A PREDICTION-BASED BEHAVIORAL MODEL FOR FINANCIAL MARKETS

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Abstract

A feedback model for financial markets is proposed, in which the control action is an agent's decision based on his beliefs of the price dynamics and his behavior reflecting his attitude, such as risk aversion or risk preference. An adaptation mechanism is described and the condition for equilibrium is formulated as a, typically non-linear, fixed point problem for operators. A data driven stochastic approximation procedure is given for the on-line tuning of the predictor to achieve equilibrium. Simulation results are also presented.

1 Introduction

Financial markets are commonly modeled as open-loop systems where a causal relationship between variables such as the demand for a particular stock and its price is assumed. However, the discrimination between input and output is often arbitrary, as pointed out by Willems [20]. In the example above it is equally true that the price of a stock influences the demand. This mutual causal dependence gives rise to a feedback system.

The purpose of this paper is to study a particular example of a feedback system modeling the behavior of agents of financial markets. The word *agent* is used in a wide sense: it may refer to a group of economic players, such as an industrial sector or to dominant investors such as certain analysts and depositaries of economic power. In any case, the effect of the agent on the market is assumed to be macroeconomically significant. The dynamic relationship between demand as input and price as output will be fixed, this is the plant. The agent predicts the observed price process, and using these predictions will buy or sell shares according to his/her strategy or behavior which reflects his/her risk aversion, conservatism, etc. A variety of behaviors of economic players is described in Shefrin [18] and Kostolany [12]. The agent's action will then show up at the input node of the plant together with noise and thus we get a closed loop system.

A key factor in the above model is the agent's belief of the price dynamics, and his/her predictive capability. For any fixed predictor of the price process, denoted by M, we get a closed loop dynamics and a price dynamics depending on the predictor M, for which the optimal or suboptimal predictor will be typically different from M. It is then reasonable to remodel the price process and use a new, better predictor. This iterative procedure will be described and analyzed for linear systems in terms of transfer functions. An on-line, data-driven procedure will also be presented which is formally applicable to non-linear systems as well.

The paper is believed to contribute to the fruitful interaction of economics and control theory, as put forward in Hansen and Sargent [9].

2 A prediction-based behavior model

The price of a given stock at time $t \in \mathbb{Z}_+$ is denoted by p_t. If we consider a portfolio with n stocks then p_t is a vector in $\mathbb{R}^n$, its i^{th} component expressing the price of the i^{th} stock. Naturally, all components of the price process (p_t) should be non-negative. However, if we think of the price of a stock as a measure of the profitability of the company that issues it, then a negative price could mean that the company is unprofitable. Therefore, we assume that the price process p of a stock can take any values in $\mathbb{R}$.

The prices are given by the market. In general, the stock market is modeled as a dynamical system that generates prices. In this paper the market is viewed as a black box, denoted by P, relating the input process u to the output process p:

$$Pu = p\ . \tag{1}$$

It is assumed that stock prices depend on the past and present values of the demand process and on the current stochastic disturbances entering at the input node of the plant P.

The controller relates the price and the demand processes as

$$d = -Cp \tag{2}$$

where C is a dynamical system such that the demand process d depends only on the past values of the price process. The components of the demand sequence (d_t) can also take any values in $\mathbb{R}$: this time a negative value means that the agent would like to sell the stock concerned. We assume that stocks are infinitely divisible: any amount can be purchased or sold.

The interconnection of the two systems (see Figure 1) is given by the equation

$$d_t + e_t = u_t \tag{3}$$

where the stochastic disturbance (e_t) is a stationary process.

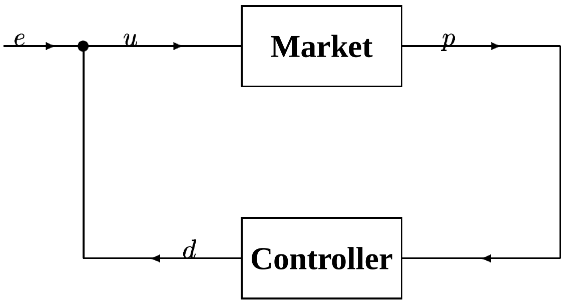

Figure 1: The closed-loop system.

Now assume that a new agent enters the stock market. He observes a stock price process p which he assumes to be stationary. Then based on his beliefs and on other side information, he constructs a price predictor M:

$$\hat{p} = Mp \tag{4}$$

where M is assumed to be a *strictly non-anticipating* predictor, meaning that $\hat{p}_t$ depends only on the values p_s for $s < t$. The letter M indicates that the agent's prediction is based on some model M of the price process.

The agent uses this predicted price to determine his own demand. We allow the possibility that he/she behaves less than fully rationally. Proponents of behavioral finance (as the set of theories based on this assumption is usually referred to) argue that psychological phenomena prevent decision makers from acting in a rational manner (see for example Greenfinch [7] and Shefrin [18]). Critics of this theory (see the works of Lucas [15] and Simon [19]) claim that the behavior of the agents is always rational from a particular perspective. In any case the strategy of the agent can be formalized as

$$B\begin{pmatrix} p \\ \hat{p} \end{pmatrix} = d \tag{5}$$

where B is assumed to be a strictly non-anticipating operator. The demand at a given time depends only on the past values of the price process. In this paper we assume that the behavior of the agent is fixed, i.e. the operator B is considered to be given.

Example 2.1 Suppose an agent at time t is trying to figure out how much of a given stock he is willing to buy. Taking into consideration all the relevant information available, he makes an estimate of the future price, denoted by $\hat{p}_{t+1}$. A reasonable agent would buy more of the stock whenever his estimate $\hat{p}_{t+1}$ is greater than the current price p_t, and would buy less if it is the other way round. Thus a simple rational behavior could be described for example by the equation

$$d_t - \alpha d_{t-1} = \begin{cases} 0 & \text{if } |\hat{p}_t - p_{t-1}| \le \delta \\ B\ \mathrm{sgn}(\hat{p}_t - p_{t-1}) & \text{if } |\hat{p}_t - p_{t-1}| > \delta \end{cases}$$

where $\delta \ge 0$ is a threshold value corresponding to transaction costs, B is the number of stocks the agent wants to purchase and $0 < \alpha < 1$, $|\alpha| \approx 1$ is a parameter expressing the faith of the agent in his past decisions.

Example 2.2 A cognitive bias that frequently occurs is the phenomenon of anchoring, also known as *conservatism*: people have in memory some reference points (anchors), for example a previous stock price or price trend. They cling excessively to prior beliefs when exposed to new evidence, they reject new facts that are contrary to their preconcieved ideas. This may be modeled as $\hat{p}_{t+1}$ being a function of $p_{t-r},\ p_{t-r-1}, \ldots$ only (i.e. the agent does not take into consideration the last r stock prices at all). For a more detailed exposition on the concept of conservatism see Edwards [4].

Example 2.3 Another psychological phenomenon extensively studied by behaviorists is the so-called *loss aversion*. Kahnemann and Tversky [10] find that even simple risk aversion can be biased: empirical evidence shows that a loss has about two and a half times the impact of a gain of the same magnitude. This behavior can be formalized by the equation

$$d_t - \alpha d_{t-1} = B(\hat{p}_t - p_{t-1})^+ - 0.4B(\hat{p}_t - p_{t-1})^-$$

where the notations of the first example are used.

Combining equations (1) - (5), we get the following diagram.

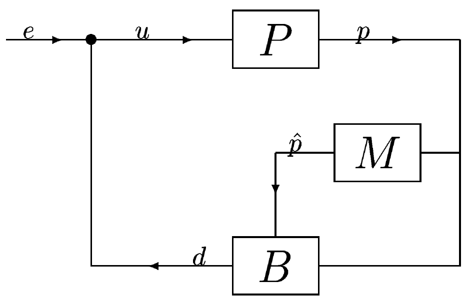

Figure 2: A prediction-based behavior model.

Fixing M, the controller $C = C(M)$ can be calculated from equations (2) and (5):

$$Cp = -B\begin{pmatrix} p \\ Mp \end{pmatrix}.$$

Assuming that the closed loop system is well-defined, a price process $p = p(M)$ is generated. It will be a stationary process with a spectrum depending on C, and therefore its least squares predictor M^* will also depend on C, say $M^* = M^*(C)$. If the market is in equilibrium then $M^*(C)$ will coincide with the initial predictor of the agent:

$$M^*(C(M)) = M. \tag{6}$$

Typically, this will not be the case. Instead, the agent notes that the price dynamics does not agree with what he assumed to be. He updates his predictor: he simply puts $M_1 := M^*(C(M))$ and uses this new predictor when determining his demand.

As long as $M^*(C(M_i)) \neq M_i$, the agent repeatedly updates his predictor. The following questions arise:

- Is there a predictor M_∞ for which the market is in equilibrium, i.e. M_∞ solves (6)?
- Let $f(M) = M^*(C(M))$. Does the iterative procedure

  $$M_{i+1} = f(M_i) \tag{7}$$

  converge?
- Let M be estimated on-line from observed values of p. Does the resulting stochastic approximation procedure converge?

To deal with these questions, we leave our general setup and turn to a mathematically more tractable class of models, namely linear models.

3 Linear Models

To have an idea of the mechanism of prediction-based behavioral models, from now on we assume that the dynamical systems $P,\ M,\ B$ defined above are all linear. In particular P is non-anticipating with an invertible constant term P_0, M is strictly non-anticipating (having no direct term) and

$$B\begin{pmatrix} p \\ \hat{p} \end{pmatrix} = B_1 p + B_2 \hat{p} \tag{8}$$

where B_1 is a strictly non-anticipating and B_2 is a non-anticipating linear operator. Thus we get

$$C = C(M) = -(B_1 + B_2 M). \tag{9}$$

The resulting closed loop system is well defined if

$$||PC||_{H_\infty} = ||P(B_1 + B_2 M)||_{H_\infty} < 1 \ .$$

Letting H be the transfer function from e to p we get

$$H = H(C) = (I + PC)^{-1} P \ .$$

Write $p = HP_0^{-1}(P_0 e)$. The optimal one-step ahead predictor of p is known to be given by

$$\hat{p} = (I - P_0 H^{-1}) p \ ,$$

assuming that H is inverse stable, i.e. P is minimum phase (see Caines [2] and Hannan and Deistler [8]). Thus we arrive at

$$M^* = I - P_0 H^{-1} \ . \tag{10}$$

Substituting C from (9) yields

$$M^* = f(M) = I - P_0 P^{-1} + P_0 B_1 + P_0 B_2 M \ . \tag{11}$$

Note that if M is non-anticipating then the resulting operator $f(M)$ is also non-anticipating. It is readily seen that equation (11) has a unique solution if $||P_0 B_2||_{H_\infty} < 1$ given by

$$M_\infty = (I - P_0 B_2)^{-1} (I - P_0 P^{-1} + P_0 B_1) \ . \tag{12}$$

In the general case P may be non-minimum phase since there is a delay in the market's response to change in demands. Then we first have to perform spectral factorization of H. Write

$$HP_0^{-1} (\bar{P}_0^{-1})^T \bar{H}^T = L \bar{L}^T \ ,$$

where $\bar{H}$ is the conjugate of H and L is stable and minimum-phase. Then the least squares predictor is obtained by

$$\hat{p} = (I - L^{-1}) p \ ,$$

and thus $f(M)$ is defined by the following sequence of equations:

$$\begin{aligned} C &= -(B_1 + B_2 M) \\ H &= (I + PC)^{-1} P \\ L\bar{L}^T &= HP_0^{-1} (\bar{P}_0^{-1})^T \bar{H}^T \\ f(M) &= I - L^{-1} \ . \end{aligned}$$

Conditions under which the above mapping f is a contraction are yet to be developed.

Now consider a model with two agents who are assumed to be parallelly connected (see Figure 3).

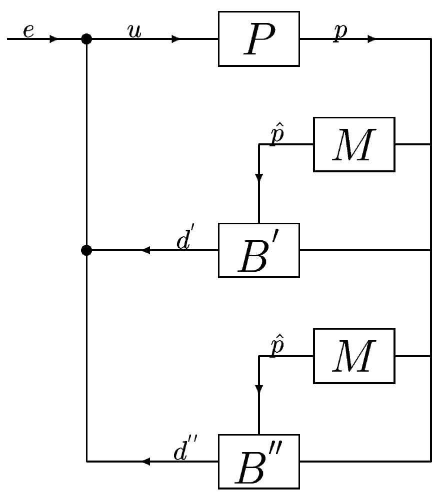

Figure 3: A model with two agents.

For simplicity, we assume that P is minimum phase. The aggregate demand is taken to be

$$d = d^{'} + d^{''} ,$$

where $d^{'}$ and $d^{''}$ denote the demands of the first and second participants, respectively.

Using linearity it is clear that the study of this system can be reduced to the study of a single-agent model with a mixed behavior $B^{'} + B^{''}$. Proceeding the same way as above, we get

$$f(M) = I - P_0 P^{-1} + P_0(B_1^{'} + B_1^{''}) + P_0(B_2^{'} + B_2^{''})M .$$

The condition for stability is now that $||P_0(B_2^{'} + B_2^{''})||_{H_\infty} < 1$. Thus the stability does not depend directly on the contribution of the individual players, but rather on their cumulative contribution. Notice that the optimal predictor contains factors of *both* behaviors $B^{'}$ and $B^{''}$; therefore, the above formula cannot be directly used by the individual agents (since they have knowledge only of their own behavior).

4 Data-driven procedures

The updating of M is easy if P is rational and minimum phase: all we need is to identify P and use equation (11). The situation is completely different in real life financial markets where P is generally non-rational and non-minimum phase. It is then more reasonable to approximate L directly from data. Write

$$p = L\nu$$

where ν is the innovation process of p for a given fixed M. In order to approximate L on the basis of observed values of p, we choose the best approximation from some parametric family $\mathcal{L} := \{ L(\theta) : \theta \in D \subseteq \mathbb{R}^k \}$. If we have little prior information, we may try to fit an $AR(k)$ model to our data. In this case

$$\mathcal{L} = \{ A \mid \mathrm{deg} A \leq k,\ A_0 = I,\ A \text{ stable} \}$$

where A is a polynomial of the shift operator and A_0 is its leading coefficient. The best kth-order estimator of the system is defined by the method of least squares: it is the one that minimizes $E|Ap|^2$ subject to $A \in \mathcal{L}_k$. The coefficients of the optimal solution can be estimated by solving the minimization problem

$$\min_{A \in \mathcal{L}_k} \sum_{n=1}^{N} (Ap)^2 ,$$

which is quadratic in the coefficients of A and thus can easily be computed. Denoting the solution by A_k^*, the predicted price process is defined by $\hat{p} = (I - A_k^*)p$.

Now let M_0 be an arbitrary fixed predictor and let $L(\eta)$ be the best $AR(k)$ approximation to the price dynamics. Let $M = I - L^{-1}(\eta)$, and let the new price process be $p(\eta)$. The best $AR(k)$ approximation for $p(\eta)$ will be denoted by $L(\theta^*(\eta))$. The mapping $M \mapsto f(M)$, see (10), now reduces to $\eta \mapsto \theta^*(\eta)$.

For fixed η, the prediction error estimator of $\theta^*(\eta)$ would be, in its simplest form,

$$\theta_{n+1}(\eta) = \theta_n(\eta) - \frac{c}{n} \nu_{\theta n}(\eta) \nu_n(\eta) \qquad (13)$$

where $\nu_n,\ \nu_{\theta n}$ are on-line estimates of

$$\bar{\nu}(\theta, \eta) := L^{-1}(\theta) p(\eta)$$

for some tentative value of θ and $\bar{\nu}_\theta(\theta, \eta)$ is the gradient of the latter and $c > 0$ is some step size. The above recursion (13) is a stochastic approximation procedure parametrized by η. Its associated ODE (see Benveniste et al. [1], Gerencsér [5], Ljung and Söderström [14]) is

$$\dot{\theta}_t(\eta) = -\frac{c}{t} W_\theta(\theta_t(\eta), \eta) \qquad (14)$$

where

$$W(\theta, \eta) = \frac{1}{2} E|\nu(\theta, \eta)|^2 .$$

It is well known that $W_\theta(\theta^*(\eta), \eta) = 0$ and $W_{\theta\theta}(\theta^*(\eta), \eta)$ is positive semidefinite. Assuming that in fact it is positive definite, the associated ODE (14) is asymptotically stable at $\theta = \theta^*(\eta)$.

In the data-driven procedure we let η to be replaced by θ_n, i.e. the prediction of p is performed using the latest model for its dynamics. Thus we finally get

$$\theta_{n+1} = \theta_n - \frac{c}{n} \nu_{\theta n} \nu_n \qquad (15)$$

where ν_n, $\nu_{\theta n}$ are on-line estimates of $\bar{\nu}(\theta_n, \theta_n)$ and $\bar{\nu}_\theta(\theta_n, \theta_n)$, respectively.

Assume that there exists $\theta^{**} \in D$ such that $\theta^*(\theta^{**}) = \theta^{**}$, i.e. θ^{**} is a fixed point of the mapping $\eta \mapsto \theta^*(\eta)$. Then the associated ODE for (15) is given by

$$\dot{\theta}_s = -c\, W_\theta(\theta_s, \theta_s) .$$

Note that stability of θ^{**} does not follow from the assumed asymptotic stability of the frozen-parameter system (14). Good conditions for the asymptotic stability of θ^{**} are still to be found.

5 Simulation results

The data-driven procedure mentioned above was simulated for various behaviors in a MATLAB environment. Figure 4 depicts the results obtained for the phenomenon of loss aversion. The market dynamics was taken to be $p_t = a^* p_{t-1} + b^* p_{t-2} + \alpha(d_t - d_{t-1}) + e_t$ and we approximated the dynamics of p_t by an $AR(1)$ model. A two-scale procedure was applied: the parameter θ was frozen for 10 steps before each updating.

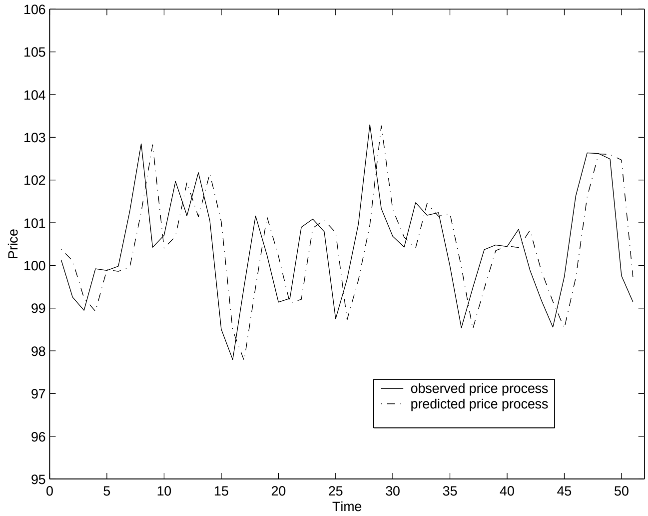

Figure 4: Observed vs. predicted prices

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