

A RELAXED CRITERION FOR CONTRACTION THEORY: APPLICATION TO AN UNDERWATER VEHICLE OBSERVER

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Abstract

On the contrary to Lyapunov theory, contraction theory studies system behavior independently from a specific attractor, thus leading to simpler computations when verifying exponential convergence of nonlinear systems. To check the contraction property, a condition of negativity on the Jacobian of the system has to be fulfilled. In this paper, attention is paid to results for which the negativity condition can be relaxed, *i.e.* the maximum eigenvalue of the Jacobian may take zero or positive values. In this issue, we present a theorem and a corollary which sufficient conditions enable to conclude when the Jacobian is not uniformly negative definite but fulfils some weaker conditions. Intended as an illustrative example, a nonlinear underwater vehicle observer, which Jacobian is not uniformly negative definite, is presented and proven to be exponentially convergent using the new criterion.

1 Introduction

Contraction theory, also called contraction analysis, is a recent tool enabling to study the stability of nonlinear systems trajectories with respect to one another, which in some cases, like tracking or observer design, may lead to a simpler analysis than with Lyapunov theory (see [11, 12] and references therein).

The original definition of contraction requires the uniform negative definiteness of the Jacobian of the system $\dot{x} = f(x, t)$ or a modified Jacobian, called generalized Jacobian F , which is obtained after a local time and state dependent transformation matrix $\Theta(x, t)$. Although there exists a converse theorem (see [11, section 3.5] stating that if a system is exponentially convergent, then there exists a local transformation matrix Θ such that the system is contracting, one may wonder whether or not it is possible to relax the negative definiteness condition of the Jacobian. An important step has already been made in this issue, which was presented in [12, section 2.3] and [9, p. 17-20] where it is shown that

under some specific conditions, systems which Jacobian are only negative *semi*-definite are also proven to be exponentially convergent.

In this paper, we will go a bit further by studying systems which Jacobian may have a temporarily positive or zero maximum eigenvalue.

Some interesting results are already available in the literature for Lyapunov stability (see for example [7, 1, 2, 13]). In the rest of this paper, the issue of adapting a result of Aeyels and Peuteman to the world of contraction theory will be first addressed in section 2. This result will be simplified in section 3 so as to study directly the maximum eigenvalue through a time integral. Finally, in section 4, a simple application to the design of an autonomous underwater vehicle nonlinear observer will be presented to illustrate the concept.

As in [11], the class of systems considered is the general deterministic continuous nonlinear systems represented by

$$\dot{x} = f(x, t) \quad (1)$$

where x is the state of the system ($x \in \mathbb{R}^n$), and f a nonlinear time and state dependent function. From (1), the virtual dynamics are written as

$$\delta\dot{x} = \frac{\partial f}{\partial x}(x, t)\delta x \quad (2)$$

where δx is a virtual displacement and $\partial f/\partial x$ is the Jacobian of the system. In the following, we will denote $\lambda_{max}(x, t)$ the largest value of the symmetric part of the above Jacobian. To obtain the generalized Jacobian F , define the local transform

$$\delta z = \Theta(x, t)\delta x \quad (3)$$

which leads to define F as

$$F = \left(\dot{\Theta} + \Theta \frac{\partial f}{\partial x} \right) \Theta^{-1} \quad (4)$$

For the definition of the original criteria enabling to conclude to contracting behavior, *i.e.* exponential convergence, the reader is referred to [11].

2 Relaxation of the negativity constraint

As the original version of contraction theory, this new criterion presents the same useful property of being independent of a specific attractor, making unnecessary the expression of an error term, as it is the case in Lyapunov theory. Therefore, the chosen point of view for this study is in a sense more general. The theorem enabling to relax the constraint of negativity can be stated as follow.

Theorem 2.1 *If the local transform Θ and the generalized Jacobian F are uniformly bounded, and if there exists an increasing sequence of t_k such that $t_k \rightarrow \infty$ when $k \rightarrow \infty$ and that $t_{k+1} \in [t_k, t_k + T]$ where $T > 0$ and for all k , such that the following condition is verified*

$$\|\delta z(t_{k+1})\|^2 - \|\delta z(t_k)\|^2 \leq -\beta \|\delta x(t_k)\|^2 \quad (5)$$

for all k and where β is a positive constant, then the system trajectories will converge exponentially to one another.

This theorem being greatly inspired by the work of Aeyels and Peuteman, only the sketch of its proof will be given, which would be sufficient however to give the reader an idea of the method. Note that the use of virtual displacements and of the notations of contraction theory renders the approach rather simple.

The proof can be obtained into two main steps. The first consists in demonstrating exponential convergence for all times t_k , where $k \in \mathbb{Z}$, while the second one will complete the proof by considering exponential convergence *between* times t_k .

Let us start by considering the times t_k . The fact that the local transform Θ is bounded, combined with the other fact stating that the metric $M = \Theta^T \Theta$ is uniformly positive definite means one has the following relation

$$\sigma_{min}^2 \|\delta x\|^2 \leq \|\delta z\|^2 = \delta x^T \Theta^T \Theta \delta x \leq \sigma_{max}^2 \|\delta x\|^2 \quad (6)$$

Using this last expression, the condition (5) can be changed in

$$\|\delta z(t_{k+1})\|^2 - \|\delta z(t_k)\|^2 \leq -\frac{\beta}{\sigma_{max}^2} \|\delta z(t_k)\|^2 \quad (7)$$

which gives

$$\|\delta z(t_{k+1})\|^2 \leq \left(1 - \frac{\beta}{\sigma_{max}^2}\right) \|\delta z(t_k)\|^2 \quad (8)$$

It can be noticed that if $\beta > 0$, the sequence is indeed decreasing since $1 - \frac{\beta}{\sigma_{max}^2} < 1$.

Now if, instead of t_k and t_{k+1} , we consider the distant instants t_k and t_{k+n} , where $n \in \mathbb{N}$, one will obtain

$$\|\delta z(t_{k+n})\|^2 \leq \left(1 - \frac{\beta}{\sigma_{max}^2}\right)^n \|\delta z(t_k)\|^2 \quad (9)$$

which, in terms of signal norms, gives

$$\|\delta z(t_{k+n})\| \leq \left(1 - \frac{\beta}{\sigma_{max}^2}\right)^{\frac{n}{2}} \|\delta z(t_k)\| \quad (10)$$

Letting $n = 1$ and noting that $x^y = e^{y \ln x}$, (10) becomes

$$\|\delta z(t_{k+1})\| \leq e^{-\lambda T} \|\delta z(t_k)\| \quad (11)$$

with

$$\lambda = -\frac{1}{2T} \ln \left(1 - \frac{\beta}{\sigma_{max}^2}\right) \quad (12)$$

As $t_{k+1} - t_k \leq T$, (11) can be approximated with

$$\|\delta z(t_{k+1})\| \leq e^{-\lambda(t_{k+1} - t_k)} \|\delta z(t_k)\| \quad (13)$$

for all $k \in \mathbb{Z}$.

With the same reasoning, by starting with equation (9), one would have obtained

$$\|\delta z(t_{k+n})\| \leq e^{-\lambda(t_{k+n} - t_k)} \|\delta z(t_k)\| \quad (14)$$

Thus, it has been demonstrated that for all instant of the sequence, there is an exponential convergence of the virtual displacements δz towards 0.

Now looking at the second step of the proof of the theorem, we will pay attention to what goes on between the instants of the sequence. Assume first that t lies sometime between t_{k+1} and t_{k+2} . The bound of the generalized Jacobian F , expressed as

$$\|F\| \leq K \quad (15)$$

leads to the following inequality

$$\|\delta z(t)\| \leq e^{K(t - t_{k+1})} \|\delta z(t_{k+1})\| \quad (16)$$

Then, using the decreasing exponential formulae (13), one gets

$$\|\delta z(t)\| \leq e^{K(t - t_{k+1})} e^{-\lambda(t_{k+1} - t_k)} \|\delta z(t_k)\| \quad (17)$$

After transformation, it gives

$$\|\delta z(t)\| \leq e^{-\lambda(t - t_k)} e^{(\lambda + K)T} \|\delta z(t_k)\| \quad (18)$$

Using inequality (14) one can get back to the index 0

$$\|\delta z(t)\| \leq e^{-\lambda(t - t_0)} e^{(\lambda + K)T} \|\delta z(t_0)\| \quad (19)$$

and by assuming that $t_0 \leq T$, the bound K on the generalized Jacobian can be used to write

$$\|\delta z(t_0)\| \leq e^{KT} \|\delta z(0)\| \quad (20)$$

(19) is then changed in

$$\|\delta z(t)\| \leq e^{-\lambda(t - t_0)} e^{(\lambda + K)T} e^{KT} \|\delta z(0)\| \quad (21)$$

$$= e^{-\lambda(t - t_0 + T)} e^{2(\lambda + K)T} \|\delta z(0)\| \quad (22)$$

$$\leq e^{-\lambda t} e^{2(\lambda + K)T} \|\delta z(0)\| \quad (23)$$

Finally, by letting $\delta z(t) = \delta z$ and $\delta z(0) = \delta z_0$, we obtain

$$\|\delta z\| \leq \gamma' \|\delta z_0\| e^{-\lambda t} \quad (24)$$

with

$$\lambda = -\frac{1}{2T} \ln \left(1 - \frac{\beta}{\sigma_{max}^2}\right) \quad (25)$$

and

$$\gamma' = e^{2(\lambda+K)T} \quad (26)$$

Coming back to the δx , it gives

$$\|\delta x\| \leq \gamma \|\delta x_0\| e^{-\lambda t} \quad (27)$$

with this time

$$\gamma = \frac{\sigma_{max}}{\sigma_{min}} e^{2(\lambda+K)T} \quad (28)$$

which leads to finally conclude that for all t , there is an exponential convergence of δx towards 0, and hence of the flow of trajectories towards a unique trajectory.

3 A temporarily positive eigenvalue

Using and manipulating a decreasing sequence such as the one of condition (5) may appear as not obvious or counter-intuitive, especially because this condition, as it is presented, somehow removes the continuous time aspect by including a more discrete-time type term in the left hand side of the condition. The corollary to theorem 2.1 that we propose hereafter is a simplification allowing both to study directly $\lambda_{max}(x, t)$ and to present a condition with a time integral term.

Corollary 3.1 *Let $\lambda_{max}(x, t)$ be the maximum eigenvalue of the Jacobian of system $\dot{x} = f(x, t)$. If $\partial f/\partial x$ is uniformly bounded and if there exists an increasing sequence of time t_k such that $t \in [t_k, t_k + T]$ with $T > 0$, that verifies the inequality*

$$\int_{t_k}^{t_{k+1}} \lambda_{max}(x, t) dt \leq -\alpha \quad (29)$$

for all k and where α is a positive constant, then the system trajectories will exponentially converge to one another.

Note that the implications of the above corollary are different from those of a simple moving average (which is alluded to in [9, section 3.4, p. 16]) which would constrain the non stationary part to be periodical. In our case, as the integral interval does not move (rather, it is repeated), it allows to work on a more general class of systems.

Before showing how this result can be applied with a very simple illustrative example, we hereafter present a glimpse of its proof.

For the sake of clarity, only the case where $\Theta = I$ will be presented. The extension to the generalized Jacobian F is straightforward.

To begin with, remark that the use of virtual displacements δx , without any preliminary local transformation Θ , in condition of theorem 2.1 is somehow quite restrictive since

$$\|\delta x(t_{k+1})\|^2 - \|\delta x(t_k)\|^2 \leq -\beta \|\delta x(t_k)\|^2 \quad (30)$$

constrains β to be lower than 1. This limitation is due to the fact that for a function $\lambda_{max}(x, t)$, for which it would be possible

to have positive values, would provoke overshooting compared with a usual exponential function. This would hence induce an implicit local transformation Θ .

Accounting for this fact, introduce a scalar transform as follows

$$\|\delta z\|^2 = \sigma^2 \|\delta x\|^2 \quad (31)$$

where σ is a positive constant particularizing the local transformation Θ .

The introduction of σ gives

$$\sigma^2 \|\delta x(t_{k+1})\|^2 - \sigma^2 \|\delta x(t_k)\|^2 \leq -\beta \|\delta x(t_k)\|^2 \quad (32)$$

hence

$$\|\delta x(t_{k+1})\|^2 \leq \left(1 - \frac{\beta}{\sigma^2}\right) \|\delta x(t_k)\|^2 \quad (33)$$

Thus, for all positive β , there exists a σ such that the decreasing condition is realized.

Returning now to the proof of corollary 3.1, note that

$$\frac{d}{dt} (\delta x^T(t) \delta x(t)) = \delta x^T(t) \left(\frac{\partial f^T}{\partial x}(x, t) + \frac{\partial f}{\partial x}(x, t) \right) \delta x(t) \quad (34)$$

$$\leq 2\lambda_{max}(x, t) \|\delta x(t)\|^2 \quad (35)$$

in the time interval $[t_k, t_{k+1}]$ leads to

$$\|\delta x(t_{k+1})\| \leq \|\delta x(t_k)\| e^{\int_{t_k}^{t_{k+1}} \lambda_{max}(x, \tau) d\tau} \quad (36)$$

which, given inequality (29), implies

$$\|\delta x(t_{k+1})\| \leq \|\delta x(t_k)\| e^{-\alpha} \quad (37)$$

thus proving convergence for all t_k of the sequence.

Then, noticing that there exists a positive λ such that $\alpha \geq \lambda T \geq \lambda(t_{k+1} - t_k)$ and taking into account the boundedness assumption on $\partial f/\partial x$, the end of the proof of corollary 3.1 follows the same line as theorem 2.1 starting from expression (15).

Example 3.1 *Given the system*

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2x_1 - x_1^3 \\ -\frac{1}{2}x_2 + \cos(t)x_2 \end{pmatrix} \quad (38)$$

Its virtual dynamics can be written as

$$\begin{pmatrix} \delta \dot{x}_1 \\ \delta \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 - 3x_1^2 & 0 \\ 0 & -\frac{1}{2} + \cos(t) \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \delta x_2 \end{pmatrix} \quad (39)$$

From this, deduce

$$\lambda_{max}(x, t) = -\frac{1}{2} + \cos(t) \quad (40)$$

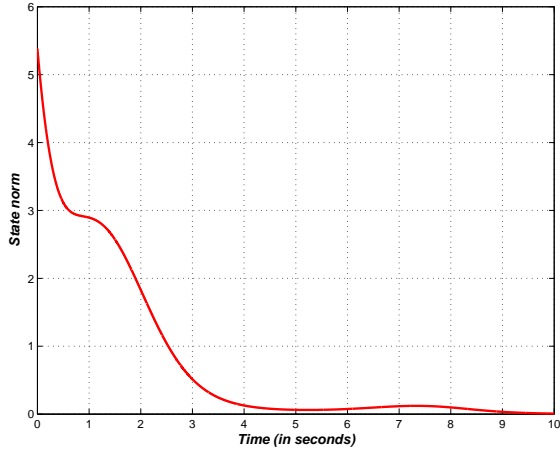


Figure 1: Evolution of the state norm of system (38)

which is positive periodically.

Choosing the sequence $t_k = 2k\pi$, one finds

$$\int_{t_k}^{t_{k+1}} \lambda_{max}(x, t) dt \leq \int_{2k\pi}^{2(k+1)\pi} \left(-\frac{1}{2} + \cos(t) \right) dt \quad (41)$$

$$= \left[-\frac{1}{2}t + \sin(t) \right]_{2k\pi}^{2(k+1)\pi} \quad (42)$$

$$= -\pi < 0 \quad (43)$$

to conclude to exponential convergence of system trajectories.

Simulation results of system (38) are represented in figure 1 with initial conditions $x_0 = (5, 2)^T$. Note the different behavior from the one that would be obtained with an always negative maximum eigenvalue.

4 Application to the design of an underwater vehicle observer

Contraction analysis was demonstrated to be very useful for the design of nonlinear observers (see for example [10]). Among the applications that have been considered, let us single out the example of an autonomous underwater vehicle (AUV). A possible model including thruster dynamics for an AUV moving on a single horizontal axis would be described by [14]

$$\begin{cases} J_\omega \dot{\omega} = -D_\omega \omega |\omega| + \tau \\ T_\omega = K_\omega \omega |\omega| \\ M_v \dot{v} = -D_v v |v| + T_\omega \end{cases} \quad (44)$$

where ω and v represent the angular velocity of the propeller and the vehicle speed respectively. T_ω is the thrust provided to the vehicle by the propeller, and τ the propeller control voltage. J_ω , M_v , D_ω , D_v and K_ω are constant positive parameters standing for, respectively, a parameter proportional to the inertia of the propeller, the mass of the AUV, the propeller nonlinear damping coefficient, the drag parameter of the vehicle and the thrust coefficient.

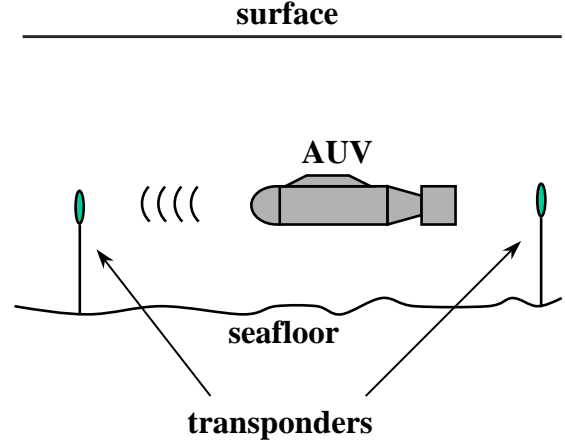


Figure 2: AUV and LBL navigation system

If only the position x of the vehicle (with $\dot{x} = v$) and the angle of the propeller α ($\dot{\alpha} = \omega$) are measured, noticing that the system (44) is a hierarchy would help us to design a simple reduced-order observer estimating ω and v , as in [11].

However, a first practical consideration will lead us to design a slightly different observer. Indeed, while one may consider that α is not too much corrupted with noise as it is measured internally in the AUV, this is not the case for the measurement x which is obtained through acoustic sensing [8]. Taking into account the higher sensitivity to noise of reduced-order observers, we design a full-state observer for the vehicle dynamics subsystem to obtain the following equations:

$$\begin{cases} \dot{\hat{\omega}} = -\frac{D_\omega}{J_\omega} \hat{\omega} |\hat{\omega}| + \frac{1}{J_\omega} \tau + k_\alpha (\hat{\alpha} - \alpha) \\ \dot{\hat{v}} = -\frac{D_v}{M_v} \hat{v} |\hat{v}| + \frac{K_\omega}{M_v} \hat{\omega} |\hat{\omega}| + k_v (\hat{x} - x) \\ \dot{\hat{x}} = \hat{v} + k_x (\hat{x} - x) \end{cases} \quad (45)$$

where the implementation of the $\hat{\omega}$ subsystem is made as in [10] through the transform $\bar{\omega} = \hat{\omega} + k_\alpha \alpha$. If k_α is tuned so that $\hat{\omega}$ is contracting, then this part of the observer will represent a time varying and exponentially decaying disturbance $T_\omega(t)$ for the $(\hat{v}, \hat{x})^T$ dynamics. Computing the virtual displacements of this subsystem as follows

$$\begin{pmatrix} \delta \hat{v} \\ \delta \hat{x} \end{pmatrix} = \begin{pmatrix} -2 \frac{D_v}{M_v} |\hat{v}| & k_v \\ 1 & k_x \end{pmatrix} \begin{pmatrix} \delta \hat{v} \\ \delta \hat{x} \end{pmatrix} \quad (46)$$

we see that for the case $\hat{v} \neq 0$, (46) is uniformly negative definite (u.n.d.) if $k_x < 0$ and $k_v = -1$, by virtue of the feedback combination property of contracting systems. Note additionally that the constraints on parameters induced by this combination property can be eased through the use of a constant scalar change of coordinates for $\delta \hat{x}$, *i.e.* by defining $\delta \hat{z} = \theta \delta \hat{x}$ (see [5]). When $\hat{v} = 0$ and with the above tuning for k_v and k_x , we have

$$F = \begin{pmatrix} 0 & -1 \\ 1 & k_x \end{pmatrix} \quad (47)$$

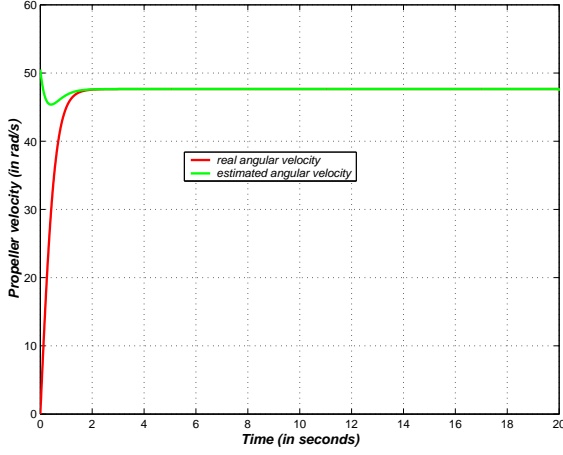


Figure 3: Propeller velocity

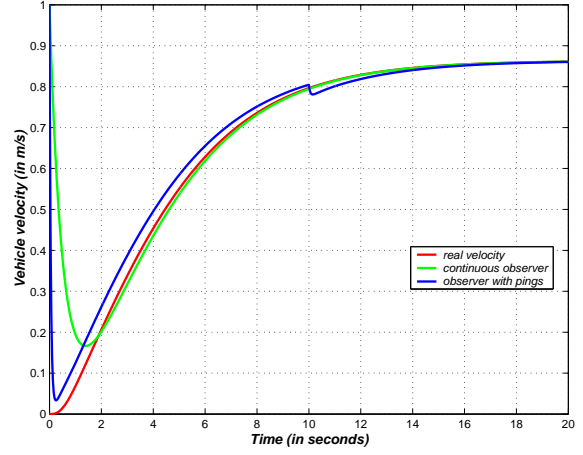


Figure 5: Underwater vehicle velocity

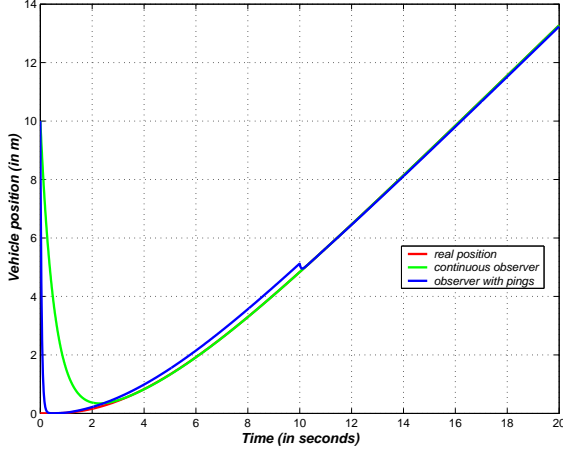


Figure 4: Underwater vehicle position

which is u.n.d. after a suitable local constant change of coordinates Θ which is easily computed. Finally, contracting behavior and therefore convergence to the real system trajectories can be concluded by joining the two cases $\hat{v} = 0$ and $\hat{v} \neq 0$ with a reasoning similar to section 4.9 of [9].

As a second practical consideration that one may consider, let us mention the fact that the information on the position x is constrained by the physical limitations of the position sensing system. Indeed, it happens that such a measurement is made using a long baseline (LBL) navigation system which consists of transponders fixed on the seafloor that the AUV interrogate with acoustic pings to estimate its position (see figure 2). Unfortunately, the update rate of LBL systems happens to go down to 0.05Hz (see [8]). Thus, one can only consider that the position information is available for a fraction of the ping period (say ten percent of the period). As a consequence, this *has* to be enough to ensure the convergence of the AUV observer, if we want it to give a correct estimate.

Instead of x , we formalise partial measurement with

$$p(t)x \quad (48)$$

where $p(t) = 1$ for $t \in [kT, T/10 + kT)$ and $p(t) = 0$ for $t \in [T/10 + kT, T + kT)$ with T being the update period of the LBL system. As this measurement will be fed into the AUV observer, and that the system equations still have to be a solution of the observer, replace (45) with

$$\begin{cases} \dot{\hat{\omega}} = -\frac{D_\omega}{J_\omega} \hat{\omega} |\hat{\omega}| + \frac{1}{J_\omega} \tau + k_\alpha (\dot{\hat{\alpha}} - \dot{\alpha}) \\ \dot{\hat{v}} = -\frac{D_v}{M_v} \hat{v} |\hat{v}| + \frac{K_\omega}{M_v} \hat{\omega} |\hat{\omega}| + k_v p(t) (\hat{x} - x) \\ \dot{\hat{x}} = \hat{v} + k_x p(t) (\hat{x} - x) \end{cases} \quad (49)$$

Note that while $t \in [T/10 + kT, T + kT)$, the observer (49) is in open-loop since the position information is not available, and that

$$F = \begin{pmatrix} -2\frac{D_v}{M_v} |\hat{v}| & 0 \\ 1 & 0 \end{pmatrix} \quad (50)$$

is *not* u.n.d.

Now using a straightforward consequence of theorem 2.1, we see that if k_v is set to -10 , by computing the integral terms $\int_{0+kT}^{T+kT} k_v p(t) dt$ and $\int_{0+kT}^{T+kT} k_x p(t) dt$ one can finally conclude to the exponential convergence of the observer.

We now present some simulation results for observer (49) where the parameters values $J_\omega = 0.0238 \text{Vs}^2$, $M_v = 340 \text{kg}$, $D_\omega = 8.8 \cdot 10^{-4} \text{Vs}^2$, $D_v = 67 \text{kg/m}$ and $K_\omega = 0.022 \text{Ns}^2$ are taken from [14]. The observer gains are tuned so that $k_\alpha = -0.5$, $k_v = -2$ and $k_x = -20$. The update period T is set to 10s .

Observer (49) was also compared to observer (45) for which continuous position measurement was assumed to be available. The gains of this observer were set to $k_\alpha = -0.5$, $k_v = -0.2$ and $k_x = -2$. The two observers were set with the same initial conditions $\hat{\omega}(0) = 50 \text{rad/s}$, $\hat{v}(0) = 1 \text{m/s}$ and

$\hat{x}(0) = 10 \text{ m}$ while the initial conditions of the AUV were set to $\omega(0) = 0 \text{ rad/s}$, $\alpha(0) = 1 \text{ rad}$, $v(0) = 0 \text{ m/s}$ and $x(0) = 0 \text{ m}$. The propeller control voltage τ is set to 2 V .

Figure 3 shows the evolution of the propeller angular velocity variables. Recall that the thrust resulting from the variables is then considered as input to the $(\dot{v}, \dot{x})^T$ (resp. $(\dot{v}, \dot{\hat{x}})^T$) subsystem.

Figure 4 and 5 show respectively the evolution of the vehicle position and speed variables. Note the difference between system and observer-with-pings variables for $1 \leq t \leq 10$ due to the lack of information. Convergence is then quickly ensured as soon as x is available.

More complex models could have been used to design an AUV observer, by considering for example the influence of the axial flow velocity on the system behavior which can be quite important (for more details, see [3] and [6]). We would hopefully keep the same considerations regarding the interrupted position information in case an LBL system is used.

5 Concluding remarks

By continuing the approach that was presented in this paper, other results could be envisaged, as for example the consideration of the averaged systems so as conclude on the convergent behavior of the original systems, thus leading to an incremental version of average theory. One could also consider possible extensions to systems with external signals such as inputs and outputs (see [4]).

On the application point of view, it may be of interest to look for more application-motivated examples to verify the potentiality and the interest of such relaxed criteria.

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