A COMBINATORIAL APPROACH TO THE (POSITIVE) REACHABILITY OF 2D POSITIVE SYSTEMS

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Abstract

Local reachability of two-dimensional (2D) positive systems, by means of positive scalar inputs, is addressed. The combinatorial nature of this property allows for a graph theoretic approach. Indeed, to every 2D positive system of dimension n with scalar inputs one can associate a 2D influence graph with n vertices, one source and two types of arcs interconnecting the source and the vertices. Some results concerned with equivalent conditions for local reachability as well as upper and lower bounds on the reachability indices are provided.

1 Introduction

Recent years have seen a growing interest in two-dimensional (2D) systems that are subject to a positivity constraint on their dynamical variables. There are actually several different motivations for this interest, coming from various domains of science and technology. Positive 2D systems arise, for instance, when discretizing pollution and self-purification processes along a river stream, or when providing a discrete model for the traffic flow in a motorway. More generally, the positivity assumption is a natural one when describing, by means of 2D systems, distributed processes whose variables represent quantities that are intrinsically nonnegative, such as pressures, concentrations, population levels, etc. The first contributions to the analysis of 2D positive systems [4, 5, 7, 11] mainly focused on the free evolution of the state variables, and therefore on the investigation of the algebraic and combinatorial properties of positive matrix pairs appearing in the state equations. More recently, T. Kaczorek afforded other topics related to the forced dynamics of 2D positive systems and, specifically, control, estimation and stabilization problems [8, 9].

In this paper we address the positive local reachability property. To this end, we assume a combinatorial point of view and hence consider just the nonzero patterns of the matrices and vectors involved in the system description. 2D influence graphs (namely direct graphs which exhibit two types of arcs and two types of input flows [5, 7]) turn out to be the appropriate tools for formalizing and solving the problem. The results presented here are preliminary. The general solution of the problem seems nontrivial, as the facts and counter-examples here provided will clearly enlighten.

2D positive systems considered in this paper are driven by scalar inputs and are described by the following state-updating equation [2, 6]:

$$\begin{aligned}\mathbf{x}(h+1,k+1) &= A_1\mathbf{x}(h,k+1) + A_2\mathbf{x}(h+1,k) \\ &+ B_1 u(h,k+1) + B_2 u(h+1,k), \quad (1)\end{aligned}$$

where the local states $\mathbf{x}(\cdot,\cdot)$ and the scalar input $u(\cdot,\cdot)$ take nonnegative values, A_1 and A_2 are nonnegative $n\times n$ matrices, B_1 and B_2 are nonnegative n-dimensional column vectors, and the initial conditions are assigned by specifying the nonnegative values of the state vectors on the *separation set*

$$\mathcal{C}_0 := \{(h,k) : h,k \in \mathbb{Z}, h+k=0\},$$

namely by assigning all **local states** of the initial **global state**

$$\mathcal{X}_0 := \{\mathbf{x}(h,k) : (h,k) \in \mathcal{C}_0\}.$$

The Hurwitz products of two $n \times n$ matrices A_1 and A_2 are inductively defined as

$$\begin{aligned} A_1{}^{i}\sqcup\!\sqcup^{0} A_2 &= A_1^i, \quad i \geq 0, \\ A_1{}^{0}\sqcup\!\sqcup^{j} A_2 &= A_2^j, \quad j \geq 0, \\ A_1{}^{i}\sqcup\!\sqcup^{j} A_2 &= A_1(A_1{}^{i-1}\sqcup\!\sqcup^{j} A_2) + A_2(A_1{}^{i}\sqcup\!\sqcup^{j-1} A_2), \quad i,j > 0, \end{aligned}$$

meanwhile

$$A_1{}^{i}\sqcup\!\sqcup^{j} A_2 = 0, \qquad \text{when either } i \text{ or } j \text{ is negative.}$$

A **2D influence graph** $\mathcal{D}^{(2)}$ is a 6-tuple

$$\mathcal{D}^{(2)} = (s, V, \mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2),$$

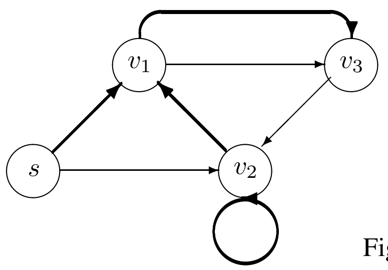
where s is the *source*, $V = \{v_1, v_2, \dots, v_n\}$ is the set of *vertices*, $\mathcal{A}_1$ and $\mathcal{A}_2$ are subsets of $V \times V$ whose elements are called $\mathcal{A}_1$-*arcs* and $\mathcal{A}_2$-*arcs*, respectively, meanwhile $\mathcal{B}_1$ and $\mathcal{B}_2$ are subsets of $s \times V$ whose elements are called $\mathcal{B}_1$-*arcs* and $\mathcal{B}_2$-*arcs*, respectively.

To every 2D positive system (1) with scalar inputs of size n we associate a 2D influence graph $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ of source s, with n vertices, $v_1, v_2, \dots, v_n$. There is an $\mathcal{A}_1$-arc (an $\mathcal{A}_2$-arc) from v_j to v_i if and only if the (i,j)th entry of A_1 (of A_2) is nonzero. There is a $\mathcal{B}_1$-arc (a $\mathcal{B}_2$-arc) from s to v_i if and only if the ith entry of B_1 (of B_2) is nonzero.

For instance, the positive system described by the following matrices

$$(A_1, A_2, B_1, B_2) = \left(\begin{bmatrix} 0 & 5 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} \right)$$

corresponds to the 2D digraph of Fig. 1.1. We have represented $\mathcal{A}_1$-arcs and $\mathcal{B}_1$-arcs by means of thick lines, while $\mathcal{A}_2$-arcs and $\mathcal{B}_2$-arcs by means of thin lines.



$\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$

Fig. 1.1

A *path* p in $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ is a sequence of adjacent arcs and, in particular, an *s-path* is a path which originates from the source s. In order to specify a path (in particular, an s-path) p one has to specify not only the extreme vertices of each arc but also the type of arc they are connected by. If we denote by $|p|_1$ the number of $\mathcal{A}_1$-arcs and $\mathcal{B}_1$-arcs, and by $|p|_2$ the number of $\mathcal{A}_2$-arcs and $\mathcal{B}_2$-arcs occurring in p, then $[|p|_1 \ \ |p|_2]$ is the *composition* of p and $|p| = |p|_1 + |p|_2$ its *length*.

A path whose extreme vertices coincide is called a *cycle*. In particular, if each vertex in a cycle appears exactly once as the first vertex of an arc, the cycle is called a *circuit*.

A 2D influence graph is *strongly connected* is for any two vertices v_i and v_j there is a path (of arbitrary composition) connecting v_i to v_j. $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ is strongly connected if and only if $A_1 + A_2$ is an irreducible matrix [1].

Two positive matrices M and N, with the same dimensions, are said to have the same *nonzero pattern* if $m_{ij} \neq 0$ implies $n_{ij} \neq 0$ and vice versa. A positive vector $\mathbf{v}$ is said to be an i**th monomial vector** if it can be expressed as $\alpha_i \mathbf{e}_i$, where $\mathbf{e}_i$ denotes the ith canonical vector and α_i is some positive real coefficient. A **monomial matrix** is a nonsingular (square) matrix whose columns are monomial vectors.

2 Reachability and positive reachability definitions

For 2D state-space models (1) two distinct definitions of reachability are usually considered [2]: local reachability and global reachability. Local reachability refers to the possibility of "reaching" an arbitrary local state $\mathbf{x}^* \in \mathbb{R}^n$, starting from zero initial conditions, meanwhile global reachability amounts to the possibility (starting, again, from zero initial conditions) of obtaining arbitrary sequences of local states $\mathbf{x}(h,k)$ on an assigned separation set

$$\mathcal{C}_t := \{(h,k) : h,k \in \mathbb{Z}, h+k = t\},$$

provided that $t \in \mathbb{N}$ is large enough.

Definition 2.1 A 2D state-space model (1) is

- **locally reachable** if, upon assuming $\mathcal{X}_0 = 0$, for every $\mathbf{x}^* \in \mathbb{R}^n$ there exists $(h,k) \in \mathbb{Z} \times \mathbb{Z}$ with $h+k > 0$ and an input sequence $\mathbf{u}(\cdot,\cdot)$ such that $\mathbf{x}(h,k) = \mathbf{x}^*$;
- **globally reachable** if, upon assuming $\mathcal{X}_0 = 0$, for every sequence $\{\mathbf{x}_h^*\}_{h \in \mathbb{Z}}$, there exists $t \in \mathbb{N}$ and an input sequence $\mathbf{u}(\cdot,\cdot)$ such that $\mathbf{x}(h, t-h) = \mathbf{x}_h^*$ for every $h \in \mathbb{Z}$.

Of course, global reachability implies local reachability. For standard (i.e., not necessarily positive) 2D systems, local reachability analysis easily reduces to the analysis of the column span of the **reachability matrix in** k **steps** [2]

$$\mathcal{R}_k = \left[(A_1{}^{i-1} \sqcup\!\sqcup^j A_2) B_1 + (A_1{}^i \sqcup\!\sqcup^{j-1} A_2) B_2 \right]_{i,j \geq 0,\ 0 < i+j \leq k}$$

as k varies over the set of positive integers. Reachable states in k steps, i.e. local states that can be reached on the separation set $\mathcal{C}_k$ starting from $\mathcal{X}_0 = 0$, constitute a linear subspace $X_k \subseteq \mathbb{R}^n$, spanned by the columns of $\mathcal{R}_k$. Clearly, the ascending chain

$$X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$$

eventually reaches stationarity and this necessarily happens, by the 2D Cayley-Hamilton theorem, in no more than n steps. As a consequence, if the system is locally reachable, the point (h,k) such that $\mathbf{x}(h,k)$ has the desired value can always be chosen on the separation set $\mathcal{C}_n$.

On the other hand, a 2D system is globally reachable if and only if

$$\text{rank}\,[\, I_n - A_1 z_1 - A_2 z_2 \ \mid \ B_1 z_1 + B_2 z_2 \,] = n, \ \forall\, z_1, z_2 \in \mathbb{C}.$$

Again, when the system is globally reachable, every global state can be reached in no more than n steps.

The definitions of **positive local reachability** and **positive global reachability** are immediately obtained from Definition 2.1, once we constrain both the input sequence and the state vectors we aim at reaching to be nonnegative. In this contribution we focus on positive local reachability for 2D positive systems with scalar inputs. A comparison with the analogous problem in the 1D setting could erroneously lead to underestimate the problem difficulty. As we shall see, most of the intuitions one may have about the problem solution are immediately disproved by very simple examples.

Once we constrain the input sequence to be nonnegative, the reachability subspaces $X_k, k \in \mathbb{N}$, are replaced by the **reachability cones** $X_k^+, k \in \mathbb{N}$. In fact, the set X_k^+ of all local states that can be reached on the separation set $\mathcal{C}_k$, by means of nonnegative inputs and starting from initial zero conditions ($\mathcal{X}_0 = 0$), obviously coincides with the set of all nonnegative combinations of the columns of $\mathcal{R}_k$, namely

$$X_k^+ = \text{Cone}\,\mathcal{R}_k.$$

As in the case of 1D positive systems (see [10]), the chain of reachability cones does not necessarily reach stationarity and, indeed, certain positive states can be reached only asymptotically. Moreover, positive local reachability is trivially equivalent to the possibility of reaching (starting from zero initial conditions) every vector of the canonical basis in $\mathbb{R}^n$ by means of nonnegative inputs, which in turn amounts to saying that there exists some $k \in \mathbb{N}$ such that the reachability matrix in k steps, $\mathcal{R}_k$, includes an $n \times n$ monomial submatrix [1]. This is, of course, a structural property of the system, by this meaning that it only depends on the nonzero patterns of the 4 system matrices and not on the specific values of their nonzero elements. However, differently from the 1D positive case, the **reachability index** I_R of a (locally reachable) 2D positive system, namely the minimum index k such that

$$X_k^+ = \text{Cone}\,\mathcal{R}_k = \mathbb{R}_+^n,$$

is not bounded by n.

Example 1 Consider the positive system described by the following matrices

$$(A_1, A_2, B_1, B_2) = \left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right),$$

which corresponds to the 2D graph of Fig. 2.1.

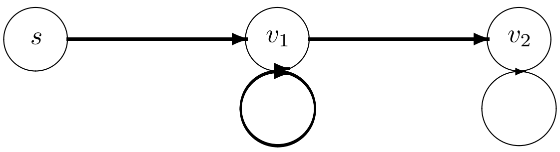


Fig. 2.1 $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ corresponding to Example 1

It is easy to verify that the system is positively locally reachable and the reachability index is $3 > 2 = n$. Indeed,

$$\begin{aligned}
\mathcal{R}_1 &= [\,B_1 \quad B_2\,] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
\mathcal{R}_2 &= [\,B_1 \quad B_2 \quad A_1B_1 \quad A_2B_1 + A_1B_2 \quad A_2B_2\,] \\
&= \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \\
\mathcal{R}_2 &= [\,B_1 \quad B_2 \quad A_1B_1 \quad A_2B_1 + A_1B_2 \quad A_2B_2 \quad A_1^2B_1 \\
&\quad ({A_1}^1 \sqcup\!\sqcup^1 A_2)B_1 + A_1^2B_2 \quad A_2^2B_1 + ({A_1}^1 \sqcup\!\sqcup^1 A_2)B_2 \\
&\quad A_2^2B_2] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Example 2 Consider the 2D positive system which corresponds to the 2D graph of Fig. 2.2.

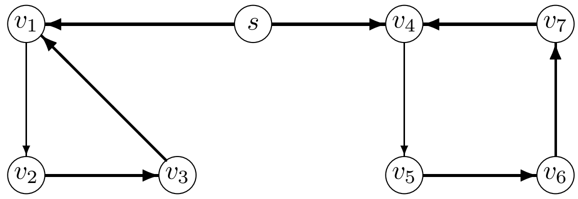


Fig. 2.2 $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ corresponding to Example 2

In this case, the reachability index proves to be 13 while the system dimension is $n = 7$. The above structure can be generalized. In fact, if the 2D influence graph of a 2D positive system has the previous structure, by this meaning that it consists of two loops including n_1 vertices and $n_1 + 1$ vertices, respectively, connected by arcs of type 1 and 2 as indicated in Fig. 2.3, then the reachability index turns out to be of the same order as $n_1 \cdot (n_1 + 1)$, namely of the same order as $n^2/4$, since $n = n_1 + (n_1 + 1)$.

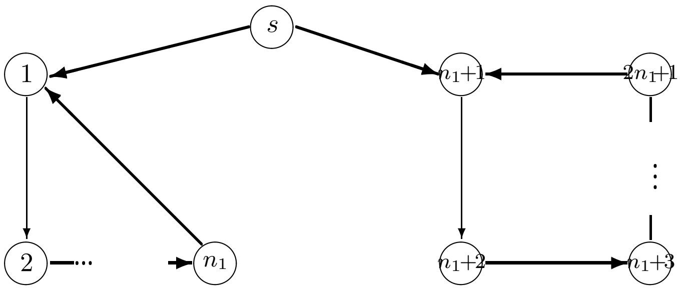


Fig. 2.3 $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ generalizing Example 2

Example 2 has proved that for a locally reachable 2D positive system the reachability index may reach the value $n^2/4$. It seems reasonable to conjecture that $n^2/4$ represents an upper bound for the reachability index of every 2D positive system. Even though several examples we have analyzed seem to confirm this conjecture, up to now a formal proof of this result is not available. A necessary condition for positive reachability is the following one.

Proposition 2.2 *If the positive system (1) is positively locally reachable then the matrix*

$$[\,A_1 \quad A_2 \quad B_1 \quad B_2\,]$$

includes an $n \times n$ monomial submatrix.

PROOF If the system is locally reachable, then there exist n nonnegative pairs $(h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+, i = 1, 2, \dots, n$, such that

$$({A_1}^{h_i - 1} \sqcup\!\sqcup^{k_i} A_2)B_1 + ({A_1}^{h_i} \sqcup\!\sqcup^{k_i - 1} A_2)B_2$$

is an ith monomial vector. If $h_i + k_i = 1$ then the ith monomial vector is a column of B_1 or of B_2. If $h_i + k_i > 1$ then the ith monomial vector is a column of A_1 or of A_2 (possibly both). ■

As for 1D positive systems, local reachability property admits an interesting and useful characterization in terms of the 2D influence graph associated with the system. Indeed, saying that

$$(A_1{}^{h_i-1}\sqcup\!\sqcup^{k_i} A_2)B_1 + (A_1{}^{h_i}\sqcup\!\sqcup^{k_i-1} A_2)B_2$$

is an ith monomial vector just means that every s-path p of composition $[|p|_1\ |p|_2] = [h_i\ k_i]$ necessarily reaches the vertex v_i alone. If so, we will say that the vertex v_i is **deterministically reached** by all s-paths of composition $[h_i\ k_i]$.

As a consequence, the 2D system (1) is positively locally reachable if and only if for every $i \in \{1, 2, \ldots, n\}$ the vertex v_i is deterministically reached by all s-paths of a given composition $[h_i\ k_i]$. Moreover the reachability index I_R coincides with

$$\max_i \min_{h_i,k_i} \{h_i + k_i \quad : \quad \text{all } s\text{-paths of composition } [h_i\ k_i] \text{ deterministically reach } v_i\}.$$

In the sequel, we will confine our attention to 2D positive systems (1) having one of the two input-to-state matrices which is zero. We will assume, without loss of generality, $B_2 = 0$ and, consequently, denote B_1 as B, for the sake of simplicity. These systems are described by the following equation:

$$\mathbf{x}(h+1, k+1) = A_1\mathbf{x}(h, k+1) + A_2\mathbf{x}(h+1, k) + Bu(h, k+1), \quad (2)$$

where A_1, A_2 are in $\mathbb{R}_+^{n\times n}$ and B is in $\mathbb{R}_+^n$.

3 2D influence graphs devoid of cycles

In this section we consider 2D positive systems (2) whose 2D influence graph is devoid of cycles. This amounts to saying that the system (1) is *finite memory* [3] or, equivalently, [4], due to the positivity assumption, that $A_1 + A_2$ is nilpotent.

Proposition 3.1 *Given a 2D positive system (1), its 2D influence graph $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ is devoid of cycles if and only if the system is finite memory.*

PROOF We first observe that since the source exhibits no incoming arcs, $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ is devoid of cycles if and only if $\mathcal{D}^{(2)}(A_1, A_2, 0, 0)$ is. On the other hand, if γ is a cycle in $\mathcal{D}^{(2)}(A_1, A_2, 0, 0)$ and the vertex v_i belongs to γ, then $[(A_1 + A_2)^{m\cdot|\gamma|}]_{ii} > 0$ for every positive integer m. So, if (1) is finite memory, namely $A_1 + A_2$ is nilpotent, then $(A_1 + A_2)^k = 0$ for every $k \geq n$. Therefore, no cycle γ can exist in $\mathcal{D}^{(2)}(A_1, A_2, 0, 0)$. Conversely, if there is a cycle γ in $\mathcal{D}^{(2)}(A_1, A_2, 0, 0)$ then condition $(A_1 + A_2)^k = 0$ for every $k \geq n$ cannot be satisfied. ■

Proposition 3.2 *If a 2D positive system (2), with 2D influence graph $\mathcal{D}^{(2)}(A_1, A_2, B, 0)$ devoid of cycles, is positively locally reachable then*

i) B is a canonical vector, and

ii) the reachability index I_R satisfies

$$\min\left\{k \in \mathbb{N} : \sum_{i=1}^{k} i \geq n\right\} \leq I_R \leq n.$$

PROOF i) Since $A_1 + A_2$ is (positive and) nilpotent, it entails no loss of generality [4] assuming that $A_1 + A_2$ (and hence A_1 and A_2, separately) is in upper triangular form with zero diagonal (in fact, we can always reduce ourselves to this situation by resorting to some suitable permutation of the state components). So, if A_1 and A_2 have the following structure

$$\begin{bmatrix} 0 & + & + \\ & \ddots & + \\ & & 0 \end{bmatrix}$$

and the system is positively locally reachable, then, by Proposition 2.2, in $[\,A_1 \quad A_2 \quad B \quad 0\,]$ there must appear also the nth canonical vector $\mathbf{e}_n$. This necessarily implies $B = \mathbf{e}_n$.

ii) Since $(A_1 + A_2)^n = 0$, all Hurwitz products $A_1{}^i\sqcup\!\sqcup^j A_2$ are zero whenever $i + j \geq n$. So, $X_{n+1}^+ = X_n^+$, and, in general, $X_k^+ = X_n^+, \forall\, k \geq n$. If $B = \mathbf{e}_n$, it is easily seen that after one step the only outgoing arc from the source reaches vertex v_n. On the other hand, due to the fact that only two types of arcs are available, paths of length 2 with a common initial arc (from the source to vertex v_n) and distinct compositions may reach deterministically at most two vertices. Again, paths of length 3 with a common initial arc and distinct compositions may deterministically reach at most three vertices, and so on. This means that the minimum number of steps required to deterministically reach each vertex is the smallest positive integer k such that $1 + 2 + 3 + \ldots + k \geq n$. ■

Examples of 2D positive systems (2) of order n, with $A_1 + A_2$ nilpotent and minimum reachability index I_R, can be easily constructed for every $n \in \mathbb{N}$. To that purpose, once we assume $B = \mathbf{e}_1$, and hence connect the source to v_1, we construct a 2D influence digraph with the structure of a binary tree, having at each level k no more than k vertices for all $k \in \mathbb{N}$. The outgoing arcs from each vertex have to be suitably chosen in order to guarantee that all s-paths of the same length (i.e., reaching vertices of the same level) have distinct compositions.

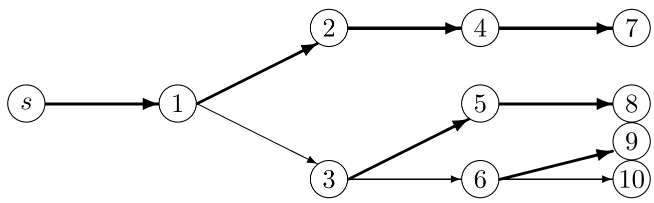


Fig. 3.1 $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ with minimum I_R

The worst case, $I_R = n$, can be obtained by simply connecting the source and all vertices along a single path.

4 2D influence graphs consisting of either one or two (disjoint) circuits

In this section we consider, first, systems (2) with 2D influence graphs consisting of a single circuit, by this meaning that all vertices $v_1, v_2, \ldots, v_n$ belong to a circuit (and each pair of adjacent vertices is connected by one single arc). This assumption amounts to saying that $A_1 + A_2$ is a permutation matrix, while $A_1 * A_2 = 0$, where $*$ denotes the Hadamard product. So, by resorting to some suitable permutation of the state components we can always obtain

$$A_1 + A_2 = \begin{bmatrix} 0 & + & 0 & & 0 \\ 0 & 0 & + & & 0 \\ & & \ddots & \ddots & \\ & & & \ddots & + \\ + & 0 & & & 0 \end{bmatrix}, \quad (3)$$

where $+$ represents a strictly positive entry and each nonzero entry $+$ appears only in one of the two matrices A_1 and A_2. Notice that vertex v_{i+1} accesses vertex v_i, for $i = 1, 2, \ldots, n-1$, while vertex v_1 accesses v_n.

We first remark that differently from the 1D case, the positive local reachability of such a system $(A_1, A_2, B, 0)$ does not require B to be a monomial vector.

Example 3 Consider the positive system described by the following matrices

$$\begin{aligned}(A_1, A_2, B_1, B_2) \;=\; & ([\mathbf{e}_2 \;\; \mathbf{e}_3 \;\; 0 \;\; 0], [0 \;\; 0 \;\; \mathbf{e}_4 \;\; \mathbf{e}_1], \\ & [\mathbf{e}_1 + \mathbf{e}_3], 0),\end{aligned}$$

and corresponding to the 2D graph of Fig. 4.1.

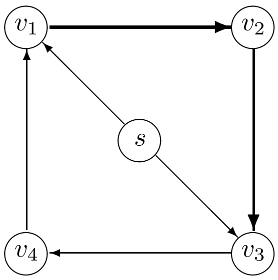


Fig. 4.1 $\mathcal{D}^{(2)}(A_1, A_2, B_1, B_2)$ a single cycle

This system is locally reachable, with reachability index $3 = \frac{n}{2} + 1$.

The situation depicted in the previous example easily generalizes to the case of $n = 2n_1$ vertices with two outgoing arcs from the source, n_1 consecutive arcs of type 1 and n_1 consecutive arcs of type 2. If the structure is the same as in the previous figure, then the system is locally reachable and $I_R = \frac{n}{2} + 1$.

As a further remark, when $\mathcal{D}^{(2)}(A_1, A_2, B, 0)$ consists of a single circuit, every monomial vector B makes $(A_1, A_2, B, 0)$ positively locally reachable with reachability index $I_R = n$. When B is the sum of k distinct monomial vectors and the system is locally reachable, the reachability index may take quite smaller values. If $k = 2$, then a lower bound for the reachability index is $\frac{n}{2} + 1$. This is a consequence of the following proposition.

Proposition 4.1 *Let $(A_1, A_2, B, 0)$ be a 2D positive system such that $\mathcal{D}^{(2)}(A_1, A_2, B, 0)$ consists of a single circuit and assume w.l.o.g. that $A_1 + A_2$ is expressed as in (3) with $A_1 * A_2 = 0$. If the system is positively locally reachable and B has k nonzero entries, of indices say $1 \le i_1 < i_2 < \ldots < i_k \le n$, then*

$$I_R \ge \max\{i_2 - i_1, i_3 - i_2, \ldots, i_k - i_{k-1}, n - i_k + i_1\} + 1.$$

PROOF Suppose, for the sake of simplicity, that $\max\{i_2 - i_1, i_3 - i_2, \ldots, i_k - i_{k-1}, n - i_k + i_1\} = i_2 - i_1$. By the ordering assumptions introduced on the system vertices and on the labels $i_1, i_2, \ldots, i_k$, it is clear that the minimum $h_1 + k_1$ such that all s-paths of composition $[h_1 \; k_1]$ deterministically reach v_{i_1} (keeping in mind that at the first step we get B and hence not a monomial vector) coincides with the length of the s-path that, starting from the source, reaches vertex v_{i_2} at the first step and later enters vertex v_{i_1} without passing through the other vertices v_{i_ℓ} for $\ell \ne 1, 2$. Such an s-path has length $i_2 - i_1 + 1$. Condition $I_R = \max_i \min_{h_i, k_i}\{h_i + k_i :$ all s-paths of composition $[h_i \; k_i]$ deterministically reach $v_i,\}$ $\ge i_2 - i_1 + 1$ completes the proof. ■

Clearly, when $k = 2$, the minimum value of

$$\begin{aligned} & \max\{i_2 - i_1, i_3 - i_2, \ldots, i_k - i_{k-1}, n - i_k + i_1\} \\ = \; & \max\{(i_2 - i_1), n - (i_2 - i_1)\}\end{aligned}$$

is just $n/2$ and this proves that the minimum value of the reachability index is $\frac{n}{2} + 1$.

We consider, now, the case of a 2D influence graph consisting of two disjoint circuits. We have the following result.

Proposition 4.2 *Let $(A_1, A_2, B, 0)$ be a 2D positive system such that $\mathcal{D}^{(2)}(A_1, A_2, 0, 0)$ consists of two disjoint circuits γ and γ' of length n and n', respectively. If the system is positively locally reachable and B has only two nonzero entries, one for each cycle, then*

$$I_R \le \text{l.c.m}\{n, n'\} + \max\{n, n'\}.$$

PROOF Assume that the vertices in γ are (ordinately) $v_1, v_2, \ldots, v_n$ while the vertices in γ' are (ordinately) $v'_1, v'_2, \ldots, v'_{n'}$. Suppose, also, that the two nonzero entries in B correspond to the vertices v_1 and v'_1. The situation is depicted in Figure 4.2.

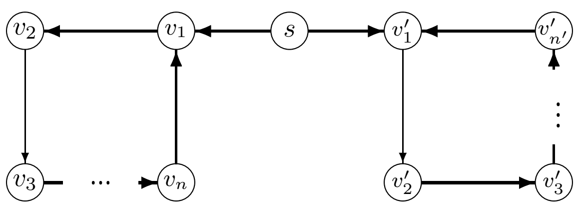


Fig. 4.2 $\mathcal{D}^{(2)}(A_1, A_2, B, 0)$ in Proposition 4.2

Due to the previous assumptions, any vertex $v_j \in \gamma$ ($v'_j \in \gamma'$) is periodically visited after $j, j+n, j+2n, \ldots$ steps ($j, j+n', j+2n', \ldots$ steps, respectively). Moreover, for every $k \in \mathbb{N}$ there exist exactly two s-paths of length k in $\mathcal{D}^{(2)}(A_1, A_2, B, 0)$, and they reach vertices $v_{k \bmod n}$ in γ and $v'_{k \bmod n'}$ in γ', respectively. Such vertices are reached deterministically if and only if the two s-paths have distinct compositions.

Let N be the $\mathrm{l.c.m.}$ of n and n' and suppose that none of the paths of length $j, j+n, \ldots, j+N$ deterministically reaches v_j. Since after $j+N$ steps we reach, at the same time and with the same composition, the two vertices v_j and v'_j just like after j steps, the subsequent evolution will periodically repeat the same nonzero pattern, thus preventing the possibility of deterministically reaching v_j.

As this reasoning applies to all vertices of γ and γ' (in particular to v_n and $v'_{n'}$), the given bound immediately follows. ■

Example 4 Consider the 2D positive system described by the following 2D influence graph.

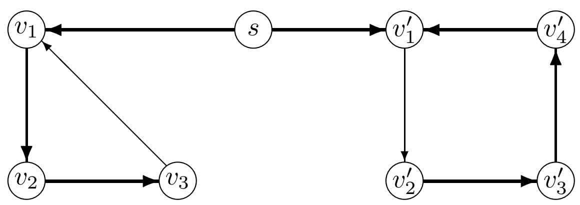


Fig. 4.3 $\mathcal{D}^{(2)}(A_1, A_2, B, 0)$ corresponding to Example 4

In this case we have two circuits: one of length $n = 3$ and the other of length $n' = 4$ and $N = \mathrm{l.c.m.}\{n, n'\} = n \cdot n'$. Simple (but tedious) calculations show that the 2D system is positively locally reachable with reachability index $I_R = n \cdot n' + n' = 16$. The last vertex to be deterministically reached (after exactly 16 steps) is v'_4.

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