Notes on the Nested Observers for Hybrid Systems

S. Di Gennaro

Università di L'Aquila – Dipartimento di Ingegneria Elettrica, 67040 Poggio di Roio, L'Aquila, Italy

Tel. +39 – 0862 – 434461, Fax +39 – 0862 – 434403; E-mail: digennar@ing.univaq.it

Abstract

In this paper a hybrid observer for hybrid systems is presented. The continuous time and the discrete event dynamics are determined in such a way that the observation problem is solved. The observer is built using a nested structure, in which the nesting is due to a refinement of those states corresponding to undecided cases. The continuous dynamics implement an estimator which extracts extra information from the continuous dynamics associated to the discrete event state. These information are based on the convergence ratio of the output estimate to the system output. The resulting continuous dynamics are not simply those of a Luenberger's observer.

Key Words—Hybrid observer, hybrid system, dynamic estimator.

1 Introduction

In output feedback control schemes the estimation of the state of the system is a central problem for the determination of a control action. In the continuous and discrete time setting, the state estimation problem is well known and easily solved. Also in the case of discrete event dynamics some results are available in literature [9]. On the contrary, for hybrid systems, namely for those systems in which continuous or discrete time dynamics coexist with discrete event dynamics, this problem has been only partially investigated and still need a deeper study. Some partial results in this direction can be found in [2] for the case of power-train control, and in [3], where a more general synthesis procedure for an observer of hybrid systems is presented. Other results regarding switching systems can be found in [1], [8].

An example in which the state estimation of a hybrid system is of paramount interest is given by the air traffic management. In this case the fault detection error in the human behavior (pilots, controllers, etc.) and in the hardware subsystems is of capital importance. Following [5], this paper tries to give some new results on the construction of a hybrid observer. On the basis of the outputs of the system and using some extra information obtained from the system dynamics, this observer enables the determination of the hybrid state of the system. In fact, the main problem in the construction of a hybrid observer is the determination of the discrete state of the system. But very often the information from the discrete output of the system is too poor for determining this discrete state. Hence, motivated by the results of [10], on fault detection and identification, and [3], on the construction of observers for hybrid systems exploiting the information derived from the analysis of the continuous time dynamics, in this paper a procedure for the design of a hybrid observer is proposed. While in [3] one builds an observer as a finite state machine which, under appropriate conditions, allows the determination of the discrete event state of the hybrid system using Luenberger's observers for resolving indecision situations, in this work the observer is presented directly as a hybrid system and the discrete event and the continuous time dynamics are determined in such a way that the observation problem is solved. Since the observer is directly expressed as a hybrid system, it is avoided the use of logic blocks, as in [3], which hide the hybrid structure of the observer. This clarifies the derivation of the observer dynamics. Moreover, a more effective mechanism is used to get extra information from the continuous dynamics associated to the discrete event state, based on the convergence ratio of the output estimate to the system output and not on its simple convergence. This renders faster the discrimination of the undecided cases. In general, the resulting continuous dynamics are not simply those of a Luenberger's observer. Another advantage of the proposed observer design is the fact that these observers are built as nested observers, in which the nesting is due to a refinement of those states corresponding to undecided cases. Finally, a feasible solution for the observer problem is given also in the case of undetectable systems.

2 The Mathematical Model

We consider a hybrid system $\mathcal{H}$ with N locations $q_1, \cdots, q_N$. Each location identifies the continuous dynamics described by the equations

$$\dot{x} = A_i x + B_i u \qquad y = C_i x \qquad i = 1, \cdots, N \tag{1}$$

with $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{p\times n}$, $x \in X \subseteq \mathbb{R}^n$ the continuous state, $y \in Y \subseteq \mathbb{R}^p$ the continuous output, and $u \in U \subseteq \mathbb{R}^m$ the system input.

Remark 1. The solution of the observation problem does not change sensibly considering systems with output $y = C_i x + D_i u$. □

The discrete event dynamics are given by a generator of formal language [11]

$$\begin{aligned} q(k+1) &\in \varphi(q(k), \sigma(k)) \\ \sigma(k) &\in \phi(q(k), x(t_k), u(t_k)) \\ \psi(k) &= \eta(q(k), \sigma(k)) \end{aligned} \tag{2}$$

with $q(k) \in Q$ the discrete location, $\psi(k) \in \Psi$ the output symbol, $\sigma(k) \in \Sigma$ the k^{th} input symbol, which takes place at t_k and force the discrete evolution. It is not possible to know these instants before the events occur. Here $Q = \{q_1, \cdots, q_N\}$, $\Psi = \{\psi_1, \cdots, \psi_r\}$, $\Sigma = \{\epsilon, \sigma_1, \cdots, \sigma_s\}$ (with ϵ the null event) are the finite sets of locations, output and input symbols. Moreover,

$$\varphi: Q \times \Sigma \to 2^Q, \quad \phi: Q \times X \times U \to \Sigma, \quad \eta: Q \times \Sigma \to \Psi$$

are the transition, the input, and the output functions (in general these are partial functions). The function ϕ specifies the possible input events σ. The functions φ, η can be extended in the usual way to accept sequences $s = \sigma_1 \cdots \sigma_{k-1}\sigma_k \in \Sigma^*$, with Σ^* the monoid on Σ [11]. For instance for φ one has $\varphi(q, \epsilon) = q$ and

$$\varphi(q, \sigma_1 \cdots \sigma_{k-1}\sigma_k) = \varphi(\varphi(q, \sigma_1 \cdots \sigma_{k-1}), \sigma_k)$$

when $\varphi(q, \sigma_1 \cdots \sigma_{k-1})!$ and $\varphi(q, \sigma_1 \cdots \sigma_{k-1}\sigma_k)!$ ("!" indicates that the partial function is defined for the given arguments)

The hybrid system $\mathcal{H}$ here considered is given by the systems described by equations (1), (2). The action of the discrete dynamics on the continuous ones is the change of the equations (1) when a location transition takes place. On the other hand, the action of the continuous dynamics on the discrete ones is the change of location when the continuous state x and/or the continuous control u belong to a certain region or when the system trajectory hits a certain boundary.

The problem addressed in this paper is the design of a hybrid observer $\mathcal{O}$ which determines the state (q, x) of $\mathcal{H}$. This is done by constructing an observer for $\mathcal{H}$, according to the following definitions which generalize the ones given in [3].

Definition 1. [5] A hybrid state $(q, x) \in Q \times X$ of $\mathcal{H}$ is observable with respect to $(q', x') \neq (q, x)$ if there exists a hybrid input $(s, u(\cdot)) \in \Sigma^* \times \mathcal{U}$ such that $\eta(q, s) \neq \eta(q, s)$ and $C_{i_q} x(t) \neq C_{i_{q'}} x'(t)$ for $k \geq \bar{k}$, $t \geq t_{\bar{k}}$, where $s = \sigma_1 \cdots \sigma_{\bar{k}} \sigma_{\bar{k}+1} \cdots \sigma_k \in \Sigma^*$, and $\mathcal{U}$ is the set of admissible continuous inputs. □

This definition reduces to the notion of current-location observability in [3] when the continuous states are observable.

Definition 2. Given a hybrid system $\mathcal{H}$ described by equations (1), (2), a hybrid observer $\mathcal{O}$ is a dynamic system of the form

$$\begin{aligned} \hat{q}(k+1, h+1) &= \hat{\varphi}(\hat{q}(k, h), \psi(k), \bar{\psi}(t_k + \delta_1)) \\ \dot{\chi}_j &= H_j \chi_j + M_j u + N_j y, \quad j \in \hat{Q}_{k+1} \\ \dot{\xi}_j &= P_j \xi_j + R_j u + S_j y + T_j \chi_j \\ \bar{\psi}(t) &= \Phi_j(\chi_j, \xi_j, u, y) \end{aligned} \tag{3}$$

$\chi_j \in \mathbb{R}^{n_1}$, $\xi_j \in \mathbb{R}^{n_2}$, $\delta_1 \in (0, t_{k+1} - t_k)$, such that

$$\hat{q}(k, h) = q(k), \text{ for } k > \bar{k}, \text{ for some positive integer } \bar{k} \text{ and some integer } h \geq 0$$

$$\lim_{t \to \infty} \|x - x_j\| = 0, \text{ for some index } j \in \hat{Q}_{k+1}.$$

Here

$$\hat{\varphi}: \hat{Q} \times \Psi' \times \bar{\Psi} \to 2^{\hat{Q}} \tag{4}$$

is a (partial) function defining the discrete event dynamics of the observer, such that

$$\hat{q}(k+1, 0) = \hat{\varphi}(\hat{q}(k, h), \psi(k), \epsilon), \qquad h \geq 0$$

$\hat{Q}$ is the state set, $\Psi' = \{\epsilon\} \cup \Psi$, $\bar{\Psi} = \{\epsilon, \bar{\psi}_1, \cdots, \bar{\psi}_s\}$ is the set of symbols generated by the functions

$$\Phi_j: \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times U \times Y \to \bar{\Psi}$$

and $\hat{Q}_{k+1} \subseteq 2^{\{1, \cdots, N\}}$ is the set of the indices of the locations belonging to $\hat{\varphi}(\hat{q}(k, 0), \psi(k), \epsilon)$. □

The function $\hat{\varphi}$ can be extended as usual to accept, in particular, strings $\bar{p}_h = \bar{\psi}(t_k + \delta_1) \cdots \bar{\psi}(t_k + \delta_{h-1}) \bar{\psi}(t_k + \delta_h) \in \bar{\Psi}^*$ of length h. On the base of the definition, the time instants $t_k + \delta_1 < \cdots < t_k + \delta_h$ are in (t_k, t_{k+1}).

In this definition $\mathcal{O}$ has discrete event dynamics specified by $\hat{\varphi}(\cdot, \cdot, \cdot)$ and continuous ones. Possibly, these latter can be absent; in this case $\mathcal{O}$ results to be a simple finite state machine $\mathcal{O}_{\mathcal{M}}$. Nevertheless, when it is necessary to extract extra information on the discrete state from the continuous output y and input u, as shown in what follows, it will contain also the continuous time dynamics as in (3).

Note that the inputs of $\mathcal{O}$ are the outputs ψ of the discrete event dynamics (along with the continuous input u and output y), but not the discrete event input σ. This is due to the fact that the discrete event input σ is considered generated by the generator of formal language (like an unobservable exosystem [7]). Note also that in the continuous time equations of (3) we considered in general $j \in \hat{Q}_{k+1}$, namely we supposed that only $|\hat{Q}_{k+1}| \leq N$ continuous dynamics are used. Clearly nothing changes, from the conceptual point of view, considering that all the continuous dynamics are used.

In what follows we recall some results about the construction of such an observer, while in Section 3 some new criteria for deriving a hybrid observer are given.

3 A Simple Hybrid Observer

A simple observer $\mathcal{O}_{\mathcal{M}}$ for $\mathcal{H}$ (a finite state machine, with no continuous dynamics) can be derived as in [9], [3], [5] applying the standard procedure for constructing an observer. This observer is a finite state machine which, under appropriate conditions, allows the determination of the discrete event state of $\mathcal{H}$. This procedure, analogous to the one used to construct a deterministic automaton from a non-deterministic one [6], is based on the iterative construction of the function $\hat{\varphi}$ in (4) as follows

$$\hat{\varphi}(\hat{q}, \psi) := \Big\{ q \in Q \mid \exists\, \bar{q} \in \hat{q},\ \sigma \in \Sigma : q \in \varphi(\bar{q}, \sigma)! \text{ and } \psi = \eta(\bar{q}, \sigma) \Big\} \tag{5}$$

which is a particularization of the discrete-event dynamics (3). In words, the function $\hat{\varphi}$ is defined for each pair $(\hat{q}, \psi)$ such that there exist at least a state $\bar{q} \in \hat{q}$ and a state transition from $\bar{q}$ to q labeled σ (namely such that $\varphi(\bar{q}, \sigma)!$) and such that the resulting output is ψ. The initial state of the observer is $\hat{q}(0) = Q$.

The conditions under which such an observer $\mathcal{O}_{\mathcal{M}}$ exists are the following [9], [3], [4], [5].

Definition 3. A cycle is a sequence of states $q_{i_1} q_{i_2} \cdots q_{i_{k-1}} q_{i_k}$ such that $q_{i_1} = q_{i_k}$. A primary cycle is a cycle which does not contain other cycles. □

Proposition 1. $\mathcal{O}_{\mathcal{M}}$ is an observer for $\mathcal{H}$ if

1. $Q \cap \hat{Q}$ is nonempty and invariant with respect to the dynamics of $\hat{\varphi}$ (namely $\hat{\varphi}(\hat{q}, \psi) \in Q \cap \hat{Q}$ for all $\hat{q} \in Q \cap \hat{Q}$ and $\psi \in \Psi$ such that $\hat{\varphi}(\hat{q}, \psi)!$)
2. every primary cycle $Q_c^i \subset \hat{Q}$ is such that $Q_c^i \cap Q \neq \phi$. □

These conditions are the formalization of intuitive conditions: the first condition requires that $\mathcal{O}_{\mathcal{M}}$ have a state set constituted by single states of Q (namely its states $\hat{q}$ have cardinality equal to 1), and that the discrete event dynamics do not bring the state outside this set; the second condition requires that all the primary cycles can bring to a state $\hat{q} = \{q_i\}$ with cardinality equal to 1. It is clear that these are necessary and sufficient conditions ensuring that, after a transient, one can individuate the precise discrete event state of $\mathcal{H}$.

When the conditions given by Proposition 1 are violated it is hence clear that one can not know the discrete event state of $\mathcal{H}$ for k greater than a certain positive integer $\bar{k}$, at least with a pure discrete event-driven observer. This is due to the fact that $Q \cap \hat{Q}$ is not invariant, namely $\hat{\varphi}$ brings to a state $\hat{q} = \{q_{i_1}, \cdots, q_{i_r}\}$ with cardinality greater then 1, and/or there exists a primary cycle $Q_c^i \cap Q = \phi$, so that it is not possible to reach states $\hat{q}$ with cardinality equal to 1.

In this case one can exploit the idea contained in [3], namely one can use some knowledge coming from the continuous dynamics to create further labels (called "signatures" in [3]) giving extra information, so discriminating some of the cases in which Proposition 1 does not apply. Clearly, these extra information must be "rich enough" to determine an observer respecting Proposition 1.

In order to exploit this quite straightforward idea, in [3] one uses Luenberger's observers as generators of signals which, fed into a decision function block, give these signatures $\bar{\psi}_1, \cdots, \bar{\psi}_s$. Each label $\bar{\psi} \in \bar{\Psi} = \{\bar{\psi}_1, \cdots, \bar{\psi}_s\}$ is a signature characteristic of a specific location $\bar{q}$ and is added as output to the arcs entering $\bar{q}$. With this change in $\mathcal{H}$ one can obtain a finite state machine $\mathcal{O}_{\mathcal{M}}$, respecting Proposition 1. Note that this sums up in rendering different the output labels $\psi \in \Psi$ which render unobservable a pair (q, x), (q', x'), substituting to these labels the labels in $\bar{\Psi}$. In some sense, with these extra labels $\bar{\psi}_1, \cdots, \bar{\psi}_s$ one renders trivial the observation problem and reduces substantially to the problem of designing an observer $\mathcal{O}_{\mathcal{M}}$ (with no continuous dynamics).

Since the signature $\bar{\psi} \in \bar{\Psi}$ is not known until the Luenberger's observers converge, a location identification logic block is used. What does this block is to wait a time δ until $\bar{\psi}$ is produced; then the transition in the observer $\mathcal{O}_{\mathcal{M}}$ happens. The original contribution of [3] is the determination of the gains in the Luenberger's observers such that $\bar{\psi}$ is produced in the prescribed time δ.

In the following Section it is shown how to construct a hybrid observer $\mathcal{O}$ (with continuous dynamics) using the extra information $\bar{\psi}_1, \cdots, \bar{\psi}_s$ in a quite different way. The main difference with the observer in [3] is its construction. In fact, a finite state machine $\mathcal{M}$ (which, in general, will not be an observer since will violate Proposition 1) will be constructed. Hence, a refinement of the states of $\mathcal{M}$ is introduced so obtaining an observer $\mathcal{O}$ for $\mathcal{H}$. Another difference with [3] will be the generation mechanism of the extra labels $\bar{\psi}_1, \cdots, \bar{\psi}_s$, not based on the simple asymptotic convergence.

4 Nested Hybrid Observers

In this Section we propose an observer procedure design based on the refinement of the states of a finite state machine $\mathcal{M}$ which mimics the hybrid system $\mathcal{H}$ on the base of the outputs ψ [5]. First we present the design of the discrete event dynamics of the observer and the refinement mechanism; then, the continuous time dynamics will be derived. The resulting observer will have the structure (3).

A finite state machine $\mathcal{M}$ which mimics the hybrid system $\mathcal{H}$ on the base of the outputs ψ can be obtained following the procedure illustrated in Section 2, using equation (5), but in general it will not result to be an observer $\mathcal{O}_{\mathcal{M}}$ since Proposition 1 does not apply.

Example 1. Let us consider the system shown in Figure 1, where on the arcs are the outputs $\psi \in \Psi$. Applying the procedure shown in Section 2 one obtains the finite state machine $\mathcal{M}$, Figure 2. This is not an observer due to the presence of the state $\{q_1, q_2\}$. □

Exploiting the idea of [10], [3], one needs to produce extra signals $\bar{\psi} \in \bar{\Psi}$ which allow the discrimination of those cases in which Proposition 1 does not apply. These extra information must be produced from the estimation of the state system, using u, y and ψ as inputs of the observer $\mathcal{O}$. In order to extract information from the continuous dynamics, for the continuous systems (1) the following is supposed to hold.

(H_1) *Systems (1) are detectable.* □

These information, such as the signatures of the previous Section, can be used to induce a refinement on those states on $\mathcal{M}$ composed of more than one state of Q.

Example 2. The refinement of the state $\hat{q} = \{q_1, q_2\}$ in $\mathcal{M}$, Figure 2, can be obtained considering signals $\bar{\psi}_1$, corresponding to the negation of being in q_1, and $\bar{\psi}_2$, corresponding to the negation of being in q_2. The refinement is given in Figure 3. □

More in general, if the state of the finite state machine $\mathcal{M}$ obtained applying the procedure of Section 2 is

$$\hat{q} = \{q_{i_1}, \cdots, q_{i_h}\} \in \hat{Q}$$

one has to discriminate among the dynamics (1) and individuate the right location $i \in \{i_1, \cdots, i_h\}$. To this aim, in this paper one considers a generation mechanism of the labels $\bar{\psi}$ which forces the observer to evolve on a refinement of the state $\hat{q}$. This refinement is nested into the state $\hat{q}$ of $\mathcal{M}$. Note that the resulting system is of the form (3). Moreover, if the events $\bar{\psi}$ occur instantaneously, $\mathcal{M}$ would respect Proposition 1, so constituting an observer $\mathcal{O}$ for $\mathcal{H}$. It is hence clear that Proposition 1 has to be weakened to consider events $\bar{\psi}$ which occur in finite (bounded) time. This is done by the following.

Theorem 1. Let δ a positive real number and let us suppose that for the hybrid system $\mathcal{H}$ given by (1), (2) one has $t_{k+1} \geq t_k + \delta$, for all $k \geq 0$. Then, $\mathcal{O}$ given by (3) is an observer for $\mathcal{H}$ if

1. $Q \cap \hat{Q} \neq \emptyset$ and, for all $\hat{q} \in Q \cap \hat{Q}$ and $\psi \in \Psi$ such that $\hat{\varphi}(\hat{q}, \psi, \epsilon)!$, there exists a string $\bar{p}_h = \bar{\psi}(t_k + \delta_1) \cdots \bar{\psi}(t_k + \delta_h) \in \bar{\Psi}^*$, $h \geq 0$, with $\delta_1 < \cdots < \delta_h < \delta$, such that $\hat{\varphi}(\hat{q}, \psi, \bar{p}_h) \in Q \cap \hat{Q}$;

2. every primary cycle $Q_c^i \subset \hat{Q}$ is such that $Q_c^i \cap Q \neq \emptyset$. □

Proof. This results can be proved by considering that after each discrete event transition due to an output label ψ such that $\hat{\varphi}(\hat{q}, \psi, \epsilon) \notin Q \cap \hat{Q}$ there exists a sequence $\bar{p}_h$ which ensures that $\hat{\varphi}(\hat{q}, \psi, \bar{p}_h) \in Q \cap \hat{Q}$, namely that $\hat{\varphi}(\hat{q}, \psi, \bar{p}_h) \in Q \cap \hat{Q}$ is a set formed by a single state. Therefore, Proposition 1 can be applied and the result is readily shown. □

This result is quite obvious. When entering a state $\{q_{i_1}, \cdots, q_{i_h}\} \in 2^Q$ (which does not allow the precise determination of the state of $\mathcal{H}$), if the transition in the nested refinement is fast enough (namely if it occurs before of the next discrete event transition in $\mathcal{H}$) then it is possible to determine the state q_i of $\mathcal{H}$ by using the information $\bar{\psi}_j$ coming from the continuous dynamics (1).

We are ready to build the discrete event dynamics of the observer (3). This is done using the following defining rules

$$
\begin{aligned}
\hat{\varphi}(\hat{q}, \psi, \varepsilon) &:= \Big\{ q \in Q \mid \exists\, \bar{q} \in \hat{q},\ \sigma \in \Sigma :\ q \in \varphi(\bar{q}, \sigma)! \\
&\qquad\qquad \text{and } \psi = \eta(\bar{q}, \sigma) \Big\} \\
\hat{\varphi}(\hat{q}, \varepsilon, \bar{\psi}) &:= \Big\{ q \in \hat{\varphi}(\hat{q}, \psi, \varepsilon) \mid \bar{\psi} \wedge q = 1 \Big\}
\end{aligned}
\tag{6}
$$

where we write $\bar{\psi} \wedge q = 1$ to mean that the signal $\bar{\psi}$ is compatible with the dynamics corresponding to the state q. The second of (6) specifies the construction of the nested discrete event dynamics.

Definition 4. Given a hybrid system $\mathcal{H}$, expressed by (1), (2), a system (3) with discrete event dynamics specified by (6) and satisfying Theorem 1 is called a nested observer. □

Example 3. The nested observer for the hybrid system of Figure 1 is given by Figure 4. □

The remaining of this Section will be devoted to the determination of the signals $\bar{\psi} \in \bar{\Psi}$. The discrimination among the states in $\hat{q} = \{q_{i_1}, \cdots, q_{i_h}\}$ can be done by using state estimators, giving the signals $\hat{y}_j = C_j \hat{x}$, $j = i_1, \cdots, i_h$, and checking when the differences between the output $y = C_i x$ and its estimates $\hat{y}_j$ go to zero (or, better, do not go to zero, so eliminating some of the possible cases). The condition to be checked is therefore $e_{ij} = C_i(A_i x + B_i u) - \dot{\hat{y}}_j > 0$, $\forall\, t \in [t_k, t_{k+1})$, for at least a $j \in \hat{Q}_{k+1}$, namely we need to build *estimators giving* e_{ij} *as outputs.*

The fact that the state estimators we will propose are based on the convergence ratio of the output estimate to the system output (and not on its simple convergence) renders faster the discrimination of the undecided cases. The problem is that in general $\dot{y} = C_i(A_i x + B_i u)$ is not available, except in the trivial case in which x is measurable. Possible solutions are discussed in the following [5].

4.1 Dynamic Estimators

If x is not available to get e_{ij} we need to build an estimation of this signal which asymptotically converges to e_{ij}. This is done as follows. Let us suppose that the discrete state for $\mathcal{H}$ is q_i at time t_k, and let us consider the dynamic estimators

$$
\left.
\begin{aligned}
\dot{\hat{x}}_j &= (A_j - K_j C_j)\hat{x}_j + B_j u + K_j y \\
\dot{\zeta}_j &= F_j \zeta_j + G_j \dot{y} + H_{0,j} u + H_{1,j} \dot{u} + \kappa_j
\end{aligned}
\right. \quad j \in \hat{Q}_{k+1}
\tag{7}
$$

$\zeta_j \in \mathbb{R}^p$, where $F_j, G_j \in \mathbb{R}^{p \times p}$, $H_{0,j}, H_{1,j} \in \mathbb{R}^{p \times m}$ are matrices to be determined, as well as the functions $\kappa_j(t) \in \mathbb{R}^p$, while K_j are such that the matrices $(A_j - K_j C_j)$ are Hurwitz, which is possible thanks to hypothesis (H_1). The hypothesis of availability of $\dot{y}$ and $\dot{u}$ will be removed later on. One considers the estimation errors $x_i - \hat{x}_j$ and $e_{ij} = \dot{y} - \zeta_j = C_i(A_i x_i + B_i u) - \zeta_j$, whose dynamics are

$$
\begin{aligned}
\dot{x}_i - \dot{\hat{x}}_j &= A_i x_i + B_i u - (A_j - K_j C_j)\hat{x}_j - B_j u - K_j C_i x_i \\
\dot{e}_{ij} &= C_i A_i (A_i x_i + B_i u) + C_i B_i \dot{u} \\
&\quad - (F_j \zeta_j + G_j \dot{y} + H_{0,j} u + H_{1,j} \dot{u} + \kappa_j(t)).
\end{aligned}
$$

Hence, setting

$$
\begin{aligned}
G_j &= -F_j, & H_{0,j} &= C_j A_j B_j, \\
H_{1,j} &= C_j B_j, & \kappa_j(t) &= C_j A_j^2 \hat{x}_j
\end{aligned}
\tag{8}
$$

one works out

$$
\begin{aligned}
\dot{x}_i - \dot{\hat{x}}_j &= (A_i - K_j C_i) x_i - (A_j - K_j C_j)\hat{x}_j \\
&\quad + (B_i - B_j) u \\
\dot{e}_{ij} &= C_i A_i^2 x_i - C_j A_j^2 \hat{x}_j + (C_i A_i B_i - C_j A_j B_j) u \\
&\quad + (C_i B_i - C_j B_j)\dot{u} + F_j(\dot{y} - \zeta_j).
\end{aligned}
$$

For the i^{th} estimator one has

$$
\begin{pmatrix} \dot{x}_i - \dot{\hat{x}}_i \\ \dot{e}_{ii} \end{pmatrix} = \begin{pmatrix} A_i - K_i C_i & 0 \\ C_i A_i^2 & F_i \end{pmatrix} \begin{pmatrix} x_i - \hat{x}_i \\ e_{ii} \end{pmatrix}.
$$

Hence, choosing the matrices F_j, to be Hurwitz, one has $\lim_{t \to \infty} \zeta_i = \dot{y}$ and $\lim_{t \to \infty} (\zeta_i - \hat{\zeta}_i) = \dot{y} - \hat{\zeta}_i$, where

$$
\hat{\zeta}_j = C_j (A_j \hat{x}_j + B_j u), \qquad j \in \hat{Q}_{k+1}.
\tag{9}
$$

Therefore, with positions (8) and definitions (9), the dynamic estimators (7) assume the form

$$\begin{aligned}\dot{\hat{x}}_j &= (A_j - K_jC_j)\hat{x}_j + B_ju + K_jy \\ \dot{\zeta}_j &= -F_j(\dot{y} - \zeta_j) + C_jA_jB_ju + C_jB_j\dot{u} + C_jA_j^2\hat{x}_j \\ \gamma_j &= \zeta_j - C_j(A_j\hat{x}_j + B_ju)\end{aligned} \tag{10}$$

$j \in \hat{Q}_{k+1}$, with the output γ_j converging asymptotically to the error e_{ii} when $j = i$. The necessity of $\dot{y}$, $\dot{u}$ are now removed. It suffices to note that for the following system

$$\left.\begin{aligned}\dot{\hat{x}}_j &= (A_j - K_jC_j)\hat{x}_j + B_ju + K_jy \\ \dot{\xi}_j &= F_j\zeta_j + C_jA_jB_ju + C_jA_j^2\hat{x}_j \\ \zeta_j &= -F_jy + C_jB_ju + \xi_j \\ \gamma_j &= \zeta_j - C_j(A_j\hat{x}_j + B_ju)\end{aligned}\right. \quad j \in \hat{Q}_{k+1} \tag{11}$$

the expression of $\dot{\zeta}_j$ results to be the same as in (10). The estimators (11) can be eventually rewritten as

$$\begin{aligned}\dot{\hat{x}}_j &= (A_j - K_jC_j)\hat{x}_j + B_ju + K_jy \qquad j \in \hat{Q}_{k+1} \\ \dot{\xi}_j &= F_j(-F_jy + C_jB_ju + \xi_j) + C_jA_jB_ju + C_jA_j^2\hat{x}_j \\ \gamma_j &= -F_jy + C_jB_ju + \xi_j - C_j(A_j\hat{x}_j + B_ju)\end{aligned} \tag{12}$$

The first two equations of (12) are in the form of the second and third of (3), with $\hat{x}_j = \chi_j$, $n = n_1$, $p = n_2$. Moreover, function Φ_j in (3) are defined as

$$\Phi_j(\chi_j, \xi_j, u, y) := \Phi_j(\gamma_j) = \begin{cases} \epsilon & \text{if } e_{ij} < 0 \\ \bar{\psi}_j & \text{otherwise} \end{cases} \tag{13}$$

namely a signal $\bar{\psi}_j$ is generated if the output estimate does not converge to the output of the system.

4.2 Reduced–Order Estimators

The dynamic estimators (12) have a state vector $\begin{pmatrix} \hat{x}_j \\ \xi_j \end{pmatrix} \in \mathbb{R}^{n+p}$. In order to reduce the dimensions of these estimators, one can consider in (7) a reduced order observer instead of a full-order one. To this aim, it is well known that one can use appropriate state vectors $z_i = \begin{pmatrix} z_{i,1} \\ z_{i,2} \end{pmatrix} = T_ix$ such that the systems (1) are given by

$$\begin{aligned}\dot{z}_{i,1} &= A_{i,11}z_{i,1} + A_{i,12}z_{i,2} + B_{i,1}u \\ \dot{z}_{i,2} &= A_{i,21}z_{i,1} + A_{i,22}z_{i,2} + B_{i,2}u \\ y &= z_{i,1}\end{aligned} \tag{14}$$

$i = 1, \cdots, N$, with T_i such that $C_iT_i^{-1} = (\, I \quad 0 \,)$, where I is the $p \times p$ identity matrix. Hence, repeating the same passages of the previous Section, and supposing $\mathcal{H}$ in q_i, one considers the dynamic estimators

$$\begin{aligned}\dot{\chi}_j &= (A_{j,22} - L_jA_{j,12})\chi_j + (A_{j,21} - L_jA_{j,11} \\ &\quad + A_{j,22}L_j - L_jA_{j,12}L_j)y + (B_{j,2} - L_jB_{j,1})u \\ \dot{\xi}_j &= F_j(-F_jy + B_{j,1}u + \xi_j) + (A_{j,11}^2 + A_{j,12}A_{j,21})y \\ &\quad + (A_{j,11}B_{j,1} + A_{j,12}B_{j,2})u \\ &\quad + (A_{j,11}A_{j,12} + A_{j,12}A_{j,22})(\chi_j + L_jy) \\ \gamma_j &= -F_jy + B_{j,1}u + \xi_j - (A_{j,11}y + A_{j,12}(\chi_j + L_jy) \\ &\quad + B_{j,1}u)\end{aligned}$$

$j \in \hat{Q}_{k+1}$, which have a state vector $\begin{pmatrix} \chi_j \\ \xi_j \end{pmatrix} \in \mathbb{R}^n$. The construction of the functions Φ_j is as in (13).

4.3 Derivative of the Output Signal

A further solution to the problem of generating the signals e_{ij} is the approximate derivation of the output y. The advantage of this solution is that in this case the continuous estimation dynamics are much smaller compared with the previous cases, since one has to consider only one p–dimensional system. Another advantage is the fact that this approach can be used also when hypothesis (H_1) is not verified. The drawbacks are obviously that this estimator is noisy and is capable to follow only signals up to a certain frequency.

Supposing that the signal y is affected by noise only at frequencies high with respect to the integration range, we build the estimation dynamics as follows. Supposing for the moment $\dot{y}$ available, one considers the system

$$\dot{\zeta} = \frac{1}{\varepsilon}(-\zeta + \dot{y})$$

$\varepsilon > 0$, which gives as output the signal $\dot{y}$. Now, since $\dot{y}$ is not available, one considers the change of variable $\zeta = \dot{\xi}$ so that

$$\begin{aligned}\ddot{\xi} &= \tfrac{1}{\varepsilon}(-\dot{\xi} + \dot{y}) \\ \zeta &= \dot{\xi}.\end{aligned}$$

Hence, one obtains a p–dimensional system

$$\begin{aligned}\dot{\xi} &= \tfrac{1}{\varepsilon}(-\xi + y) \\ e_{ij} &= \tfrac{1}{\varepsilon}(-\xi + y) - C_jA_j\hat{x}_j - C_jB_ju \qquad j \in \hat{Q}_{k+1}.\end{aligned} \tag{15}$$

Clearly, no proof of convergence for e_{ij} can be done. The construction of the functions Φ_j is as in (13).

4.4 The Convergence Velocity

It is clear that a key point is the respect of Theorem 1, namely the production of a sequence

$$\bar{p}_h = \bar{\psi}(t_k + \delta_1) \cdots \bar{\psi}(t_k + \delta_h) \in \bar{\Psi}^*, \qquad h \geq 0$$

with

$$\delta_1 < \cdots < \delta_h < \delta \tag{16}$$

such that $\hat{\varphi}(\hat{q}, \psi, \bar{p}_h) \in Q \cap \hat{Q}$. In order to obtain signals $\bar{\psi}$ respecting condition (16) for a given a time interval δ, one needs to be capable to fix the observers' convergence velocity. Therefore, hypothesis (H_1) has to be substituted by

(H_2) *Systems (1) are observable.* □

The determination of the observers' gains can be done exploiting the results in [3].

As previously noted, system (15) which realizes the derivative of the output can be used also when hypotheses (H_1), (H_2) are not verified. This is a clear advantage in realizing the observer $\mathcal{O}$ making use of (15).

Conclusions

A hybrid observer has been presented for the estimation of the discrete event state and the continuous time state of a hybrid system $\mathcal{H}$. This observer is built considering first a finite state machine accepting the discrete event output of $\mathcal{H}$, and then refining the states with cardinality greater than one. The resulting observer has a nested structure. The dynamics in these refinements are driven by information coming from the continuous time dynamics, based on the convergence ratio of the output estimate to the system output. This renders faster the discrimination of the undecided cases.

Acknowledgement

The author is in debt with Salvatore Monaco, who addressed him to this research field and supported him with his technical suggestions.

This work has been supported by the European Project "Hybridge".

References

[1] G. A. Ackerson and K. S. Fu, On State Estimation in Switching Environments, *IEEE Transactions on Automatic Control*, Vol. 15, pp. 10-17, 1970.

[2] A. Balluchi, L. Benvenuti, M.D. Di Benedetto, and A. L. Sangiovanni Vincentelli, A Hybrid Observer for the Driveline Dynamics, *Proceedings of the European Control Conference 2001*, pp. 613-623, Porto, Portugal, 2001.

[3] A. Balluchi, L. Benvenuti, M. D. Di Benedetto and A. L. Sangiovanni–Vincentelli, Design of Observers for Hybrid Systems, *Proceedings of the* 5^{th} *International Workshop, HSCC 2002*, Stanford, CA, USA, March 2002, pp. 76–89, in Lecture Notes in Computer Science, vol. 2289, C. J. Tomlin and M. R. Greenstreet Eds., Hybrid Systems: Computation and Control, Springer–Verlag, Berlin Heidelberg, 2002.

[4] M. Cüneyt, C. M. Özveren and A.S. Willsky, Stability and Stabilizability of Discrete Event Dynamic Systems, *Journal of the Association for Computing Machinery*, Vol. 38, No. 3, pp. 730-752, July 1991.

[5] S. Di Gennaro, Nested Observers for Hybrid Systems, *Proceedings of the Latin-American Conference on Automatic Control – CLCA 2002*, December 3-6, Guadalajara, México, 2002.

[6] J. E. Hopcroft, J. D. Ullman, *Introduction to Automata Theory, Languages and Computation*, Addison-Wesley, Reading, MA, 1979.

[7] A. Isidori, *Nonlinear Control Systems*, Third edition, Springer-Verlag, London, 1995.

[8] Y. Liu, Switching Observer Design for Uncertain Nonlinear Systems, *IEEE Transactions on Automatic Control*, Vol. 42, pp. 1699-1703, 1997.

[9] C. M. Özveren, and A. S. Willsky, Observability of Discrete Event Dynamic Systems, *IEEE Transactions on Automatic Control*, Vol. 35, pp. 797-806, 1990.

[10] M. -A. Massoumnia, G. C. Verghese and A. S. Willsky, Failure Detection and Identification, *IEEE Transactions on Automatic Control*, Vol. 34, No. 3, pp. 316-321, March 1989.

[11] P. J. Ramadge, W. M. Wonham, Supervisory Control of a Class of Discrete-Event Processes" *SIAM Journal of Control and Optimization*, Vol. 25, No. 1, pp. 206-230, Jan. 1987.

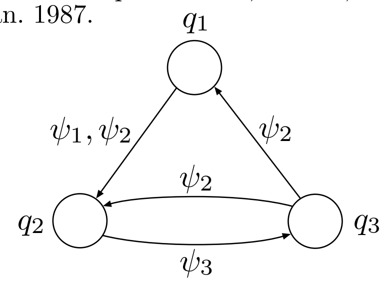

Figure 1: Hybrid system $\mathcal{H}$

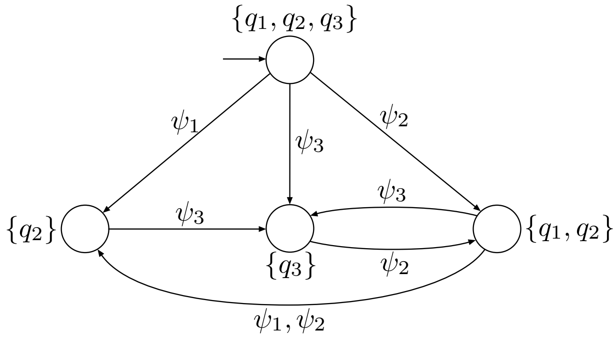

Figure 2: Finite state machine $\mathcal{M}$

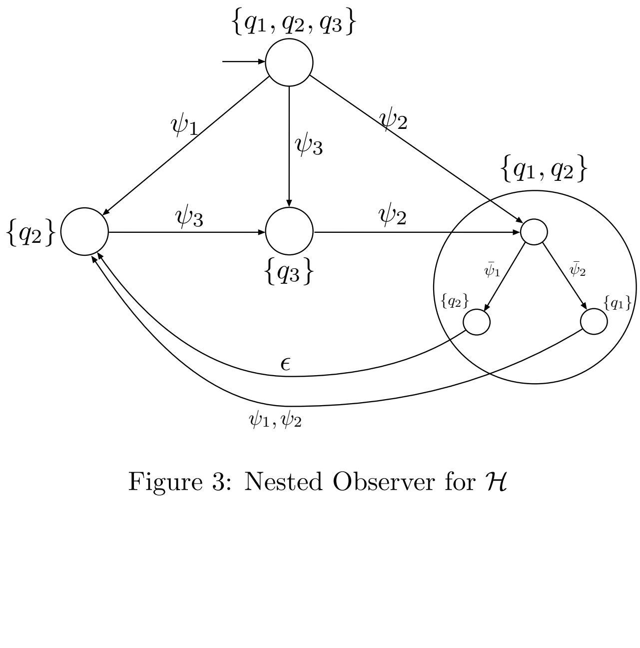

Figure 3: Nested Observer for $\mathcal{H}$