CAPTURE INTO RESONANCE: A NOVEL METHOD OF EFFICIENT CONTROL

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Abstract

In the present paper we propose a method to use *capture into resonance* to control the behavior of a certain class of dynamical systems. In many dynamical systems the coupling between the unperturbed system and weak periodic perturbations (wave) can be reduced to a purely resonant interaction occurring in the vicinity of a certain surface in the phase space. While resonance interaction can change invariants of the unperturbed system (e.g. energy), it is random in nature, and, consequently, is rather inefficient as a mechanism of regular transport. We propose a method to structure the resonance interaction with little additional cost. When the nominal dynamics brings the system close to a resonance surface we apply a short control pulse to force the capture of a phase point into the resonance with the wave. A captured point is transported by the wave across the energy levels. We apply the second pulse to release a phase point from the resonance when the desired energy level is achieved. As a model problem we consider dynamics of a charged particle in an electromagnetic field. We compare the cost of the proposed control with other methods and discuss possible applications of this technique.

1 Introduction

In a variety of near-integrable Hamiltonian dynamical systems significant simplification can be achieved by reducing the coupling between the unperturbed system and weak periodic perturbations (waves) to purely resonant interactions occurring in the vicinity of a certain surface in the phase space. A wide range of applications of this technique includes energy exchange between coupled oscillators, [2], mixing in fluids [9], celestial mechanics, [1], billiards, [4], Josephson junctions, [12], dynamics of charged particles in electromagnetic fields [3, 11]. Mathematically accurate theory of the most prominent resonance phenomena, *scattering on resonance* and *capture into resonance* was developed in [7, 8]

Recently lot of attention was paid to near-integrable Hamiltonian systems where small "perturbations" (like the presence of additional planets) can be used to reduce the cost of control, [5, 1, 6], see also [10] for general discussion of control of Hamiltonian systems. One of classical control objectives for near-integrable systems is to move a phase point from one invariant manifold of underlying integrable system to another, for example, to change the energy of a particle. By itself, resonance interaction can change the energy, but, as it is random in nature, is rather inefficient as an acceleration mechanism.

We propose a method to structure the resonance interaction with little additional cost. When the internal dynamics brings the system close to that surface we apply a short control pulse to force the capture of a phase point into the resonance with the wave. A captured point is transported by the wave across the energy levels. We apply the second pulse to release a phase point from the resonance when the desired energy level is achieved.

As a model problem we consider dynamics of a charged particle in a uniform magnetic field and a weak electrostatic wave. We compare the cost of the proposed control with other methods and discuss possible applications of this technique. The obtained results may be interesting not just for wave-particle interactions, but for a variety of problems, where resonant interaction is important.

The structure of the paper is as follows. In Sects. 2 – 4 we describe the nominal dynamics. In Sect. 2 we discuss the basic equations and approximations that govern dynamics of a charged particle in electromagnetic fields. In Sect. 3 we consider the averaged system and show in what parts of the phase space the method of averaging does and does not work. The nominal dynamics in the vicinity of a resonance is discussed in Sect. 4, in particular scattering on a resonance (Subsect. 4.1) and capture into a resonance (Subsect. 4.2). In Sects. 5 and 6 we discuss the control algorithm itself.

2 Charged particles in electromagnetic field.

Let a charged particle move in a uniform magnetic field, $\mathbf{B}$, directed along the z-axis in the presence of an electrostatic wave, that generates electric field, $\mathbf{E}$, propagating along the y-axis:

$$\mathbf{B} = B\mathbf{e}_z, \quad \mathbf{E} = -E\cos(ky - \omega t)\mathbf{e}_y,$$

where k and ω are the wave vector and the frequency of the wave, respectively, t is the time. The Hamiltonian of a charged particle has the form

$$H = \frac{1}{2m}\left\{P_x^2 + P_z^2 + \left(P_y - \frac{e}{c}Bx\right)^2\right\} + \frac{eE}{k}\sin(ky - \omega t),$$

where $\mathbf{P} = (P_x, P_y, P_z)$, m and e are the generalized momentum, the mass and the charge of a particle, respectively. As H does not depend on z explicitly, P_z is an integral of motion and it can be set to 0. Introduce a phase of the wave φ:

$$\varphi = ky - \omega t, \quad P_\varphi = P_y$$

Dimensionless Hamiltonian of a charged particle is

$$h = \frac{1}{2}P_x^2 - \omega P_\varphi + \frac{1}{2}\left(-P_\varphi + \kappa x\right)^2 + \kappa\varepsilon \sin\varphi. \tag{1}$$

In (1), $\varepsilon = cE/vB$, $\kappa = 1/k\rho$, where $\rho = cmv/(eB)$ is characteristic Larmor radius, v is typical velocity, and

$$h = H/(mv^2) - \omega P_\varphi. \tag{2}$$

We are interested in the following range of parameters:

$$\kappa \ll 1, \quad \omega, \varepsilon \sim 1.$$

Note, that the characteristic value of x is of order $1/\kappa$.

3 The averaged system and the structure of a resonance.

Hamiltonian (1) possesses two degrees of freedom. The variable φ is fast and the variables x, P_x, P_φ are slow. The new "energy", h (that differs from the usual energy by a quantity ωP_φ), is an integral of motion.

As the phase φ changes much faster than x, in the first approximation we can average the motion over fast φ-oscillations. This approximation is valid everywhere except for a small part of the phase space where $\dot\varphi \approx 0$. The averaging corresponds to omitting the wave term in (1) and reduces the Hamiltonian h to

$$h_{av} = \frac{1}{2}P_x^2 - \omega P_\varphi + \frac{1}{2}\left(-P_\varphi + \kappa x\right)^2.$$

In averaged system P_φ and the energy H are integrals of motion and the problem becomes integrable.

The equation $\dot\varphi = 0$ defines a parallel to the P_x-axis plane in the $(\kappa x, P_x, P_\varphi)$ space, that we call the *resonant surface*, or the *resonance*, and denote by R:

$$\dot\varphi = \frac{\partial h_{av}}{\partial P_\varphi} = -\omega + P_\varphi - \kappa x = 0. \tag{3}$$

On R the projection of the averaged velocity of a particle on the direction of the wave vector is equal to the phase speed of the wave. This phenomena is similar to the classical Cherenkov-type resonance $\omega = (\mathbf{k}, \widehat{\mathbf{v}})$.

Motion of particles near the resonance and far from it is drastically different. Near a resonance φ is not fast compared with x. Hence, the value of the integral of the averaged system, P_φ, may change in the process of a passing through the vicinity of the resonance. As a result, we can not expect the averaged system to approximate the exact system adequately. The change in

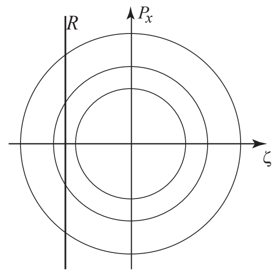

Figure 1: The phase portrait of the system with the Hamiltonian h_{av} on the (ζ, P_x) plane (see (1)). On the whole phase plane $P_\varphi = P_{\varphi,0}$. The vertical solid line is the intersection of $P_\varphi = P_{\varphi,0}$ plane and the resonant surface.

P_φ leads to the corresponding change in the energy of a particle (see (2)). In terms of a new variable,

$$\zeta = -P_\varphi + \kappa x,$$

Hamiltonian (1) and resonance condition (3) have the form

$$h = \frac{1}{2}P_x^2 - \omega P_\varphi + \frac{1}{2}\zeta^2 - \kappa\varepsilon\sin\varphi, \quad \zeta_{res} = -\omega.$$

Figure 1 presents the phase portrait of the system with the Hamiltonian h_{av} on the (ζ, P_x) plane defined by $P_\varphi = P_{\varphi,0} =$ const. The vertical straight line is the intersection of the resonant surface and the plane $P_\varphi = P_{\varphi,0}$.

4 Dynamics in the vicinity of the resonance.

A vicinity of the resonant surface, where $|\dot\varphi| <$ const$\sqrt{\kappa\varepsilon}$, is called a *resonant zone*. In resonant zone

$$h = h_x + \kappa h_\varphi, \tag{4}$$

where

$$h_x = \frac{1}{2}\bar{P}_x^2 - \omega\kappa x - \frac{1}{2}\omega^2, \tag{5}$$

$$h_\varphi = b\varphi + \frac{1}{2}\frac{1}{\kappa}\bar{P}_\varphi^2 + \varepsilon\sin\varphi. \tag{6}$$

In ((4) - (6)), we used a notation

$$P_x = \bar{P}_x + \kappa\varphi, \quad P_\varphi = \bar{P}_\varphi + P_{\varphi,res}, \quad b = P_{x,res},$$

where $P_{x,res}$ and $P_{\varphi,res}$ are the values of P_x and P_φ, respectively, on R. Equations ((4) - (6)) describe the motion of a particle inside the resonant zone.

A characteristic trajectory of exact system looks as follows. A particle approaches the resonant zone with the value of P_φ oscillating with a small amplitude, $\sim (\kappa\varepsilon)$, near some value P_φ^-. When in the process of the motion it arrives to the resonant zone is either captured into the resonance, or crosses the resonant zone without being captured. After the passage through the resonant zone (and far from the resonance) the value of P_φ

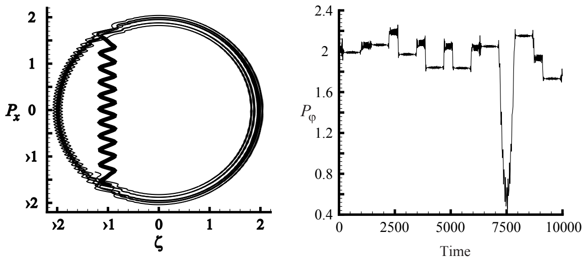

Figure 2: Several consecutive scatterings and the following capture and release in nominal dynamics: (a) Projection on the (ζ, P_x) plane; P_φ as a function of time. Small jumps and a big drop correspond to scatterings and the capture, respectively.

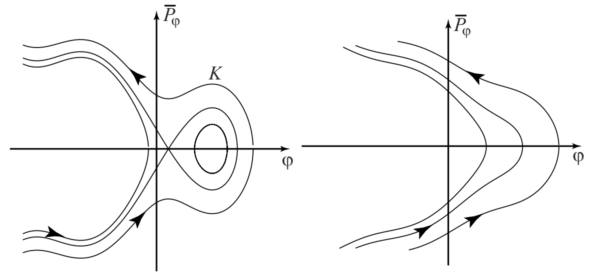

Figure 3: The schematic phase portrait on the $(\varphi, \overline{P}_\varphi)$ plane: (a) $|\varepsilon| \geq |b|$, (b) $|\varepsilon| < |b|$

oscillates near some other value, P_φ^+, again with a small amplitude $\sim (\kappa\varepsilon)$ (see Figure 2 below).

In the case of *capture into resonance*, upon the arrival to the resonant zone a phase point drifts for a long time (of order $\sim 1/(\kappa\varepsilon)$) along the resonant surface. As a result, P_φ changes by a value of order 1. Among all the particles only a small part, of order $\sim \sqrt{\kappa\varepsilon}$, is captured. Initial conditions for particles that are or are not captured are mixed. Therefore, it is reasonable to consider capture as probabilistic phenomenon. For a particular particle the probability to be captured is small, of order $\sim \sqrt{\kappa\varepsilon}$.

The particles that cross the resonant zone without being captured typically pass through this zone in time of order $\sim 1/\sqrt{\kappa\varepsilon}$ (see [7, 8]). The major resonant phenomenon for such particles is the *scattering on resonance*. The magnitude of the jump in P_φ, $\Delta P_\varphi = P_\varphi^+ - P_\varphi^-$, is typically of order $\sim \sqrt{\kappa\varepsilon}$, and is referred to as the amplitude of the scattering. This value is very sensitive to small changes of the initial conditions. Hence, it is reasonable to consider the scattering as a random process.

Figure 2 shows a single trajectory of the exact system. Figure 2a presents the projection of a characteristic phase curve onto the (ζ, P_x) plane. Near-circular part is the motion far from the resonance, a wavy near-vertical line is the captured motion. Figure 2b illustrates the time evolution of P_φ. Small jumps and a big drop correspond to scatterings and the capture, respectively. Note, that the capture into the resonance is a probabilistic process: the capture occurs only once in several consecutive crossings of R.

Properties of the dynamics in the vicinity of a resonance depend on the shape of the phase portrait in the $(\varphi, \bar{P}_\varphi)$ plane. A phase portrait can be of one of two types: with or without the oscillatory domain. If

$$|\varepsilon| \geq |P_{x,res}|, \tag{7}$$

the phase portrait looks like the one shown in Fig. 3a. If (7) does not hold, the phase portrait looks like the one shown in Fig. 3b.

4.1 Scattering on resonance.

In general case, every crossing of the resonant zone leads to a jump in P_φ (see [7, 8]):

$$\Delta P_\varphi = -2s\varepsilon \frac{\sqrt{\kappa}}{\sqrt{|b|}} \int_{-s\infty}^{\varphi_*} \frac{\cos\varphi \, d\varphi}{\sqrt{2\left|2\pi\xi - \frac{\varepsilon}{b}\sin\varphi - \varphi\right|}},$$

where φ_* is the value of φ on R (i.e. at the resonance crossing) and $s = \text{sign}(b)$. The quantity ξ, that characterizes the crossing and is given by $2\pi\xi = \varphi_* + (\varepsilon/b)\sin\varphi_*$, is very sensitive to the initial conditions and can be treated as a random variable, uniformly distributed on the $(0, 1)$ interval (see [8]).

4.2 Capture into resonance.

The other phenomenon that affects the dynamics of a particle in the vicinity of a resonance is capture into resonance. Capture is possible only if the phase portrait in the $(\varphi, \bar{P}_\varphi)$ plane looks like the one shown in Fig. 3a, in other words, if (7) is satisfied and there is a separatrix on $(\varphi, \bar{P}_\varphi)$ phase plane.

Capture into resonance can be described as follows. The area under the separatrix loop in the $(\varphi, \bar{P}_\varphi)$ plane, S_R, changes while a particle moves along a phase trajectory. Suppose the area under the separatrix loop increases. Then a particle that comes very close to the separatrix may cross it and, as a result, be caught in the oscillatory domain within the separatrix loop. The capture happens in the upper half-plane if $\omega < 0$ and in the lower half-plane if $\omega > 0$. A captured particle starts moving in the vicinity of the resonant surface $\zeta = -\omega$ and its dynamics is regular and governed by the Hamiltonian h_x (see (5)):

$$\kappa\dot{x} = \dot{\bar{P}}_\varphi = \kappa\bar{P}_x, \quad \dot{\bar{P}}_x = \kappa\omega. \tag{8}$$

One can see that regardless of the sign of ω captured motion is directed "into" the orbit of the averaged system (see Fig. 2a).

Introduce an action variable of $(\varphi, \bar{P}_\varphi)$ motion as normalized by 2π area under a phase trajectory in the $(\varphi, \bar{P}_\varphi)$ plane:

$$J = \frac{1}{2\pi} \oint \bar{P}_\varphi \, d\varphi. \tag{9}$$

J is conserved during the captured motion:

$$J = J_0 = S_0/2\pi = \text{const}, \tag{10}$$

where S_0 is the value of $S_R(P_\varphi)$ when the particular particle was caught. The value of P_φ changes in the process of the captured motion and the minimum value is achieved when the trajectory crosses the axis $P_x = 0$.

The fate of a captured particle depends on the behaviour of S_R as a function of P_x. As the particle moves, $S_R(P_\varphi)$ changes along the trajectory and, if it returns to S_0, the particle is released from the resonance. In contrast to that, if $S_R(P_\varphi)$ keeps increasing, the captured particle accelerates unboundedly until it exits the system. In the particular model discussed in the present paper, the section of R by a plane $P_\varphi = \text{const}$ is a straight line that is parallel to the P_x-axis. Hence, it follows from the symmetry of the phase portrait of x-motion with respect to the axis of abscissas that a capture into the resonance and the consequent release from the resonance occur at approximately the same value of P_x.

5 Controlling the capture

In this section we discuss how the capture into the resonance can be used to control the motion of particles. Suppose we need to move a particle from some initial state with the "real" energy H_i to some final state with the energy H_f. Note, that we consider both constant magnetic field and the electromagnetic wave as parts of the nominal system. Therefore, the energy of the wave is not included in the cost of control. Standard methods require either applying an energy of order of the difference between H_f and H_i or waiting for a very long time to let the diffusion of P_φ take the system to H_f (recall, that P_φ is related to H, see (2)). While the second method has the advantage of being cost-free, there are a couple of setbacks: one must wait for a very long time and meanwhile the whole dynamics of the particles becomes chaotic.

A natural way to overcome the problems stated above is to use capture into the resonance. But, as the capture is a probabilistic phenomena one must wait for a long time before it happens and also a captured particle is transported to some new energy level, defined by $S_R(P_\varphi)$, that might not (and probably will not) be near the target one. Therefore, we need to implement control both at the entry and exit moments to enforce a quick capture and to secure a timely release.

Let us start with capturing. When a particle comes close to the separatrix we apply a short pulse pushing a particle into the oscillatory domain inside the separatrix loop (see Fig. 4 below). A possible example of such a pulse is a short impulse in y-direction. As $P_\varphi = P_y$, we have $\Delta P_\varphi = F\Delta t$, where F and Δt are the amplitude and the duration of the impulse, respectively.

The captured dynamics was described in Subsect. 4.2. After the capture a particle oscillates near the resonance. It's dynamics is regular and is governed by equations of motion (8) and conservation law (10). The capturing impulse puts a particle on a certain level of J: $J = J_c$ (see (9)). Clearly, the deeper is a trajectory into the oscillatory domain, the smaller is the value of J. The values $J = 0$ and $J = S_0/2\pi$ correspond to the elliptic stationary point and the separatrix, respectively. If no additional impulses that either push a particle deeper into the oscillatory domain or kick it out from the oscillatory domain are applied a particle will stay captured as long as $S_R(P_\varphi) \geq 2\pi J_c$.

When a particle reaches the target value of P_φ we apply another impulse to push the particle from the resonance.

The new "energy", h, is the integral of the nominal system and is related to the "real" energy H as $H \propto (h + \omega P_\varphi)$, see (2). Pulses change h, but these changes are very small, as it is shown by numerical simulations below. Therefore, in what follows we consider h to be constant throughout the evolution.

5.1 Controllability and reachable domain

The minimum value of P_φ that is reachable by this method, $P_{\varphi,min}$, corresponds to the curve that touches R:

$$h + \omega P_{\varphi,min} = \omega^2/2.$$

The trajectories with smaller values of P_φ do not cross R. The maximum reachable value of P_φ corresponds to $|P_{x,res}| = \varepsilon$ as for larger values of $|P_{x,res}|$ there is no separatrix on the (φ, P_φ) phase plane, see (7):

$$h + \omega P_{\varphi,max} = \varepsilon^2/2 + \omega^2/2.$$

In terms of P_φ, reachable set cover the entire interval between $P_{\varphi,min}$ and $P_{\varphi,max}$. For every pair of P_φ and h all the trajectories of the averaged system are circles:

$$h + \omega P_\varphi = P_x^2/2 + \left(-P_\varphi + \kappa x\right)^2/2.$$

Therefore, the total reachable set is a cut oblique cone in the $(P_\varphi, P_x, \kappa x)$ space.

In principle, one pulse is sufficient to move a particle from any initial value of $P_{\varphi,in}$ to any final value $P_{\varphi,final} > P_{\varphi,in}$ within the reachable interval. To achieve this the pulse must put a particle on the level set $J_c = S_R(P_{\varphi,final})/2\pi$. In this case the "nominal" release happens at $P_\varphi = P_{\varphi,final}$. The larger is the value of $P_{\varphi,final}$ the closer captured trajectory must be to the elliptic fixed point on the $(\varphi, \bar{P}_\varphi)$ phase plane. In the case $P_{\varphi,final} < P_{\varphi,in}$ there are no requirements on the magnitude of the initial pulse, provided it puts a particle in the oscillatory domain. To release a particle from the capture the second pulse must be applied.

One can apply a sequence of pulses to put a particle on the proper level of J. This control method is robust in a sense that there is no need for extreme accuracy of the magnitude of either capturing or releasing pulses. The first, capturing, pulse must be just strong enough to put a particle inside the oscillatory domain and not too strong in order not to overshoot over the separatrix loop. If the resulting value of J_c is smaller than $J_f = S_f/2\pi$ (where S_f is the value of S that corresponds to $P_{\varphi,final}$) no adjustments before the second pulse are necessary. Otherwise, additional pulse(s) are required to push a trajectory deeper into the oscillatory domain. The exiting pulse must

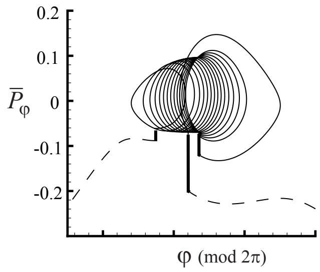

Figure 4: Projection of the phase curve onto the (φ, P_φ) plane: forced capture and release.

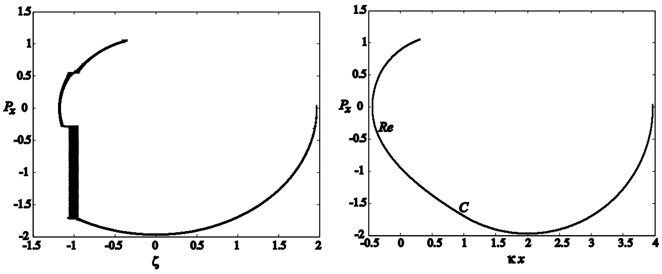

Figure 5: Forced capture and release: (a) projection of the phase curve onto the (ζ, P_x) plane; (b) projection of the phase curve onto the $(\kappa x, P_x)$ plane. The points C and Re denote locations of the capture and release, respectively. The initial conditions are $\kappa x_0 = 3.97$, $y_0 = 0.04$, $P_{\varphi,0} = 2$.

be just strong enough to kick the particle away from the loop. Characteristic controlled captured dynamics is shown in Figure 4. The dashed lines to the right and to the left correspond to nominal motion before and after the capture, respectively. The thin solid line is captured motion. Vertical thick solid lines are control impulses. The first pulse (the vertical line in the middle) is the original pulse that forced a capture. It follows from the shape of the first loop of the captured curve in Fig. 4, that it is close to the separatrix. The second pulse (the right line) was be applied to put a particle deeper into the oscillatory domain and the last pulse (the left line) released a particle from the resonance when a target value of P_φ is reached. Note, that although the second pulse put a particle relatively deep inside the separatrix loop, by the release time the area under S decreased and quite a weak pulse was sufficient for release.

6 Numerical simulations

It was stated before that resonance phenomena is probabilistic in nature: tiny changes in initial conditions may result quite different behaviour of trajectories near resonance. Although the forced capture designed to somewhat straighten things out, in controlled system there is a certain residual degree of stochastisity inherited from the nominal dynamics.

All the problems come from the fact that we want not to follow individual particles but to propose "one size fits all"-type of control algorithm. In the former case we could use the method described in the previous section to apply capturing and releasing pulses at exactly prescribed moments for a given particle. On the other hand, in the latter case we must settle for an "approximate" method with kicks being localized in a certain place in the real (physical) space.

To analyze the properties of the proposed control mechanism, we performed a set of numerical simulations. Initial conditions were chosen to be $\kappa x_0 \in (3.9, 4.1)$, $y_0 \in (-0.1, 0.1)$, $P_{\varphi,0} = 2$. The values of other parameters are: $\omega = 1$, $\varepsilon = 12$, $\kappa = 0.0001$. Our objective was to change the value of P_φ from 2.0 to 0.65. Total time of the evolution was $T_f = 50000$. In this case R is a vertical line $\kappa x = 1$ (or, in terms of ζ, $\zeta = -1$). For a kick we implemented a rectangular pulse that gives an additional term in equation for $\dot{P}_\varphi$:

$$\dot{P}_\varphi = -\kappa\varepsilon \cos\varphi + p_j f_p(\kappa x),$$

where $f_p(\kappa x) = 1$ for $\kappa x \in (\kappa x_{k,j} - \delta_{k,j}, \kappa x_{k,j} + \delta_{k,j})$ and $f_p(\kappa x) = 0$ otherwise and $j = \{c, r\}$ correspond to capturing and releasing pulses, respectively. We call κx_k, δ_k and p location, duration and magnitude of the kick, respectively. A limitation on δ_k is that time a particle spends in a "pulse" zone is much smaller than time it takes a particle to travel around the separatrix loop. A characteristic captured trajectory is shown in Fig. 5. The points C and Re denote locations of the capture and release, respectively. The total applied energy in this simulation is approximately equal to 0.015, which is nearly nearly two orders of magnitude less then the difference between the final and initial values of P_φ, 0.665 and 2.0, respectively.

The simulations show that the percentage of captured particles is rather sensitive to the location of the capturing kick. In our simulations particles approach R from below on $(\varphi, \bar{P}_\varphi)$ phase plane. A natural moment to apply the kick is when a particle is right above the separatrix loop on $(\varphi, \bar{P}_\varphi)$ phase plane (point K in Fig. 3a). For different initial conditions particle arrive to the respective points K at different values of κx. Therefore, the kick is most effective (in terms of the percentage) when it is applied in the middle of a range of κx_K for a given set of initial conditions. In terms of the values of the parameters used in simulations, the best results were achieved with $\kappa x_k \approx 1.02$. For $\kappa x_k = 1.02$, 87% of trajectories were captured (see below for a detailed description of the simulations). For $\kappa x_k = 1.01$ and $\kappa x_k = 1.03$, the percentage of captured trajectories was 53% and 38%, respectively.

The effectiveness of a kick is somewhat less sensitive to the duration and the magnitude of the kick, $\delta_{k,c}$ and p_c. The value of $\bar{P}_\varphi$ at the moment of the kick is different for different initial conditions. Hence, the pulse must nether overshoot nor undershoot for most of initial conditions. So, $(\delta_k \times p)$ should be approximately equal to the height of the separatrix loop in Fig. 3a, which, in turn, is of order of $\sqrt{\kappa\varepsilon}$ (see Sect. 4).

Note, that as the values of state variables right before the kick are different (most importantly in φ and $\bar{P}_\varphi$), kicks put particles on different trajectories of captured motion (in other words, the values of J_c, see Sect. 5, are different). Consequently, although the captured motion is always regular, exact properties, like the

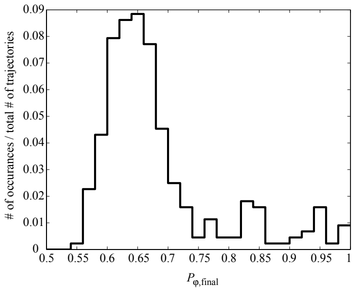

Figure 6: Distribution of the final values of P_φ for captured and released trajectories.

time it takes a particle to arrive to the "release" zone and the final value of P_φ, may (slightly) vary.

Mechanism of release is more robust – if the pulse is large enough the release definitely occur. On the other hand, the value of P_φ after the pulse does depend on the value of the resonant phase at the moment of release. A characteristic value of $(\delta_{k,r} \times p_r)$ should again be of order of $\sqrt{\kappa\varepsilon}$.

After the release particles proceed along an averaged trajectory. Before the time of simulations runs out, they undergo an additional scattering on resonance. In that region the area under the separatrix decreases and the natural capture is impossible.

We performed 441 runs with $\kappa x_{k,c} = 1.02$, $\delta_{k,c} = 0.001$ and $p_c = -0.003$ for capture and $\kappa x_{k,r} = -0.4$, $\delta_{k,r} = 0.001$ and $p_r = 0.001$ for release. Of these 441 runs, 87% of trajectories were captured. All the captured trajectories (for which the releasing kick was applied) were released. Captured trajectories are distributed quite uniformly over the box of initial conditions. Figure 6 shows a distribution of the final values of P_φ for captured and released trajectories. All the factors (capture, release and later scattering) contribute to a certain dispersion of P_φ, but all the values are way away from the initial value.

6.1 Separation of particles.

One of the possible applications of the control via capture is to separate particles of two different types (for example, isotopes). Let isotopes differ by mass only: m_1 and m_2 with $\beta = m_2/m_1$. Typically isotopes come from an accelerating device with the same energies. We performed a set of simulations with $\beta = 1.05$. For the type "2" of the particles the resonance is located at $\zeta = 1/\beta_2$. As a result, the pulse, that is synchronized with isotope 1, is applied at a "wrong" moment. Consequently, only a tiny bit of particles are captured due not to a control pulse, but a natural dynamics discussed in Subsect. 4.2. Distribution of the final values of P_φ in this case has a single narrow peak localized near the original value of 2.

Conclusions

In the present paper we proposed a method to use capture into resonance as an efficient tool to control the behavior of near-integrable dynamical systems where interaction between an unperturbed system and a weak periodic perturbation can be reduced to a purely resonant interaction. We showed that weak control pulses can structurize resonance interaction allowing it to be used for regular transport across invariant manifolds of the unperturbed system.

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