# EVENT-BASED OBSERVER DESIGN FOR OBSERVABLE NONLINEAR SYSTEMS WITH BAD INPUT POINTS

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## Abstract

The aim of this paper is to provide a methodology for the design of practical continuous high gain event-based observers for nonlinear systems with an almost everywhere injective $r$-observability map. As opposed to other high gain approaches, injectivity is allowed to be lost for a nonempty set of bad input points. An example with simulations illustrates the procedure.

## 1 Introduction

The following forced single output nonlinear systems are considered

$$\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x}, \boldsymbol{u}) = \mathbf{f}_{\boldsymbol{u}}(\boldsymbol{x}), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0, \qquad y = h(\boldsymbol{x}) \tag{1}$$

where $\boldsymbol{x} \in \mathcal{X} \subset \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}$ is the output, and $\boldsymbol{u}(t)\colon [0, \infty) \to \mathcal{U} \subset \mathbb{R}^p$ is a sufficiently smooth time function that acts as an external input to system (1). Furthermore, the vector field $\mathbf{f}$ and the function $h$ are assumed to be sufficiently smooth.

During the last ten years, the high gain approach has become a standard procedure to design observers for nonlinear systems (1) [1, 5]. However, its use has been limited to the cases where the nonlinear system is uniformly observable for every input [5], i.e. with a globally diffeomorphic observability map for any admissible input. In this paper, forced nonlinear systems (1) are considered, whose so-called *suspension map* [8] —an input dependent observability map plus an identity map on time derivatives of the inputs— is not injective everywhere. Therefore there exists a subset of *bad input points* [12] where no inverse exists.

The paper is organized as follows. In the next section, the suspension map is defined and the property of $r$-observability for a forced nonlinear system (1) is characterized. Bad input points and the $n$-observability form are also defined. Section 3 introduces the pseudo-observability form in order to compensate for the lack of a true $n$-observability form for these systems; events and $\delta_M$-admissible trajectories are also defined here. Section 4 presents the main result, which is the construction of continuous high gain event-based observers under trajectory restrictions. This is followed by an illustrating example, ending with some conclusions.

## 2 Observability and bad input points

For nonlinear systems *bad input functions* $\boldsymbol{u}^*(t)$ may exist for pairs of initial conditions $(\boldsymbol{x}_0, \bar{\boldsymbol{x}}_0)$ with $\boldsymbol{x}_0 \neq \bar{\boldsymbol{x}}_0$, such that their respective outputs are identical for all time. Systems for which no bad input functions exist are called uniformly observable for every input [5].

Although observability is a generic property [10], not all systems are uniformly observable for every input and in fact the existence of bad input functions, or respectively bad input points, is quite common. Some observer design strategies have already been proposed for systems with bad inputs, mainly for bilinear [4], state-affine [7], or somehow detectable systems [3]. In order to characterize a wider class of systems, consider first a vector $\underline{\boldsymbol{u}} \in \underline{\mathcal{U}} \subset \mathbb{R}^{p(w+1)}$,

$$\underline{\boldsymbol{u}} = \begin{bmatrix} \boldsymbol{u}^T, & \dot{\boldsymbol{u}}^T, & \ldots, & \overset{(w)}{\boldsymbol{u}}{}^T \end{bmatrix}^T \tag{2}$$

with $w$ large enough. Define the $r$-th order *suspension map* [8] whose domain is the state-input space $\mathcal{X}_{\underline{\mathcal{U}}} = \mathcal{X} \times \underline{\mathcal{U}}$

$$\mathbf{Q}_r\colon \mathcal{X}_{\underline{\mathcal{U}}} \to \mathbb{R}^r \times \underline{\mathcal{U}}, \qquad \mathbf{Q}_r\left(\begin{bmatrix} \boldsymbol{x} \\ \underline{\boldsymbol{u}} \end{bmatrix}\right) = \begin{bmatrix} \mathbf{q}_r(\boldsymbol{x}, \underline{\boldsymbol{u}}) \\ \underline{\boldsymbol{u}} \end{bmatrix} \tag{3}$$

with $\mathbf{q}_r$ is the input dependent $r$-observability map [16]

$$\mathbf{q}_r(\boldsymbol{x}, \underline{\boldsymbol{u}}) = \begin{bmatrix} L^0_{\mathbf{f}_{\boldsymbol{u}}} h \\ L^1_{\mathbf{f}_{\boldsymbol{u}}} h \\ \vdots \\ L^{r-1}_{\mathbf{f}_{\boldsymbol{u}}} h \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ \overset{(r-1)}{y} \end{bmatrix} \tag{4}$$

where $L_{\mathbf{f_u}}^k h$ are input dependent Lie derivatives of $h$ along $\mathbf{f}_{\boldsymbol{u}}$, defined by

$$L_{\mathbf{f_u}}^0 h(\boldsymbol{x}, \underline{\boldsymbol{u}}) = h(\boldsymbol{x}), \tag{5}$$

$$L_{\mathbf{f_u}}^k h(\boldsymbol{x}, \underline{\boldsymbol{u}}) = \frac{\partial L_{\mathbf{f_u}}^{k-1} h}{\partial \boldsymbol{x}} \cdot \mathbf{f_u} + \frac{\partial L_{\mathbf{f_u}}^{k-1} h}{\partial \underline{\boldsymbol{u}}} \cdot \frac{\mathrm{d}\underline{\boldsymbol{u}}}{\mathrm{d}t}, \quad k \geq 1. \tag{6}$$

Notice that $w$ in (2) is such that the input $\boldsymbol{u}$ appears explicitly until the $(r-w)$-th Lie derivative (6).

Injectivity of this map is a crucial property in the course of observability analysis and observer design. Evidently, the map (3) will never be injective if $u(t)$ is a bad input function $\boldsymbol{u}^*(t)$. However, even for "good" input functions *bad input points* may be encountered in the corresponding trajectory. These are points where the map (3) loses injectivity.

**Definition 1** *Consider a finite number $r \geq n$. The points $(\boldsymbol{x}^*, \underline{\boldsymbol{u}}^*) \in \mathcal{X}_{\underline{\mathcal{U}}}$ where the $r$-th order suspension map $\mathbf{Q}_r$ given by (3) is not injective are called* bad input points *of the map $\mathbf{Q}_r$. The set of bad input points $\mathcal{X}_{\underline{\mathcal{U}}}^B$ for a given $r \geq n$ is therefore defined by*[1]

$$\mathcal{X}_{\underline{\mathcal{U}}}^B = \big\{(\boldsymbol{x}^*, \underline{\boldsymbol{u}}^*) \colon \exists (\boldsymbol{x} \neq \boldsymbol{x}^*) \in \mathcal{X},\ \textit{s.t.}\\ \mathbf{q}_r(\boldsymbol{x}, \underline{\boldsymbol{u}}^*) = \mathbf{q}_r(\boldsymbol{x}^*, \underline{\boldsymbol{u}}^*)\big\}. \tag{7}$$

Considering these bad input points the following characterization of observability may be given:

**Definition 2** *System (1) is called* $r$-observable *if the complement $\mathcal{X}_{\underline{\mathcal{U}}} \setminus \mathcal{X}_{\underline{\mathcal{U}}}^B$ of the set of bad input points $\mathcal{X}_{\underline{\mathcal{U}}}^B$ of the map $\mathbf{Q}_r$ is dense in $\mathcal{X}_{\underline{\mathcal{U}}}$ with respect to the usual $\mathbb{R}^{n+p(w+1)}$ topology.*

**Remark 1** The density condition implies that the bad input point set $\mathcal{X}_{\underline{\mathcal{U}}}^B$ is a "small" set when compared to the state-input space $\mathcal{X}_{\underline{\mathcal{U}}}$. If $\mathcal{X}_{\underline{\mathcal{U}}}^B$ is a lower dimensional manifold in $\mathcal{X}_{\underline{\mathcal{U}}}$, it automatically fulfills the requirement.

If the $n$-th order suspension map (3) is globally injective, then the input dependent $n$-observability map (4) can be used as state transformation $\boldsymbol{z} = \mathbf{q}_n(\boldsymbol{x}, \underline{\boldsymbol{u}})$, $\boldsymbol{x} = \mathbf{q}_n^{-1}(\boldsymbol{z}, \underline{\boldsymbol{u}})$. System (1) is thus transformed to the $n$-observability form

$$\dot{\boldsymbol{z}} = \mathbf{A}_n \boldsymbol{z} + \mathbf{B}_n \varphi(\boldsymbol{z}, \underline{\boldsymbol{u}}), \qquad \boldsymbol{z}(0) = \mathbf{q}_n(\boldsymbol{x}_0, \underline{\boldsymbol{u}}(0)), \tag{8}$$

$$y = \mathbf{C}_n \boldsymbol{z} \tag{9}$$

with the characteristic nonlinearity $\varphi$ defined by

$$\varphi(\boldsymbol{z}, \underline{\boldsymbol{u}}) = L_{\mathbf{f_u}}^n h\left(\mathbf{q}_n^{-1}(\boldsymbol{z}, \underline{\boldsymbol{u}}), \underline{\boldsymbol{u}}\right) \tag{10}$$

and the matrices $\mathbf{A}_n$, $\mathbf{B}_n$, and $\mathbf{C}_n$ given by

$$\mathbf{A}_r = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ 0 & \mathbf{0}^T \end{bmatrix}, \quad \mathbf{B}_r = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}, \quad \mathbf{C}_r = \begin{bmatrix} 1, & \mathbf{0}^T \end{bmatrix} \tag{11}$$

[1]The superscript $B$ denotes that it is the set of *bad* input points.

where $\mathbf{0} \in \mathbb{R}^{r-1}$ is a vector of zeros and $\mathbf{I}$ is the identity matrix of dimension $r-1$. The $n$-observability form (8)–(11) is therefore a chain of integrators and all the nonlinearities are concentrated on the single term $\varphi$, which describes the right hand side of the dynamic equation for $z_n$, the $(n-1)$-th time derivative of the output $y$.

## 3 Events and the pseudo-observability form

Throughout the rest of the paper it is assumed that system (1) is $n$-observable[2] and the bad input point set $\mathcal{X}_{\underline{\mathcal{U}}}^B$ is not empty. Since injectivity of the $n$-observability map (4) of system (1) is not satisfied for some set of bad input points $\mathcal{X}_{\underline{\mathcal{U}}}^B$, it is not possible to construct an $n$-observability form (8)–(9) using (10). A natural question to ask is whether it is possible to construct something similar whose trajectories resemble those of the output $y(t)$ and its derivatives. The *pseudo-observability form* is proposed as a positive answer to this question.

First restrict the domain of $\mathbf{Q}_n$ to the complement of the bad input point set[3] $\mathcal{X}_{\underline{\mathcal{U}}}^G = \mathcal{X}_{\underline{\mathcal{U}}} \setminus \mathcal{X}_{\underline{\mathcal{U}}}^B$. Then there exists a (left) inverse $\mathbf{Q}_n^{-1} \colon \mathbf{Q}_n(\mathcal{X}_{\underline{\mathcal{U}}}^G) \to \mathcal{X}_{\underline{\mathcal{U}}}^G$ with a restricted domain, which is the image of $\mathcal{X}_{\underline{\mathcal{U}}}^G$ under $\mathbf{Q}_n$. Now construct an *open* neighborhood $\mathcal{N}_\epsilon$ to the set of bad input points $\mathcal{X}_{\underline{\mathcal{U}}}^B$. For example, consider the following $\epsilon$-neighborhood[4] to the bad input point set $\mathcal{X}_{\underline{\mathcal{U}}}^B$

$$\mathcal{N}_\epsilon = \left\{(\boldsymbol{x}, \underline{\boldsymbol{u}}) \in \mathcal{X}_{\underline{\mathcal{U}}} \colon d\left(\mathcal{X}_{\underline{\mathcal{U}}}^B, (\boldsymbol{x}, \underline{\boldsymbol{u}})\right) < \epsilon\right\} \tag{12}$$

where $d(\mathcal{W}, \boldsymbol{w})$ is a distance function from a point $\boldsymbol{w}$ to the set $\mathcal{W}$. Build the complement $\mathcal{N}_\epsilon^{comp} = (\mathbb{R}^n \times \underline{\mathcal{U}}) \setminus \mathcal{N}_\epsilon$, which is *closed* by definition. Consider a *compact* subset $\mathcal{X}_{\underline{\mathcal{U}}}^C$ of $\mathcal{X}_{\underline{\mathcal{U}}}$ and build

$$\mathcal{K} = \mathcal{X}_{\underline{\mathcal{U}}}^C \cap \mathcal{N}_\epsilon^{comp}. \tag{13}$$

The set $\mathcal{K}$ is compact, because $\mathcal{X}_{\underline{\mathcal{U}}}^C$ is compact and $\mathcal{N}_\epsilon^{comp}$ is closed, thus $\mathbf{Q}_n(\mathcal{K})$ is compact. These sets are schematically represented in Figure 1. Furthermore, since $\mathcal{K} \subset \mathcal{X}_{\underline{\mathcal{U}}}$, it does not contain any bad input points. Therefore $\mathbf{Q}_n$ with domain restricted to $\mathcal{K}$ is injective and the inverse $\mathbf{Q}_n^{-1}$ restricted to $\mathbf{Q}_n(\mathcal{K})$ is continuous. Calling this restricted inverse $\mathbf{Q}_{n,\mathcal{K}}^{-1}$, then

$$\mathbf{Q}_{n,\mathcal{K}}^{-1} \colon \mathbf{Q}_n(\mathcal{K}) \to \mathcal{K}, \qquad \mathbf{Q}_{n,\mathcal{K}}^{-1} \in \mathcal{C}^0. \tag{14}$$

If only the set $\mathcal{K}$ is considered, then $\varphi$ in (8) can be uniquely defined, i.e.

$$\varphi_{\mathcal{K}} \colon \mathbf{Q}_n(\mathcal{K}) \to \mathbb{R}, \qquad \varphi_{\mathcal{K}} = L_{\mathbf{f_u}}^n h \circ \mathbf{Q}_{n,\mathcal{K}}^{-1}. \tag{15}$$

[2]If not, then it is assumed that the system can previously be immersed in an $r$-observable system of order $r > n$ as shown in [11] and [13].

[3]$\mathcal{X}_{\underline{\mathcal{U}}}^G$ corresponds to the **good** input points; hence the superscript $G$.

[4]The notation $\mathcal{N}_\epsilon$ is here used to denote any open neighborhood, although here it refers to an $\epsilon$-neighborhood.

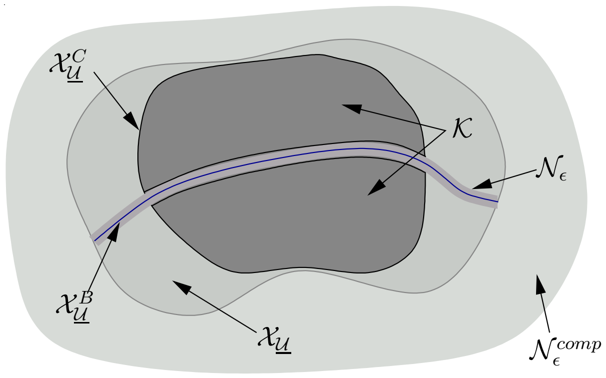


Figure 1: Schematic representation of the sets $\mathcal{X}_{\underline{\mathcal{U}}}$, $\mathcal{X}_{\underline{\mathcal{U}}}^{B}$, $\mathcal{N}_\epsilon$, $\mathcal{N}_\epsilon^{comp}$, $\mathcal{X}_{\underline{\mathcal{U}}}^{C}$, and $\mathcal{K}$.

The domain of $\varphi$ will now be extended to include also points outside the set $\mathbf{Q}_n(\mathcal{K})$ using the following theorem due to Tietze [14].

**Theorem 1 (Tietze)** *Let $\mathcal{X}$ be a metric space, $\mathcal{Y}$ a closed subset of $\mathcal{X}$, and $\mathrm{f}\colon \mathcal{Y} \to [0,1]$ a continuous function. Then $\mathrm{f}$ has a continuous extension $\mathrm{g}\colon \mathcal{X} \to [0,1]$.*

It follows from Tietze's theorem that, since $\mathbf{Q}_n(\mathcal{K})$ is compact (i.e. closed and bounded) and the inverse $\mathbf{Q}_{n,\mathcal{K}}^{-1}$ is continuous, it is possible to "complete" $\mathbf{Q}_{n,\mathcal{K}}^{-1}$ and propose a continuous function $\mathbf{Q}_n^I$ with the following properties:

$$\mathbf{Q}_n^I\colon \mathbb{R}^n \times \underline{\mathcal{U}} \to \mathcal{X}_{\underline{\mathcal{U}}}, \quad \mathbf{Q}_n^I \in \mathcal{C}^0, \quad \mathbf{Q}_n^I\big|_{\mathbf{Q}_n(\mathcal{K})} = \mathbf{Q}_{n,\mathcal{K}}^{-1}. \tag{16}$$

A nonlinear function $\bar{\varphi}$ can now be constructed as

$$\bar{\varphi}\colon \mathbb{R}^n \times \underline{\mathcal{U}} \to \mathbb{R}, \qquad \bar{\varphi} = L_{\mathbf{f}_{\boldsymbol{u}}}^n h \circ \mathbf{Q}_n^I. \tag{17}$$

**Remark 2** Tietze's theorem can also be used to propose $\bar{\varphi}$ without necessarily extending the restricted inverse $\mathbf{Q}_{n,\mathcal{K}}^{-1}$ and then using equation (17). Construct $\varphi_{\mathcal{K}}$ using equation (15) and then extend its domain; the theorem can be used since $\mathbf{Q}_n(\mathcal{K})$ is compact and $\varphi_{\mathcal{K}}$ is continuous. Then $\bar{\varphi}|_{\mathbf{Q}_n(\mathcal{K})} = \varphi_{\mathcal{K}}$.

Assume that $\bar{\varphi}$ can be constructed Lipschitz continuous everywhere and if not, that it can be approximated by such a function. Consider the following system

$$\dot{\boldsymbol{\zeta}} = \mathbf{A}_n\boldsymbol{\zeta} + \mathbf{B}_n\bar{\varphi}(\boldsymbol{\zeta}, \underline{\boldsymbol{u}}), \quad \boldsymbol{\zeta}(0) = \boldsymbol{\zeta}_0, \quad y = \mathbf{C}_n\boldsymbol{\zeta} \tag{18}$$

with $\mathbf{A}_n$, $\mathbf{B}_n$, and $\mathbf{C}_n$ given by (11) and $\bar{\varphi}$ given by (15). If the solution $\boldsymbol{x}(t; \boldsymbol{x}_0, \boldsymbol{u}(t))$ of (1) remains in $\mathcal{K}$ for $t \in [0, T]$, $T > 0$, and furthermore $\boldsymbol{\zeta}_0 = \mathbf{q}_n(\boldsymbol{x}_0, \underline{\boldsymbol{u}}(0))$, then the trajectory $\boldsymbol{\zeta}(t; \boldsymbol{\zeta}_0, \underline{\boldsymbol{u}}(t))$ of (18) corresponds to $\underline{\boldsymbol{y}}(t)$ with

$$\underline{\boldsymbol{y}} = \left[y, \quad \dot{y}, \quad \ldots, \quad \overset{(n-1)}{y}\right]^T \tag{19}$$

during $t \in [0, T]$, but not afterwards. However, it is known that the output and its derivatives are continuous functions of time, so consider the time function

$$\varrho\colon \mathbb{R}_+ \to \mathbb{R}, \quad \varrho \in \mathcal{C}^0, \quad \varrho(t) = L_{\mathbf{f}_{\boldsymbol{u}}}^n h(\boldsymbol{x}(t), \underline{\boldsymbol{u}}(t)), \tag{20}$$

and propose the system

$$\begin{aligned} \dot{\boldsymbol{z}}(t) &= \mathbf{A}_n\boldsymbol{z}(t) + \mathbf{B}_n\varrho(t), \\ \boldsymbol{z}(0) &= \mathbf{q}_n(\boldsymbol{x}_0, \underline{\boldsymbol{u}}(0)), \\ y(t) &= \mathbf{C}_n\boldsymbol{z}(t). \end{aligned} \tag{21}$$

Viewing $\varrho(t)$ as an external signal, generated using system (1) and equation (20), it is clear that the solutions $\boldsymbol{z}(t; \mathbf{q}_n(\boldsymbol{x}_0, \underline{\boldsymbol{u}}(0)), \varrho(t))$ of (21) are equal to the trajectories $\mathbf{q}_n(\boldsymbol{x}(t; \boldsymbol{x}_0, \boldsymbol{u}(t)), \underline{\boldsymbol{u}}(t))$, which correspond to $\underline{\boldsymbol{y}}(t)$. Now define the signal

$$\delta\colon \mathbb{R}_+ \to \mathbb{R}, \qquad \delta(t) = \varrho(t) - \bar{\varphi}(\boldsymbol{z}(t), \underline{\boldsymbol{u}}(t)). \tag{22}$$

It is quite easy to verify that $\delta\colon \mathbb{R}_+ \to \mathbb{R}$ is (absolutely) continuous everywhere and that $\delta(t) = 0$ when $(\boldsymbol{x}(t), \underline{\boldsymbol{u}}(t)) \in \mathcal{K}$.

Using (22), another way of writing (21) is

$$\begin{aligned} \dot{\boldsymbol{z}}(t) &= \mathbf{A}_n\boldsymbol{z}(t) + \mathbf{B}_n\left[\bar{\varphi}(\boldsymbol{z}(t), \underline{\boldsymbol{u}}(t)) + \delta(t)\right] \\ \boldsymbol{z}(0) &= \mathbf{q}_n(\boldsymbol{x}_0, \underline{\boldsymbol{u}}(0)), \\ y(t) &= \mathbf{C}_n\boldsymbol{z}(t). \end{aligned} \tag{23}$$

This form is called the *pseudo-observability form* of system (1). Its trajectories reproduce exactly those of $\underline{\boldsymbol{y}}(t)$. Notice that (23) is similar to the usual $n$-observability form (8)–(11), but the "defects" in the nonlinearity $\bar{\varphi}$ are corrected by an external signal $\delta(t)$ with special properties.

A trajectory that remains near the bad input points $\mathcal{X}_{\underline{\mathcal{U}}}^{B}$ or outside the set $\mathcal{K}$ generates a nonzero signal $\delta(t)$ in the pseudo-observability form (23). It is then desirable to *restrict* trajectories of the system (1), such that they generate a signal $\delta(t)$ which is at least bounded. The motivation for this restriction follows the *persistence of excitation* condition found in adaptive control and observation theory [2, 9], or the condition of a *regularly persistent input* needed in observer design for bilinear and state-affine systems [4, 6, 7]. Trajectories $(\boldsymbol{x}(t), \underline{\boldsymbol{u}}(t))$ are not allowed to remain at bad input points for prolonged periods of time.

**Definition 3** *For a given compact set $\mathcal{K}$ and a nonlinearity $\bar{\varphi}$, a trajectory $\boldsymbol{x}(t; \boldsymbol{x}_0, \boldsymbol{u}(t))$ of system (1) is called $\delta_M$-*admissible *with a finite $\delta_M > 0$ if $\delta(t)$ in the pseudo-observability form is bounded by $\delta_M$, i.e.*

$$|\delta(t)| \le \delta_M \quad \textit{for all} \quad t \ge 0. \tag{24}$$

The bound $\delta_M$ on the signal $\delta(t)$ somehow implies that trajectories must not remain outside $\mathcal{K}$ (even outside $\mathcal{X}_{\underline{\mathcal{U}}}^{C}$) for prolonged periods of time, for then $\delta(t)$ may become too large. The periods outside $\mathcal{K}$ are called *events*. The $i$-th event $\Delta_i^* = (t_i^{\text{in}}, t_i^{\text{out}})$ is the time interval such that

$$(\boldsymbol{x}(t), \underline{\boldsymbol{u}}(t)) \in \mathcal{K}^{comp} = \mathcal{X}_{\underline{\mathcal{U}}} \setminus \mathcal{K} \quad \text{for all} \quad t \in \Delta_i^*. \tag{25}$$

**Remark 3** If only trajectories for which events are finite are considered, i.e. $t_i^{\text{out}} - t_i^{\text{in}}$ is finite, and the event interval contains no finite escape time, then these trajectories are $\delta_M$-admissible for some $\delta_M > 0$. To see this, recall the continuity property of $\delta(t)$ and the fact that $\delta(t_i^{\text{in}}) = \delta(t_i^{\text{out}}) = 0$. It implies that $|\delta(t)|$ has a maximum $\delta_i^M$ during $\Delta_i^*$; then $\delta_M = \max_i \delta_i^M$.

# 4 Continuous high gain event-based observer

As in the continuous observer approach [15], a dynamic part in observability coordinates is considered, while an algebraic part is used to obtain the estimate $\hat{\boldsymbol{x}}(t)$ in original coordinates. The *continuous high gain event-based observer* uses an approximate $n$-observability form using $\bar{\varphi}$ given by (17). It leads to an approximate high gain observer that estimates $\underline{\boldsymbol{y}}(t)$ (see (19)). The algebraic part may be implemented using a generalized inverse of the suspension map $\mathbf{Q}_n^I$.

Consider the system

$$\begin{aligned} \dot{\hat{\boldsymbol{z}}} &= \mathbf{A}_n\hat{\boldsymbol{z}} + \mathbf{B}_n\bar{\varphi}(\hat{\boldsymbol{z}}, \underline{\boldsymbol{u}}) + \tfrac{1}{2}\mathbf{S}_\theta^{-1}\mathbf{C}_n^T \cdot [y - \mathbf{C}_n\hat{\boldsymbol{z}}] , \\ \hat{\boldsymbol{z}}(0) &= \hat{\boldsymbol{z}}_0 \end{aligned} \tag{26}$$

with the matrices $\mathbf{A}_n$, $\mathbf{B}_n$, and $\mathbf{C}_n$ as in (11), $\mathbf{S}_\theta$ the matrix solving the Lyapunov equation [5]

$$\theta\mathbf{S}_\theta + \mathbf{A}_n^T\mathbf{S}_\theta + \mathbf{S}_\theta\mathbf{A}_n = \mathbf{C}_n^T\mathbf{C}_n, \tag{27}$$

and $\bar{\varphi} \in \mathcal{C}^0\colon \mathbb{R}^n \times \underline{\mathcal{U}} \to \mathbb{R}$ constructed as explained in Section 3 (see (17)).

**Theorem 2** *Consider an $n$-observable system (1) and some $\delta_M > 0$. For any open neighborhood $\mathcal{N}_\epsilon$ of the set of bad input points, some compact $\mathcal{K}$ from (13), and a corresponding approximation $\bar{\varphi}$ from* (17), *there exists $\theta > 1$ sufficiently large for system (26), such that the error $\mathbf{e}(t) = \hat{\boldsymbol{z}}(t) - \underline{\boldsymbol{y}}(t)$ converges to a ball of arbitrary radius $\varepsilon > 0$ in finite time for every $\delta_M$-admissible trajectory, i.e. the error trajectories are uniformly ultimately bounded.*

**Proof:** The proof follows closely that of the conventional high gain observer [5]. Assume the state $\boldsymbol{z}(t)$ of the pseudo-observability form (23) resembles $\underline{\boldsymbol{y}}(t)$. Define the error $\boldsymbol{e} = \hat{\boldsymbol{z}} - \boldsymbol{z}$, whose dynamics are

$$\dot{\boldsymbol{e}} = \left(\mathbf{A}_n - \tfrac{1}{2}\mathbf{S}_\theta^{-1}\mathbf{C}_n^T\mathbf{C}_n\right)\boldsymbol{e} + \mathbf{B}_n\widetilde{\varphi}(\boldsymbol{z}, \hat{\boldsymbol{z}}, \underline{\boldsymbol{u}}) + \mathbf{B}_n\delta \tag{28}$$

with $\widetilde{\varphi}(\boldsymbol{z}, \hat{\boldsymbol{z}}, \underline{\boldsymbol{u}}) = \bar{\varphi}(\hat{\boldsymbol{z}}, \underline{\boldsymbol{u}}) - \bar{\varphi}(\boldsymbol{z}, \underline{\boldsymbol{u}})$ and consider the Lyapunov function $V(\boldsymbol{e}) = \boldsymbol{e}^T\mathbf{S}_\theta\boldsymbol{e}$. Use equation (28) and the Lyapunov equation (27) to obtain

$$\dot{V}(\boldsymbol{e}) = -\theta\boldsymbol{e}^T\mathbf{S}_\theta\boldsymbol{e} + 2\boldsymbol{e}^T\mathbf{S}_\theta\mathbf{B}_n\widetilde{\varphi}(\boldsymbol{z}, \hat{\boldsymbol{z}}, \underline{\boldsymbol{u}}) + 2\boldsymbol{e}^T\mathbf{S}_\theta\mathbf{B}_n\delta. \tag{29}$$

Express it in terms of the norm $\|\boldsymbol{e}\|_{\mathbf{S}_\theta}^2 = \boldsymbol{e}^T\mathbf{S}_\theta\boldsymbol{e}$ and use the Schwartz inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{e}\|_{\mathbf{S}_\theta} \leq -\tfrac{1}{2}\theta\|\boldsymbol{e}\|_{\mathbf{S}_\theta} + \|\mathbf{B}_n\widetilde{\varphi}(\boldsymbol{z}, \hat{\boldsymbol{z}}, \underline{\boldsymbol{u}})\|_{\mathbf{S}_\theta} + \|\mathbf{B}_n\delta\|_{\mathbf{S}_\theta}, \tag{30}$$

such that using properties of the norm $\|\cdot\|_{\mathbf{S}_\theta}$ (see [5]) and the Lipschitz property, it becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{e}(t)\|_{\mathbf{S}_\theta} \leq -\gamma\|\boldsymbol{e}(t)\|_{\mathbf{S}_\theta} + \frac{\sqrt{\mathbf{S}_{1n,n}}}{\theta^{n-\frac{1}{2}}}\,|\delta(t)| \tag{31}$$

where $\gamma = \frac{1}{2}\theta - \kappa$ with $\kappa = \frac{k\sqrt{\mathbf{S}_{1n,n}}}{c_1}$ ($k$ is the Lipschitz constant of $\bar{\varphi}$, $\mathbf{S}_1$ is the solution of (27) for $\theta = 1$, and $c_1^2 = \lambda_{\min}[\mathbf{S}_1]$). An explicit expression for an upper bound on $\|\boldsymbol{e}(t)\|_{\mathbf{S}_\theta}$ can be obtained

$$\begin{aligned} \|\boldsymbol{e}(t)\|_{\mathbf{S}_\theta} \leq{} & \exp\left[-\gamma t\right]\|\boldsymbol{e}(0)\|_{\mathbf{S}_\theta} + \\ & + \frac{\sqrt{\mathbf{S}_{1n,n}}}{\theta^{n-\frac{1}{2}}} \underbrace{\int_0^t \exp\left[-\gamma(t-\tau)\right]|\delta(\tau)|\mathrm{d}\tau}_{s(t)} . \end{aligned} \tag{32}$$

From properties of the norm $\|\cdot\|_{\mathbf{S}_\theta}$, it can also be expressed using the Euclidean norm (additionally with $c_2^2 = \lambda_{\max}[\mathbf{S}_1]$)

$$\|\boldsymbol{e}(t)\| \leq \left(\frac{c_2}{c_1}\right)\theta^{n-1}\exp\left[-\gamma t\right]\|\boldsymbol{e}(0)\| + \frac{\sqrt{\mathbf{S}_{1n,n}}}{c_1}s(t). \tag{33}$$

The first term of (33) decays exponentially to zero, although some initial overshoot may occur. The second term is a bounded continuous time function because $|\delta(t)|$ is bounded. Furthermore, it is the first order linear response to the forcing function $|\delta(t)|$. So therefore $s(t) \leq \delta_M/\gamma$ for all $t \geq 0$ and finally

$$\|\boldsymbol{e}(t)\| \leq \left(\frac{c_2}{c_1}\right)\theta^{n-1}\exp\left[-\gamma t\right]\|\boldsymbol{e}(0)\| + \frac{\sqrt{\mathbf{S}_{1n,n}}}{c_1}\cdot\frac{\delta_M}{\gamma} . \tag{34}$$

For every $\|\boldsymbol{e}(0)\|$, there thus exists $T_\varepsilon \geq 0$ such that

$$\|\boldsymbol{e}(t)\| < \varepsilon \quad \text{for all} \quad t \geq T_\varepsilon. \tag{35}$$

The value of $\varepsilon$ can be made arbitrarily small by increasing the value of $\gamma$. To finish the observer design, make $\theta = 2(\gamma + \kappa)$. $\square$

**Remark 4** For a bigger compact $\mathcal{K}$ and a smaller $\epsilon$ in the open neighborhood $\mathcal{N}_\epsilon$, the set of $\delta_M$-admissible trajectories becomes bigger, which seems beneficial. However, usually the Lipschitz constant $k$ of the approximation $\bar{\varphi}$ also grows, and consequently the gain $\theta$ needs to be larger.

**Remark 5** The calculations of $\varphi$ and $\mathbf{Q}_{n,\mathcal{K}}^{-1}$, together with the constructions of $\bar{\varphi}$ and $\mathbf{Q}_n^I$ may become increasingly complex for high order systems, so this approach should be taken with care.

Theorem 2 allows building the dynamic part of an observer for system (1), but an algebraic part is still needed to obtain the estimate $\hat{\boldsymbol{x}}(t)$ of $\boldsymbol{x}(t)$. Since the inverse $\mathbf{Q}_n^{-1}$ is uniformly continuous in $\mathbf{Q}_n(\mathcal{K})$ (because $\mathcal{K}$ is compact), if used as algebraic part then also $\hat{\boldsymbol{x}}(t) - \boldsymbol{x}(t) \to \mathbf{0}$, at least in $\mathcal{K}$. But during the events, nothing can be guaranteed.

An algebraic part could be constructed using $\mathbf{Q}_n^I$ defined in (16)[5], but this could be problematic. A more practical approach could be to use $\mathbf{Q}_{n,\mathcal{K}}^{-1}$ until an event occurs, then maintain a constant value of $\hat{\boldsymbol{x}}(t)$ during the event, and afterwards return to using $\mathbf{Q}_{n,\mathcal{K}}^{-1}$ as follows:

$$\hat{\boldsymbol{x}}(t) = \begin{cases} \mathbf{Q}_{n,\mathcal{K}}^{-1}(\hat{\boldsymbol{z}}(t), \underline{\boldsymbol{u}}(t)) & \text{when} \quad t \notin \Delta_i^* \\ \hat{\boldsymbol{x}}(t_i^{\text{in}}) & \text{when} \quad t \in \Delta_i^*. \end{cases} \tag{36}$$

## 5 An example

Consider the forced nonlinear system of second order

$$\dot{x}_1 = (x_1 - u)x_2, \qquad \dot{x}_2 = -x_1, \qquad y = x_1 \tag{37}$$

with $\mathcal{X} = \mathbb{R}^2$ and $\underline{\mathcal{U}} = \mathbb{R}^2$. The system is $r$-observable with $r = n = 2$ and the suspension map is

$$\mathbf{Q}_2(\boldsymbol{x}, \underline{\boldsymbol{u}}) = \begin{bmatrix} x_1, & x_2(x_1 - u), & u, & \dot{u} \end{bmatrix}^T. \tag{38}$$

Bad input points are given by $\mathcal{X}_{\underline{\mathcal{U}}}^B = \{(\boldsymbol{x}, \underline{\boldsymbol{u}}) \in \mathbb{R}^4 : x_1 = u\}$ with $\boldsymbol{x} = [x_1, x_2]^T$ and $\underline{\boldsymbol{u}} = [u, \dot{u}]^T$. An $\epsilon$-neighborhood is given by $\mathcal{N}_\epsilon = \{(\boldsymbol{x}, \underline{\boldsymbol{u}}) \in \mathbb{R}^4 : |x_1 - u| < \epsilon\}$. Since $L_{\mathbf{f}_{\boldsymbol{u}}}^2 h = [(x_1 - u)x_2 - \dot{u}]\, x_2 - x_1^2 + u x_1$, then

$$\mathbf{Q}_{2,\mathcal{K}}^{-1} = \begin{bmatrix} y, & \frac{\dot{y}}{y-u}, & u, & \dot{u} \end{bmatrix}^T \tag{39}$$

implies that

$$\varphi_{\mathcal{K}}(\underline{\boldsymbol{y}}, \underline{\boldsymbol{u}}) = L_{\mathbf{f}_{\boldsymbol{u}}}^2 h \circ \mathbf{Q}_{2,\mathcal{K}}^{-1} = \frac{\dot{y}(\dot{y} - \dot{u})}{y - u} - y^2 + uy. \tag{40}$$

To construct $\bar{\varphi}$, propose a continuous approximation $\psi_\epsilon(y, u)$ to $1/(y - u)$, e.g.

$$\psi_\epsilon(y, u) = \begin{cases} 1/(y - u) & \text{if} \quad |y - u| \geq \epsilon \\ (y - u)/\epsilon^2 & \text{if} \quad |y - u| < \epsilon\ . \end{cases} \tag{41}$$

Then $\bar{\varphi}$ can be constructed as

$$\bar{\varphi}(\boldsymbol{z}, \underline{\boldsymbol{u}}) = (z_2^2 - \dot{u} z_2) \cdot \psi_\epsilon(z_1, u) - z_1^2 + u z_1. \tag{42}$$

The dynamic part of a continuous high gain event-based observer is implemented as

$$\begin{aligned} \dot{\hat{\boldsymbol{z}}} &= \begin{bmatrix} \hat{z}_2 \\ \bar{\varphi}(\hat{\boldsymbol{z}}, \underline{\boldsymbol{u}}) \end{bmatrix} + \begin{bmatrix} \theta \\ \theta^2/2 \end{bmatrix} \cdot [y - \hat{z}_1]\,, \\ \hat{\boldsymbol{z}}(0) &= \begin{bmatrix} \hat{x}_{1,0}, & (\hat{x}_{1,0} - u(0))\, \hat{x}_{2,0} \end{bmatrix}^T. \end{aligned} \tag{43}$$

Evidently, as the $\epsilon$-neighborhood is made smaller and the continuous approximation $\psi_\epsilon(y, u)$ to $1/(y - u)$ is made more exact, the Lipschitz constant of $\bar{\varphi}$ grows, thereby needing a higher value of $\theta$. On the other hand, a smaller $\epsilon$ makes more trajectories be considered $\delta_M$-admissible for a given $\delta_M$. A clear compromise takes place.

The algebraic part could be implemented calculating $\hat{x}_1 = \hat{z}_1$, $\hat{x}_2 = \hat{z}_2 \cdot \psi_\epsilon(\hat{z}_1 - u)$. However, every time $\hat{z}_1 = u$, then $\psi_\epsilon = 0$ (see (41)) and thus $\hat{x}_2$ is forced to zero. The other approach is to proceed as in (36); consider $\hat{\epsilon} > 0$ and the set $\widehat{\mathcal{N}}_\epsilon = \{(\hat{\boldsymbol{z}}, \underline{\boldsymbol{u}}) \colon |\hat{z}_1 - u| < \hat{\epsilon}\}$. Use $\hat{x}_2 = \hat{z}_2/(\hat{z}_1 - u)$ until the trajectory enters the set $\widehat{\mathcal{N}}_\epsilon$. This value is then retained until the trajectory leaves $\widehat{\mathcal{N}}_\epsilon$ (notice $\widehat{\mathcal{N}}_\epsilon \neq \mathcal{N}_\epsilon$). The result is a discontinuous estimate $\hat{\boldsymbol{x}}(t)$, with small "jumps" every time the trajectory leaves the set $\widehat{\mathcal{N}}_\epsilon$.

Simulation results are shown in Figure 2 with initial condition $\boldsymbol{x}_0 = [1, 1]^T$ and input $u(t) = -\cos t$. The gain is $\theta = 20$ and the nonlinearity $\bar{\varphi}$ is built as in (42) using (41) with $\epsilon = 0.02$. The error $e_2(t) = \hat{z}_2(t) - \dot{y}(t)$ in observability coordinates is shown on the top figure, while the bottom one shows the estimation $\hat{x}_2(t)$ of $x_2(t)$ using $\hat{\epsilon} = 0.14$. Notice $\hat{\epsilon} > \epsilon$, such that some time is allowed for $\hat{z}_2/(\hat{z}_1 - u)$ to begin converging to $x_2$ after an event.

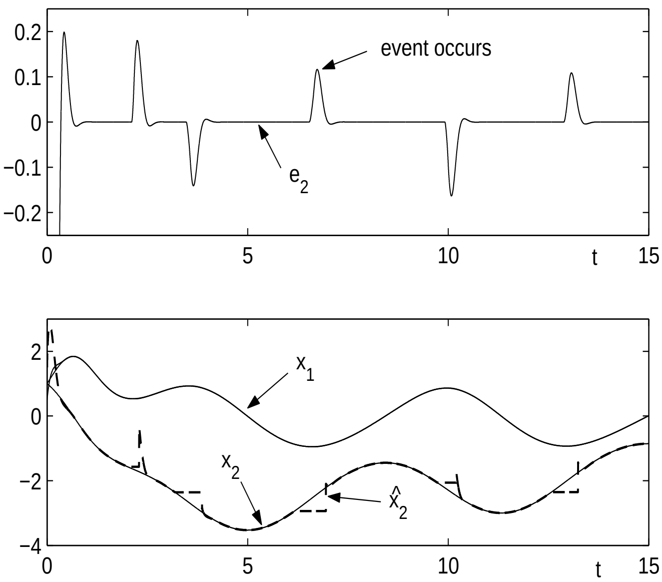


Figure 2: Error $e_2(t)$ and state estimation $\hat{\boldsymbol{x}}(t)$ (- -) of original coordinates $\boldsymbol{x}(t)$ (—) for a continuous high gain event-based observer (41)–(43) for system (37).

## 6 Conclusions

Continuous high gain event-based observers have been proposed for nonlinear systems with bad input points. If trajectories are restricted not to remain in a neighborhood of the subset of bad input points for long periods of time, the dynamic part gives an approximate estimate of the output and a certain number of its time derivatives, with a uniform ultimate bound on error trajectories. The algebraic part is implemented with the inverse of the transformation to observability coordinates when it exists, and with other (heuristic) approaches when not.

[5] Some abuse of notation is being used here, since $\mathbf{Q}_n^I$ also includes the identity map of $\underline{\boldsymbol{u}}$.

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## References

[1] A. Atassi and H. Khalil. Separation results for the stabilization of nonlinear systems using different high-gain observer designs. *Syst. Control Lett.*, 39:183–191, 2000.

[2] G. Besançon. Remarks on nonlinear adaptive observer design. *Syst. Control Lett.*, 41:271–280, 2000.

[3] G. Besançon and H. Hammouri. On uniform observation of nonuniformly observable systems. *Syst. Control Lett.*, 29:9–19, 1996.

[4] F. Celle, J.-P. Gauthier, D. Kazakos, and G. Sallet. Synthesis of nonlinear observers: A harmonic-analysis approach. *Math. Syst. Theory*, 22:227–237, 1989.

[5] J.-P. Gauthier, H. Hammouri, and S. Othman. A simple observer for nonlinear systems. Applications to bioreactors. *IEEE Trans. Automatic Control*, 37:875–880, 1992.

[6] A. Glumineau and V. López-Morales. Transformation to state affine system and observer design. In H. Nijmeijer and T. Fossen, editors, *New Directions in Nonlinear Observer Design*, number 244 in Lecture notes in control and information sciences, pages 59–71. Springer, London, 1999.

[7] H. Hammouri and J. De León Morales. Observer synthesis for state-affine systems. In *Proc. 29th Conference on Decision and Control*, Honolulu, Hawaii, December 1990.

[8] P. Jouan and J.-P. Gauthier. Finite singularities of nonlinear systems. Output stabilization, observability and observers. *J. Dyn. Control Syst.*, 2:255–288, 1996.

[9] S. Sastry and M. Bodson. *Adaptive Control — Stability, Covergence and Robustness*. Prentice-Hall, Englewood Cliffs, 1989.

[10] H. Sussmann. Single input observability of continuous time systems. *Math. Syst. Theory*, 12:371–393, 1979.

[11] A. Vargas. *Observer design for nonlinear systems with reduced observability properties*. Number 999 in Fortschritt-Berichte VDI Reihe 8. VDI-Verlag, Düsseldorf, 2003.

[12] A. Vargas, J. Moreno, and M. Zeitz. Nonlinear observer design strategies via observability forms. In *5th IFAC Symposium Nonlinear Control Systems (NOLCOS)*, pages 539–544, Saint Petersburg, Russia, July 2001.

[13] A. Vargas, J. Moreno, and M. Zeitz. Order extension of nonlinear systems for observer design under reduced observability properties. In *15th IFAC World Congress*, Barcelona, Spain, July 2002.

[14] B. von Querenburg. *Mengentheoretische Topologie*. Springer, Berlin, 3rd edition, 2000.

[15] X. Xia and M. Zeitz. On nonlinear continuous observers. *Int. J. Control*, 66:943–954, 1997.

[16] M. Zeitz. Observability canonical (phase-variable) form for nonlinear time-variable systems. *Int. J. Systems Science*, 15:949–958, 1984.