A NOTE ON GRAMIAN-BASED INTERACTION MEASURES

Wolfgang Birk†, Alexander Medvedev‡

† Vehicle Control, Volvo Cars Corporation, SE-405 31, Gothenburg, Sweden, eMail: wbirk@volvocars.com
‡ Information Technology, Uppsala University, Box 337, SE-751 05, Uppsala, Sweden, eMail: alexander.medvedev@it.uu.se

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Abstract

This note deals with the choice of measurement/actuator pairs for decentralized control, where the controller remains unspecified. The theoretical background of gramian based interaction measures is clarified and a geometrical interpretation is given. Moreover, a generalization of the Hankel interaction index array is proposed and it is shown that the introduction of weighted gramians makes the criteria more flexible compared to the augmenting with additional filter dynamics.

1 Introduction

Prediction of interaction present in a control system from an open-loop perspective has been dealt with for a long time. Both steady state and dynamic measures for interaction have been suggested.

The first proposed interaction measures were the *Rijnsdorp interaction measure* [15] and the *relative gain array* (RGA) [3] with its extention to a dynamic measure [10]. For the steady-state gain case, these measures are related to each other via a non-linear map [7].

In order for the RGA to be applicable, a decentralized controller has to be used and the steady state control error has to be zero which assumptions are generally not fulfilled. This led *e.g.* to the introduction of the block relative gains (BRG) [13] and partial relative gains (PRG) [8]. Thereby, it became possible to get indications for interactions in multivariable control systems with control structures different from decentralized control.

Newly developed measures are a measure based on Hankel singular values [5] and the *Hankel interaction index array* [19], which makes use of the Hankel norm of the scalar subsystems. Both measures are gramian based and analyze scalar sub-systems of the multivariable system in order to get insight into the system structure and thus draw a-priori conclusions on interaction in the closed loop system.

Gramian based measures judge the overall dynamic behavior of a system and therefore indications of these measures are realistic even for transient behavior. Since available publications do not clarify the geometrical aspects of the gramian based measures, the paper reviews these aspects to get a better understanding of the measures when no specific controller structure is assumed.

The paper is structured as follows. Starting out from a general description of linear multivariable systems, the notion of gramians is introduced and utilized to derive subspaces of the state space. It is illustrated how intersections in subspaces are related to interaction in multivariable systems. Thereafter, the connection between gramians and the Hankel operator is discussed and based on the results generalizations of the Hankel interaction index array are derived. Then, weighted gramians are introduced into the interaction measures and their relation to filtering of process dynamics is analyzed. Finally, an industrial example is given followed by conclusions and outlook.

2 Preliminaries

Interaction concerns input/output behavior of multivariable systems, namely the relation between different channels. A channel is defined as the path from an input to an output. The output of a linear multivariable system is given as the convolution of an input signal with its impulse response function matrix

$$y(t) = \int_{t_1}^{t_2} g(t-\tau)u(\tau)d\tau \tag{1}$$

where $y(t) \in \mathcal{L}_2^p[t_1, t_2]$ and $u(t) \in \mathcal{L}_2^m[t_1, t_2]$. The associated function spaces are denoted $\mathcal{Y}$ and $\mathcal{U}$, respectively. The impulse response function has to be bounded and thus $g(t) \in \mathcal{L}_1^{p\times m}[t_1, t_2]$. The restriction to the time interval $[t_1, t_2]$ makes it possible to consider unstable multivariable systems as long as $g(t)$ is bounded in the interval. For stable systems the interval can be set to $[-\infty, \infty]$.

The frequency domain representation of (1) can be obtained by applying the Laplace transform and is given by

$$Y(s) = G(s)U(s) \tag{2}$$

where s denotes the differential operator and $Y(s)$, $U(s)$ denote the Laplace transform of $y(t)$ and $u(t)$, respectively. $G(s)$ is referred to as the transfer function matrix, which is the Laplace transform of the impulse response function $g(t)$.

Now, the standard inner product of two vectors $\phi(t)$ and $\xi(t)$ in the function space $\mathcal{L}_2^n[t_1, t_2]$ is defined as

$$\langle \phi(t), \xi(t) \rangle = \int_{t_1}^{t_2} \phi^T(t)\xi(t)dt \tag{3}$$

For this inner product a Gram matrix, also referred to as gramian, can be derived by computing the inner product of all combinations of the vector elements $\phi_q(t)$ and $\xi_r(t)$. The elements in the gramian are given by

$$[\Gamma]_{qr} = \langle \phi_q(t), \xi_r(t) \rangle = \int_{t_1}^{t_2} \phi_q(t)\xi_r(t)dt$$

Consequently, the gramian can be written as

$$\Gamma = \int_{t_1}^{t_2} \phi(t)\xi^T(t)dt \tag{4}$$

From the properties of the scalar product, it follows that the gramian is symmetric and positive semidefinite. Thus, the eigenvalues of the gramian are all real and non-negative.

The gramian is the matrix of the inner product relative to the basis in which the vectors ϕ and ξ are expressed, [6]. When the basis is changed using a non-singular transformation T,

$$\phi'(t) = T\phi(t)\,, \quad \xi'(t) = T\xi(t)$$

the gramian for the transformed vectors is then found as

$$\Gamma' = \int_{t_1}^{t_2} \phi'(t){\xi'}^T(t)dt = T\Gamma T^T \tag{5}$$

Thence, the gramian is not invariant to basis changes. Following [6], two vectors are linearly independent if the gramian is positive definite or, in other words, has only strictly positive eigenvalues. Gramians have a close relation to the associated function spaces which is stated in the following lemma.

Lemma 1. *A vector $\xi(t)$ lies in the range space of the impulse response function $g(t)$ if and only if it lies in the range space of*

$$\Gamma = \int_{t_1}^{t_2} g(t)g^T(t)dt$$

Proof. A proof is given in [4]. □

Thus, using the eigenvector/eigenvalue decomposition of the gramian an orthonormal basis for the range of $g(t)$ can be derived, namely the eigenvectors associated with the non-zero eigenvalues constitute a basis. An important generalization to the decomposition of gramians is the operator decomposition, namely the derivation of singular values and Schmidt pairs of a compact operator and its adjoint [20].

Often $g(t)$ is expressed in terms of a state space realization,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) && \text{(6a)} \\ y(t) &= Cx(t) + Du(t) && \text{(6b)} \end{aligned}$$

where $A \in \mathrm{R}^{n\times n}$, $B \in \mathrm{R}^{n\times m}$, $C \in \mathrm{R}^{p\times n}$ and $D \in \mathrm{R}^{p\times m}$. All state space realizations of a system can be obtained by the similarity transform

$$\begin{aligned} z(t) &= Tx(t) && \text{(7a)} \\ \dot{z}(t) &= TAT^{-1}z(t) + TBu(t) && \text{(7b)} \\ y(t) &= CT^{-1}z(t) + Du(t) && \text{(7c)} \end{aligned}$$

The solution of (6) is given by

$$y(t) = Ce^{A(t-t_1)}x(t_1) + \int_{t_1}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \tag{8}$$

where $x(t_1)$ is the initial condition. Since interaction concerns the input/output relationship, the initial condition $x(t_1) = 0$ can be assumed without loss of generality. Then, (8) equals (1) for all impulse response functions with $g(t) = 0,\ t < 0$. In the frequency domain, $G(s) = C(sI - A)^{-1}B + D$.

Each of the pm scalar subsystems, which describes the behavior from input j to output i, is then defined by either the triple (A, B_j, C_i), $G_{ij}(s)$ or $g_{ij}(t)$. There, B_j is the j^{th} column vector in B and C_i is the i^{th} row vector in C.

3 Subspaces of $\mathcal{X}$

Subspaces are a geometrical construct which is used to divide up a vector space according to properties of its elements like controllability and observability. Subsystems of a multivariable system are also associated with subspaces and thus, the same framework can be used to analyze the relationship between subsystems in terms of system properties. Thence, subspaces can be used in interaction analysis of multivariable systems.

In order to characterize subspaces that are associated with subsystems of (6), the indices i and j are used, *e.g.* $\mathcal{X}_{12}$ would denote the state space of the scalar subsystem from input 2 to output 1.

3.1 Controllable subspace of $\mathcal{X}$

The state vector $x(t)$ is governed by the differential equation (6a). Thus the state impulse response matrix can be given as

$$X(t) = [X_1(t)\ X_2(t)\ \ldots\ X_m(t)] \tag{9}$$

where $X_j(t) = e^{At}B_j$ is the response to a Dirac delta impulse in $u_j(t)$ with $x(t_1) = 0$. Accordingly, the state impulse response matrix in the frequency domain is obtained as

$$X(s) = (sI - A)^{-1}B$$

According to [14], the controllable subspace $\mathcal{X}_c$ is the subspace of $\mathcal{X}$ of least dimension containing the range of the state impulse response function $X(t)$. The controllable subspace that is associated with input j is denoted $\mathcal{X}_{cj}$.

Applying the definition of the gramian (4) to $X(t)$ the controllability gramian is obtained as

$$\Gamma_c = \int_{t_1}^{t_2} e^{At}BB^Te^{A^Tt}dt \tag{10}$$

For stable systems, the controllability gramians in time domain and frequency domain are connected via Parseval's equality yielding for individual inputs

$$\begin{aligned} \Gamma_{cj} &= \int_{t_1}^{t_2} e^{At}B_jB_j^Te^{A^Tt}dt \\ &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}(sI - A)^{-1}B_jB_j^T(\bar{s}I - A^T)^{-1}ds \end{aligned}$$

where $\bar{s}$ denotes the complex conjugate.

From Lemma 1 it follows that a basis for $\mathcal{X}_{cj}$ can be found via the eigenvector/eigenvalue decomposition of the controllability gramian Γ_{cj}, where the eigenvectors are denoted a_j. Only the eigenvectors associated with the non-zero eigenvalues are considered and they are sorted according to the size of the eigenvalues.

$$\mathcal{X}_{cj} = \text{span}\,\{a_{jr} | r = 1 \ldots dim(\mathcal{X}_{cj})\}$$

According to (5), the controllability gramians of two state space realization are connected via the transform matrix T as

$$\Gamma'_{cj} = T\Gamma_{cj}T^T$$

where Γ'_{cj} is the controllability gramian of the transformed system.

3.2 Observable subspace of $\mathcal{X}$

The characterization of the observable subspace can be done in a similar manner as for the controllable subspace. The output impulse response matrix to an initial condition $x_i(t_1) = 1$ with $u(t) = 0$ can be found as

$$Y(t) = [Y_1(t)\ Y_2(t)\ \dots\ Y_n(t)] \tag{11}$$

where $Y_i(t) = C_i e^{At}$. Clearly, the Laplace transform of $Y(t)$ is given by $Y(s) = C(sI - A)^{-1}$.

In the case of observability, the row space of $Y(t)$ or $Y(s)$ has to be analyzed. Again according to [14], the observable subspace $\mathcal{X}_o$ is the subspace of least dimension containing the row space of the output impulse response matrix $Y(t)$. The observable subspace that is associated with an individual output is denoted $\mathcal{X}_{oi}$.

Analogously, the observability gramians are given by

$$\begin{aligned}
\Gamma_o &= \int_{t_1}^{t_2} e^{A^T t} C^T C e^{At} dt \\
&= \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} (\bar{s}I - A^T)^{-1} C^T C (sI - A)^{-1} ds \\
\Gamma_{oi} &= \int_{t_1}^{t_2} e^{A^T t} C_i^T C_i e^{At} dt \\
&= \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} (\bar{s}I - A^T)^{-1} C_i^T C_i (sI - A)^{-1} ds
\end{aligned}$$

Applying the eigenvector/eigenvalue decomposition the eigenvectors b_i of Γ_{oi} are found and a basis for $\mathcal{X}_{oi}$ is constituted by the b_i associated with the r non-zero eigenvalues

$$\mathcal{X}_{oi} = \text{span}\,\{b_{ir} | r = 1 \dots dim(\mathcal{X}_{oi})\}$$

Again, after the similarity transform, Γ'_{oi} of the transformed system is

$$\Gamma'_{oi} = T^{-T} \Gamma_{oi} T^{-1}$$

3.3 Controllable and observable subspace $\mathcal{X}_{co}$

The controllable and observable subspace is the intersection of the observable subspace and the controllable subspace, namely $\mathcal{X}_{co} = \mathcal{X}_c \cap \mathcal{X}_o$.

As already known from the minimal realization theory [12] the subspace $\mathcal{X}_{co}$ is associated with a realization of (6) that has the same input-output behavior with least order. Using *e.g.* the Kalman canonical decomposition [21] a realization with $\mathcal{X}_{co}$ as state space can be extracted.

The subspace $\mathcal{X}_{coij}$ is then related to a minimal realization of the scalar subsystem between input j and output i. A characterization of the subspace can be obtained from the associated controllable and observable subspaces.

The subspaces $\mathcal{X}_{cj}$ and $\mathcal{X}_{oi}$ are spanned by $\{a_{j1}, \dots, a_{jq}\}$ and $\{b_{i1}, \dots, b_{ir}\}$, respectively. Vectors that lie in the intersection of the two vector spaces can be parameterized in both bases simultaneously.

$$x(t_p) = \sum_{\alpha=1}^{q} a_{j\alpha} \lambda_\alpha = \sum_{\beta=1}^{r} a_{j\beta} \mu_\beta, \quad t_1 \le t_p \le t_2$$

The condition can be rewritten to

$$\underbrace{[a_{j1}\ \dots\ a_{jq}\ b_{i1}\ \dots\ b_{ir}]}_{\Upsilon} \begin{bmatrix} \lambda \\ -\mu \end{bmatrix} = 0 \tag{12}$$

According to [16], a basis for the intersection can be found by deriving a set of linearly independent vectors that lie in the null space of Υ. Thus, (12) needs to have non-trivial solutions.

If $\mathcal{X}_{coij} \neq 0$ then the output i is affected by the input j. Furthermore, if there is an input r with $r \neq j$ for which $\mathcal{X}_{coir} \neq 0$, then there are two channels that affect each other in the multivariable system and thus, interaction is present. Consequently, the intersection of controllable and observable subspaces contain information on interaction.

4 Gramians and the Hankel operator

Generally, the input/output behavior of a linear causal system without direct term can be described by a Hankel operator. It is defined by

$$\begin{aligned}
\Psi_g &: \mathcal{L}_2(-\infty, 0] \mapsto \mathcal{L}_2[0, \infty) \\
\Psi_g u(t) &= \begin{cases} \int_{-\infty}^{0} C e^{A(t-\tau)} B u(\tau) d\tau & t \ge 0 \\ 0 & t < 0 \end{cases}
\end{aligned} \tag{13}$$

Applying the bilateral Laplace transform to (13) a frequency domain representation of the Hankel operator can be derived. The Hankel operator then becomes a strictly proper transfer function matrix.

When a general transfer function matrix G is given, the noncausal and direct part of the transfer function matrix need to be removed. Introducing the orthogonal projection $P : \mathcal{L}_2(-\infty, \infty] \mapsto \mathcal{L}_2[0, \infty)$ and applying it to G the Hankel operator can be stated as

$$\begin{aligned}
\Psi_G &: \mathcal{H}_2^\perp \mapsto \mathcal{H}_2 \\
\Psi_G U &= P(GU), \text{ with } U \in \mathcal{H}_2^\perp
\end{aligned}$$

There, the spaces $\mathcal{H}_2^\perp$ and $\mathcal{H}_2$ are the counterparts of $\mathcal{L}_2(-\infty, 0]$ and $\mathcal{L}_2[0, \infty)$ in the frequency domain, respectively.

Furthermore, the adjoint of the Hankel operator can be defined as

$$\begin{aligned}
\Psi_g^* &: \mathcal{L}_2[0, \infty) \mapsto \mathcal{L}_2(-\infty, 0] \\
\Psi_g^* y(t) &= \begin{cases} \int_{0}^{\infty} B^T e^{A^T(t-\tau)} C^T y(\tau) d\tau & \tau \le 0 \\ 0 & \tau > 0 \end{cases}
\end{aligned} \tag{14}$$

Singular values and Schmidt pairs of the Hankel operator can now be derived from $\Psi_g^*\Psi_g$. The Schmidt pairs are a generalization of eigenvectors and $\Psi_g^*\Psi_g$ can be interpreted as a generalization of a gramian. Thus, two operators can be defined in terms of the Hankel operator and its adjoint

$$\begin{aligned}\Gamma_\Psi &: \mathcal{L}_2(-\infty,0] \mapsto \mathcal{L}_2(-\infty,0] \\ \Gamma_\Psi u &= \Psi_g^*\Psi_g u \qquad (15a)\\ \Gamma_\Psi^* &: \mathcal{L}_2[0,\infty) \mapsto \mathcal{L}_2[0,\infty) \\ \Gamma_\Psi^* u &= \Psi_g\Psi_g^* u \qquad (15b)\end{aligned}$$

When (15a) or (15b) are applied on a Dirac delta impulse function and the resulting functions are evaluated at 0, the following gramians can be obtained.

$$\begin{aligned}\Gamma_{in} &= \int_0^\infty B^T e^{A^T t} C^T C e^{At} B dt = B^T \Gamma_o B \\ \Gamma_{out} &= \int_0^\infty C e^{At} B B^T e^{A^T t} C^T dt = C\Gamma_c C^T\end{aligned}$$

These gramians can be used to derive bases for the range or row space of the impulse response function $g(t)$. Descriptions for the frequency domain can be obtained as well, but are omitted here.

In contrary to the controllability and observability gramians, Γ_{in} and Γ_{out} do not depend on the choice of the state vector and are directly related to the $\mathcal{H}_2$-norm via the relationship

$$||G(s)||_2 = \sqrt{tr(B^T\Gamma_o B)} = \sqrt{tr(C\Gamma_c C^T)}$$

When scalar subsystem are analyzed, the gramians Γ_{out} and Γ_{in} reduce to scalars. Naturally, if $\Gamma_{out,ij} = C_i\Gamma_{cj}C_i^T$ is different from zero, then the output i is affected by the input j. Thence, interaction can be expected if there is at least another input r with $\Gamma_{out,ir} \neq 0$. Similar statements can be formulated for Γ_{in}.

5 Quantification of interaction

In decentralized controller design, the choice of the measurement/actuator pair is a crucial task. The achievable performance of the closed loop system usually depends on this choice.

In [19] and [5] the proposed gramian based measures are applied on the scalar subsystems of a multivariable process. Each measurement/actuator combination is tested for its viability to control the measured output.

Two well-known system norms that can be directly derived from the gramians are the Hankel-norm and the $\mathcal{H}_2$-norm

$$\begin{aligned}||G(s)||_H &= \sqrt{\rho(\Gamma_c\Gamma_o)} \\ ||G(s)||_2 &= \sqrt{tr(B^T\Gamma_o B)} = \sqrt{tr(C\Gamma_c C^T)}\end{aligned}$$

Although the Hankel norm is derived from the gramian product, which does depend on the state vector choice, the eigenvalues of the product do not depend on the chosen state space realization. Moreover, it relates to the $\mathcal{X}_{co}$ subspaces of the state space, and thus is suited as a quantification.

Using the gramian product the $\mathcal{H}_\infty$-norm can be bounded according to [21] in the form

$$||G(s)||_H \leq ||G(s)||_\infty \leq 2\sum_i \sqrt{\sigma_i(\Gamma_c\Gamma_o)} = \overline{||G(s)||}_\infty$$

where $\sigma_i(\Gamma_c\Gamma_o)$ denotes the eigenvalues of the product.

In [19] the normalized Hankel Interaction Index Array (HIIA) was suggested to solve the measurement/actuator pairing problem for decentralized control

$$[\Sigma_H]_{ij} = \frac{||G_{ij}(s)||_H}{\sum_{q,r}||G_{qr}(s)||_H} \qquad (16)$$

Using the above results, the HIIA can be generalized to the use of different gramian based norms, namely

$$[\Sigma_2]_{ij} = \frac{||G_{ij}(s)||_2}{\sum_{q,r}||G_{qr}(s)||_2} \qquad (17)$$

$$[\Sigma_\infty]_{ij} = \frac{\overline{||G_{ij}(s)||}_\infty}{\sum_{q,r}\overline{||G_{qr}(s)||}_\infty} \qquad (18)$$

The generalized measures have the same properties as the original HIIA. Still, the interaction measures need to be validated, in order to see the effect of different norms on the measurement/actuator pairing.

As the matrix D is not considered in the computation of gramians, any gramian based measure has the drawback of ignoring a direct term. In other words, plants with $G(\infty) \neq 0$ have an HIIA which does not reflect that part of the dynamics. In cases, where the interaction is purely caused by the direct term and not by the causal part of the dynamics, the indications derived from the HIIA might be wrong.

The problem can be solved for the $\mathcal{H}_2$-norm based HIIA by considering the direct term in the computation of the system norm. The needed modification is rather easy implemented in (17)

$$[\Sigma_{\tilde{2}}]_{ij} = \frac{||G_{ij}(s)||_2 + ||G_{ij}(\infty)||_2}{\sum_{r,q}||G_{rq}(s)||_2 + ||G_{rq}(\infty)||_2} \qquad (19)$$

where the norm on the direct term is simply the 2-norm for matrices.

The new measures can now be used to find appropriate measurement/actuator pairs, according to the methodology described in [19].

5.1 Weighted gramians and filtering

Not in all application it is desired to judge the overall dynamics but instead focus on certain frequency regions. Especially, if the multivariable system has largely varying dynamics for different frequency bands.

In [18] it has also been pointed out that interaction measures should considered frequency regions where control is active. Hence, filtering should be applied and there are two different kinds of approach. Firstly, the filter dynamics can be augmented to the system dynamics, either at the input or the output side of the multivariable system. This approach is utilized in [19]. Secondly, a weighting function that corresponds to the filter dynamics can be introduced in the computation of the gramians, which is discussed here.

Assuming a scalar causal filter with the transfer function $F(s)$ is used and a minimal realization of the filter is given as

$$F(s) = C_f(sI - A_f)^{-1}B_f + D_f$$

The filtered state impulse response function can then be expressed by

$$X^f(s) = X(s)F(s) = (sI-A)^{-1}B\left(C_f(sI - A_f)^{-1}B_f + D_f\right)$$

and the resulting controllability gramian is obtained as

$$\Gamma_c^f \;=\; \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} X(s)X^H(s)|F(s)|^2 ds$$

where $X^H(s)$ denotes the complex conjugate transpose of $X(s)$. Clearly, the gramian of the filtered system can be directly obtained by introducing a weight into the controllability gramian. Thus, the weight provides an extra degree of freedom into the criteria.

The filtered controllability gramian can be evaluated by solving a block matrix Lyapunov equation [17]. It can also be rewritten as four coupled matrix equations

$$\begin{aligned} A\Gamma_c^f + \Gamma_c^f A^T &= -\left(BC_f\Gamma_{21} + \Gamma_{21}(BC_f)^T\right) &\text{(20a)}\\ A\Gamma_{12} + \Gamma_{12}A_f^T &= -BC_f\Gamma_{22} &\text{(20b)}\\ A_f\Gamma_{21} + \Gamma_{21}A^T &= -\Gamma_{22}(BC_f)^T &\text{(20c)}\\ A_f\Gamma_{22} + \Gamma_{22}A_f^T &= -B_fB_f^T &\text{(20d)} \end{aligned}$$

As $\Gamma_{12} = \Gamma_{21}^T$, only three equations have to be solved to obtain Γ_c^f. For the equations to have a unique solution, the matrix A_f of the chosen weighting function needs to fulfill certain requirements.

First, $\lambda_i(A)$ and $\mu_i(A_f)$ denote the eigenvalues of A and A_f, respectively. Then according to [9], the Lyapunov equations have a unique solution if and only if $\lambda_i(A) + \lambda_j(A) \neq 0, \forall i, j$ and $\mu_i(A_f) + \mu_j(A_f) \neq 0, \forall i, j$. A unique solution for the Sylvester equation is obtained if and only if $\lambda_i(A) + \mu_j(A_f) \neq 0, \forall i, j$. This implies, that for a stable multivariable system any stable weighting function yields a unique solution for the equations and thus for the gramians.

The use of weighted gramians as underlying structure for gramian based norms leads to different results than augmenting the multivariable system with additional filter dynamics. Additionally, through the weighting functions model uncertainties can be considered in the interaction measures, see Example 1. A drawback of the weighted gramians approach is the restriction to scalar filter functions.

6 Example

The aim of the example is to illustrate the usage of the gramian-based measures on a real-life process. The linearized physical model of a coal injection vessel, which is an example of a continuous time 2×2 servo system, is studied to find appropriate measurement/actuator pairs for decentralized control.

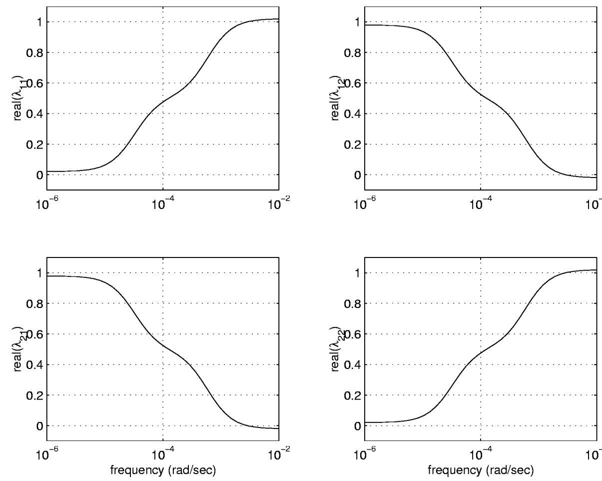

Figure 1: Real part of the RGA for the coal injection vessel over frequency.

First, the dynamic RGA for the process is analyzed and the pairs whose elements have the closest value to 1 are chosen. Then, the gramian based measures are computed with no filtering according to (16), (18) and (19). Pairs that yield the largest sum are chosen and compared with the RGA based choice. Appropriate filters are chosen and the gramian based measures are computed via weighted gramians (20). Finally, the performance of the measures is evaluated according to the correctness of the indications.

Example 1 (Coal injection vessel). The coal injection vessel is a pressurized multivariable tank system which is discussed in [11], [1] and [2]. Consult the above references for the process model. The input signals are the openings of the pressure and flow control valves. The output consists of the pressure in the vessel and the net weight.

Fig. 1 shows a plot for the real parts of the RGA elements over frequency. Clearly, the measurement/actuator pairing changes from anti-diagonal to diagonal from low to high frequencies. Drawing conclusions on the pairing from the static RGA, the anti-diagonal pairing should be favored, which means the pressure is stabilized with the flow control valve and the weight is controlled with the pressure control valve. According to knowledge of the process, this is an unconventional choice.

Computation of the gramian based interaction measures Σ_H, Σ_2, Σ_∞ yields the following arrays

$$\Sigma_H \;=\; \begin{bmatrix} 0.3740 & 0.5297 \\ 0.0500 & 0.0463 \end{bmatrix};\; \Sigma_2 = \begin{bmatrix} 0.4676 & 0.5043 \\ 0.0133 & 0.0148 \end{bmatrix}$$

$$\Sigma_\infty \;=\; \begin{bmatrix} 0.4680 & 0.4676 \\ 0.0335 & 0.0309 \end{bmatrix}$$

First of all, the gramian based measures do not differ much, although different norms were used. When a pairing decision should be taken according to Σ_H and Σ_2 the anti-diagonal pairing is favored again, while a distinction based on Σ_∞ cannot be made. Thus, more knowledge of the process is needed to make a clear decision.

Due to the character and operation of the coal injection process and model uncertainties, the linear model is not reflecting the plant dynamics in frequency ranges below 10^{-4} rad/sec. Therefore, high pass filtering should be applied.

In order to avoid non-strictly proper weighting functions, a band pass filter with the break frequencies 0.01 rad/sec and 1 rad/sec is chosen instead. Using the filter as weight in the gramians, the HIIA are recomputed

$$\Sigma_H = \begin{bmatrix} 0.5001 & 0.4931 \\ 0.0003 & 0.0065 \end{bmatrix}; \Sigma_2 = \begin{bmatrix} 0.5002 & 0.4931 \\ 0.0002 & 0.0065 \end{bmatrix}$$
$$\Sigma_\infty = \begin{bmatrix} 0.5001 & 0.4930 \\ 0.0004 & 0.0065 \end{bmatrix}$$

Obviously, the gramian based measures have changed and now $\Sigma_H \approx \Sigma_2 \approx \Sigma_\infty$. In all three cases the diagonal pairing should be chosen for decentralized control. According to knowledge on the process the diagonal pairing should be favored and has been successfully used for decentralized control of the process.

From the example it can be seen, that the generalized gramian measures can be used to improve the decision process for measurement/actuator pairing. Still knowledge of the process and model uncertainties is needed to make the correct choice for weighting functions.

7 Conclusions

The theoretical background for the gramian based interaction measures is discussed. It is shown that the gramian based approach to interaction is closely related to intersections of subspaces of function spaces. The relation between the Hankel operator and the gramian based measures is clarified and an extension to the use of different norms is given.

Moreover, it can be suggested that weighting functions in the underlying scalar product should be used to emphasize importance of certain frequency regions. Thereby, the model dynamics are not augmented with additional process states and model uncertainty can be considered.

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