$n_\theta = \frac{(d+n)!}{d!n!} \quad (8)$$

This identification can be done by using Least Squares method.

## 3 Model based estimation of the LEs

In order to estimate the LEs based on polynomial modelling, at first the idea for selecting the most suitable structure of polynomial model is presented. Then by using this structure, the LE estimation procedure is explained.

### 3.1. Determining the model structure and embedding dimension

In practical situations the dynamical equations (5) and state variables of chaotic system are not available. However, the original phase space geometry can be reconstructed by applying the method of delays by Takens' theorem [18] and the invariant measures of chaotic system, like LEs, can be calculated from this reconstructed space. In method of delays, the delay coordinates as follows are used to form a  $d$  dimensional vector space:

$$\underline{x}(t) = [y(t-(d-1)\tau), \dots, y(t-\tau), y(t)] \quad (9)$$

where  $\tau$  is the lag time and the dimension of this reconstructed space,  $d$ , is called the embedding dimension which its minimum value is looked for. There are many methods that concern this dimension including *False Neighbour* method [10] and *Singular Value Decomposition* method [4] and related papers for their modifications and using the extension for multivariate time series case [1,2,14].

In this paper, in order to model the reconstructed state space, the vector (9) is considered as the state vector. The lag time can be selected by using well known methods based on autocorrelation function or mutual information [8]. Here, the lag time is fixed and the vector (9) with normalized step is considered. Then for deriving the state equations, the function  $f(\cdot)$  is estimated by polynomial model such that:

$$y(t+1) = f(\underline{x}(t)) \quad (10)$$

By this, the number of required state variables which, is the order of autoregressive polynomial model, will also be equal  $d$ . Therefore, the minimum embedding dimension is chosen as the most suitable order of the polynomial model. The under-estimation of order causes the loss of dynamics of data generator, and the Lyapunov spectrum can not be obtained neither complete nor accurate from this model. On the other hand, the over-estimation of model order leads to spurious LEs which perturbs the decision on true LEs.

In order to select the suitable model order or embedding dimension, a criterion based on global polynomial model is presented. To show the main idea, consider a two dimensional chaotic system with the state trajectory as shown in Figure 1. The objective is to find the model as (5) by using the structure of (7). If the order of model is under-estimated to  $d=1$ , the obtained model will project the points  $(i,1) \quad i=1, \dots, 7$  to the same one step ahead value, say  $\hat{x}_{k+1}$ .

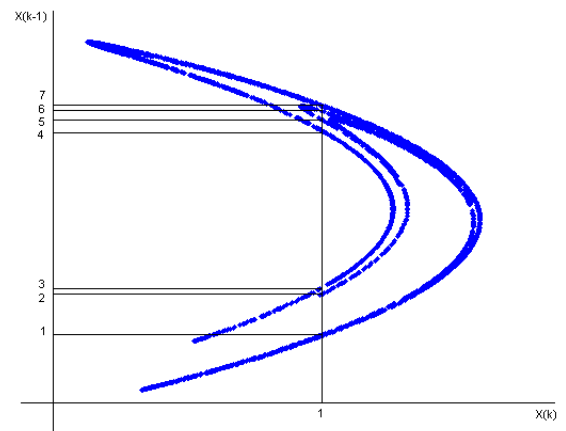


Figure 1: Attractor of a two dimensional chaotic system, if order is under-estimated to  $d=1$  all the points 1, ..., 7 on  $X(k-1)$  axis are projected on point 1 in  $X(k)$  axis.

Therefore, the first step ahead prediction error for each transition of this point is:

$$e(i,1) = \hat{x}_{k+1} - x_{k+1}(i,1) \quad i = 1, \dots, 7 \quad (11)$$

where  $x_{k+1}(i,1)$  denotes to the true first step ahead value.

These errors will be large since only one fixed projection has been considered for all points. If the order of model is selected to  $d=2$ , then for each points of  $x_{k+1}(i,1) \quad i = 1, \dots, 7$  different one step ahead value is estimated. The prediction error in this case is:

$$e(i,1) = \hat{x}_{k+1}(i,1) - x_{k+1}(i,1) \quad i = 1, \dots, 7 \quad (12)$$

The errors in this case are much smaller than the previous case since the error is only due to the capability of selected model to predict the next value. The Least Mean Squares (LMS) of these errors for all the points of attractor as follows is also different value in these two case:

$$\sigma = \left( \frac{1}{N} \left( \sum_{k=1}^N e_k^2 \right) \right)^{\frac{1}{2}} \quad (13)$$

where  $N$  is the total number of points. Therefore, for selecting the optimum model order, the value of  $\sigma$  is considered for different model order. The dimension that  $\sigma$  is decreased to a lower level and after that takes approximately the same value is the suitable model order and is denoted by  $d_{opt}$ .

After finding the  $d_{opt}$ , for this fixed order, values of  $\sigma$  for different degree of nonlinearity is calculated and the degree which  $\sigma$  takes approximately the same values is selected as the nonlinearity degree of polynomial and denoted by  $n_p$ .

*Remark 1-* The above mentioned idea for estimating the embedding dimension can be used by considering other types of models provided that the model function satisfies the continuous differentiability property.

### 3.2. LEs estimation

In this section, LEs estimation based on polynomial models by using time series (6) is presented. For this, the polynomial model (7) is considered which  $d_{opt}$  and  $n_p$  are estimated by the procedure of Section 3.1. The state vector is defined as following delay vector which delay is normalized:

$$\underline{x}(k) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{d_{opt}}(t) \end{bmatrix} = \begin{bmatrix} y(k - (d_{opt} - 1)) \\ y(k - (d_{opt} - 2)) \\ \vdots \\ y(k) \end{bmatrix} \quad (14)$$

The system can be represented in a canonical form as:

$$\underline{x}(k+1) = \begin{bmatrix} x_2(k) \\ x_3(k) \\ \vdots \\ f(\underline{x}(k)) \end{bmatrix} \quad (15)$$

Then, the Jacobian matrix  $J_k$  in each point  $k$  of the typical trajectory for this canonical representation is as:

$$J_k = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Df_1 & Df_2 & \cdots & Df_{m-1} & Df_m \end{bmatrix} \quad (16)$$

where  $Df_i = \frac{\partial f}{\partial x_i}$  and the  $Df_i \quad i = 1, \dots, d$  are computed in terms

of the parameters of the model. Since the accuracy of Jacobians depend on estimated model parameters, it is tried to improve the model fitting by using suitable structure of autoregressive polynomials. For this, a general polynomial with order  $d_{opt}$  and degree of nonlinearity  $n_p$  which have been selected in the last part is considered. To estimate the parameters of the local model, for each point of the reconstructed state space with dimension  $d_{opt}$  like (14), some nearer neighbours are selected. The number of neighbours should be more than number of parameters which should be estimated. Then the LMS method is used to estimate the model parameters in each point. The obtained local model is used to calculate the Jacobian matrix as (16). This is accomplished for all the points on the reconstructed attractor and the calculated Jacobians are used in the QR algorithm for calculating the LEs.

## 4 Simulation results

To show the effectiveness of the proposed procedure in Section 3, the algorithms are applied to some well-known chaotic systems whose characteristics are defined in Table 1 [3,17,19].

### 4.1. Univariate case

In this section a univariate time series of above mentioned systems is considered. At first, the proposed method for estimating the suitable order of model based on global polynomial modeling is implemented. For this, the developed general program of polynomial modelling is applied for various  $d$  and  $n$ , and  $\sigma$  is computed for all the cases which the results are shown in Table 2. Based on the theoretic discussions on Section 3.1, then the optimum order of model is selected in each case. According to these results, the optimum model order and nonlinearity degree of each system are estimated as the Table 3. By using the values in Table 3, an structure of polynomial model is selected for each chaotic system.

Systems	Dynamical Equation	Coeffs.	LEs
Logistic	$x_{k+1} = r \cdot x_k (1 - x_k)$	$r = 4$	0.693
Triangular	$x_{k+1} = r \cdot (1 - 2 \cdot  0.5 - x_k )$	$r = 0.91$	0.5988
Henon	$\begin{cases} x_{k+1} = 1 - a x_k^2 + y_k \\ y_{k+1} = b x_k \end{cases}$	$\begin{cases} a = 1.4 \\ b = 0.3 \end{cases}$	$\begin{cases} 0.4188 \\ -1.62 \end{cases}$
Lorenz	$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = x(R - z) - y \\ \dot{z} = xy - bz \end{cases}$	$\begin{cases} \sigma = 16 \\ R = 45.92 \\ b = 4 \end{cases}$	$\begin{cases} 1.497 \\ 0 \\ -22.45 \end{cases}$
Rössler	$\begin{cases} \dot{x} = -(y + z) \\ \dot{y} = x + ay \\ \dot{z} = b + z(x - c) \end{cases}$	$\begin{cases} a = 0.15 \\ b = 0.2 \\ c = 10 \end{cases}$	$\begin{cases} 0.0901 \\ 0 \\ -9.7734 \end{cases}$

Table 1. The characteristics of systems for simulations

Logistic map, N=500					
d	1	2	3	4	5
n					
1	0.9992	1.0016	1.0032	1.0027	1.0065
2	5.43e-6	5.69e-6	5.95e-6	5.99e-6	6.0e-6
3	2.30e-5	2.50e-5	2.72e-5	2.86e-5	3.09e-5
4	6.83e-5	7.92e-5	9.04e-5	9.72e-5	1.07e-4
Triangular map, N=500					
d	1	2	3	4	5
n					
1	0.9018	0.8570	0.8627	0.8607	0.8550
2	0.2587	0.2597	0.2599	0.2596	0.2618
3	0.2233	0.2043	0.2045	0.2049	0.2069
4	0.1337	0.1228	0.1224	0.1231	0.1245
Henon map, N=500, time series of $x$ variable					
d	1	2	3	4	5
n					
1	0.9394	0.9283	0.8763	0.8649	0.8553
2	0.2787	1.75e-7	1.52e-7	1.96e-7	3.07e-7
3	0.2748	4.13e-7	2.76e-7	4.80e-7	1.76e-6
4	0.2742	3.05e-6	9.71e-6	2.69e-6	2.62e-5
Lorenz system, N=1000, time series of $x$ variable, lag time= 0.05					
d	1	2	3	4	5
n					
1	0.4115	0.2784	0.2284	0.2062	0.1988
2	0.4115	0.2786	0.2292	0.2074	0.2004
3	0.3830	0.0781	0.0167	0.0096	0.0092
4	0.3830	0.0802	0.0105	0.0100	0.0096

Table 2: Mean squared of first step ahead prediction error of defined chaotic systems for different values of model order (d) and degree of nonlinearity (n)

The LEs, then can be estimated by the method explained in Section 3.2 which are shown in Table 3. The results for the Logistic and Henon maps that polynomial models exactly fit the dynamics, are almost the same as results obtained by computation from dynamical equations. For triangular map and Lorenz flow that the equations are not in polynomial form the results are also well acceptable. This accuracy is due to optimum selection of model structure in Section 3. To show the disadvantage of improper choice of model structure, the estimated LEs of Lorenz system for non-optimum structure are shown in Table 4. The improper model structure leads not only to inaccurate LEs but also obtains the spurious LEs.

## 4.2. Multivariate case

In some applications, e.g. meteorology, the measurements data are in the vector form. Moreover, the full dynamics of a system from a single time series may not be observable. In this case, using multivariate time series may lead to better results. To show this, time series of  $x$  and  $y$  variables of Rössler system are considered. In order to estimate the suitable model order or embedding dimension, two cases are accomplished. In the first case, the univariate time series of  $x$  and  $y$  variables are used separately. The mean squared of first step ahead prediction error are provided in Table 5. It is seen that, the embedding dimension is estimated to  $d=2$  which is not acceptable. However in the second case, if multivariate time series of joint variables  $x$  and  $y$  are used, as it is seen from the results in Table 5, the embedding dimension is estimated equal 3 which is exactly the system dimension.

## 5 Summary

In this paper, an improved method for the estimation of LEs based on polynomial models is proposed. At first, based on global polynomial model fitting to the given data, the minimum embedding dimension is estimated which has the same role as model order. The suitable nonlinearity degree of polynomials then, is calculated by evaluating the first step prediction errors. This step results in a suitable structure for local models which is used in the second step to calculate the Jacobians needed in QR algorithm for estimating the LEs. To show the effectiveness of proposed methodology, the simulation results are provided. The potential superiority of using multiple time series versus scalar case is shown by applying the proposed method to data from Rössler system.

	$d_{opt}$	$n_p$	Estimated LEs
Logistic map	1	2	0.6926
Triangular map	1	4	0.5871
Henon map	2	2	0.4134, -1.6172
Lorenz system	3	3	1.5254, 0.0128, -18.4338

Table 3. The estimated optimum order of model, nonlinearity degree of polynomial model, and LEs

$d$	$n_p$	Lyapunov Exponents
4	3	2.3044, 0.7538, -1.6386, -21.5328
3	2	1.5551, 1.1270, -12.8595

Table 4. The estimated LEs of Lorenz system with improper choice of model structure

Univariate time series of $x$ variable					
$d$	1	2	3	4	5
$n$					
1	0.5206	0.2121	0.2065	0.1939	0.1887
2	0.5173	0.1406	0.1087	0.0807	0.0790
3	0.5125	0.0890	0.0617	0.0385	0.0342
Univariate time series of $y$ variable					
$d$	1	2	3	4	5
$n$					
1	0.4916	0.0870	0.0779	0.0744	0.0737
2	0.4915	0.0735	0.0380	0.0282	0.0239
3	0.4915	0.0641	0.0210	0.0143	0.0133
Multivariate time series of $x$ and $y$ variables					
$d$	2	3	4	5	6
$n$					
1	0.1745	0.1643	0.1389	0.1379	0.1344
	0.0506	0.0459	0.0346	0.0340	0.0325
2	0.1448	0.0846	0.0558	0.0503	0.0413
	0.0443	0.0173	0.0095	0.0085	0.0067
3	0.1191	0.0419	0.0289	0.0239	0.0206
	0.0395	0.0064	0.0043	0.0037	0.0034

Table 5: Mean squared of first step ahead prediction error of Rössler system for  $N=1000$ , and  $\tau_s=0.5$ . In the third part, the first and second rows are related  $\sigma$  for  $x$  and  $y$  respectively.

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## References

- [1] M. Ataei, A. Khaki-Sedigh, B. Lohmann, and C. Lucas 'Determining embedding dimension from output time series of dynamical systems- Scalar and multiple output cases', Proceedings of the 2<sup>nd</sup> International Conference on System Identification and Control Problems, Moscow, Russia, pp. 1004-1013, (2003).
- [2] M. Ataei, A. Khaki-Sedigh, B. Lohmann, and C. Lucas 'Determination of embedding dimension using multiple time series based on singular value decomposition', Proceedings 4<sup>th</sup> International Symposium on mathematical Modeling, Austria, pp. 190-196, (2003).
- [3] K. Briggs, 'An improved method for estimating Liapunov exponents of chaotic time series', *Phys. Lett. A*, **151**, no. 1,2, pp. 27-32, (1990).
- [4] D. S. Broomhead, and G. P. King 'Extracting qualitative dynamics from experimental data', *Physica D*, **20**, pp. 217-236, (1986).
- [5] R. Brown P. Bryant, and H. D. I. Abarbanel 'Computing the Lyapunov spectrum of a dynamical system from an observed time series', *Phys. Rev. A*, **43**, no. 6, pp. 2787-2806, (1991).
- [6] L. Diect, R. D. Russell, and E. S. Van Vleck 'On the computation of Lyapunov exponents from continuous dynamical systems', *SIAM J. Numer. Anal.*, **34**, no.1, pp. 402-423, (1997).
- [7] J. P. Eckmann, S. O. Kamphorst, D. Ruelle, and S. Ciliberto 'Liapunov exponents from time series', *Phys. Rev. A*, **34**, no. 6, pp. 4971-4979, (1986).
- [8] A. M. Fraser, and H. L. Swinney 'Independent coordinates for strange attractors from mutual information', *Phys. Rev. A*, **33**, no. 2, pp. 1134-1140, (1986).
- [9] H. Kantz, 'A robust method to estimate the maximum Lyapunov exponent of a time series', *Phys. Lett. A*, **185**, pp. 77-87, (1994).
- [10] M. B. Kennel, R. Brown, and H. D. I Abarbanel 'Determining embedding dimension for phase space reconstruction using a geometrical construction', *Phys. Rev. A*, **45**, no 6., pp.3403-3411, (1992).
- [11] A. Khaki-Sedigh, M. Ataei, B. Lohmann, and C. Lucas 'Adaptive calculation of Lyapunov exponents from time series observations of chaotic time varying dynamical systems', submitted for publication.
- [12] B. Lillekjendlie, D. Kugiumtzis, and N. Christophersen 'Chaotic time series, Part II: System identification and prediction', *Modelling, Identification and Control*, **15**, no. 4, pp. 225-243, (1994).
- [13] D. F. McCaffrey, S. Ellner, A. R. Gallant, and D. W. Nychka 'Estimating the Lyapunov exponent of a chaotic system with nonparametric regression', *Journal of the Amer. Stat. Assoc.*, **87**, no. 419, pp. 682-695, (1992).
- [14] A. I. Mees, P. E. Rapp, and L. S. Jenning 'Singular value decomposition and embedding dimension', *Phys. Rev. A*, **36**, no. 1, pp. 340-346, (1987).
- [15] N. N. Oiwa, and N. Fiedler-Ferrara 'A fast algorithm for estimating Lyapunov exponents from time series', *Phys. Let. A* **246**, pp. 117-121, (1998).
- [16] M. Sano, and Y. Sawada 'Measurement of the Lyapunov spectrum from a chaotic time series', *Phys. Rev. Let.* **55**, no 10, pp. 1082-1085, (1985).
- [17] H. G. Schuster 'Deterministic Chaos: an introduction', VCH Verlagsesellschaft, Germany, (1995).
- [18] F. Takens, 'Detecting strange attractors in turbulence', In: *Lecture Notes in Mathematics*. (D.A. Rand, L.S. Young. (Ed)) **898**, pp. 366-381, Springer, Berlin, (1981).
- [19] A. Wolf, J. B. Swift, H. L. Swinney, J. A. Vastano 'Determining Lyapunov exponents from a time series', *Physica D*, **16**, pp. 285-317, (1985).