WHAT IS THE MINIMUM FUNCTION OBSERVER ORDER

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Abstract

The design of a minimal order observer which can estimate the state feedback control signal $K\mathbf{x}(t)$ with arbitrarily given observer poles and $K \in R^{pxn}$, has been tried for years, with the prevailing conclusion that it is an unsolved problem. This paper asserts the following four clear-cut claims. 1) this design problem has been simplified to a set of linear equations $K = K_z \text{diag}\{\mathbf{c}_1, ..., \mathbf{c}_r\} D$ ($\mathbf{c}_i \in R^{1xm}$, m = rank(C)) if the observer is strictly proper, where D is already determined and other parameters completely free, and r is the observer order. 2) <u>only</u> this set of linear equations can provide the unified upper bound of r, $\min\{n, v_1 + ... + v_p\}$ and $\min\{n-m, (v_1-1)+...+(v_p-1)\}$, for strictly proper and proper observers, respectively, where v_i ($v_1 \geq ... \geq v_m$) is the i-th observability index of system (A,B,C,0). 3) This bound is lower than all other existing ones and is the lowest possible general upper bound. 4) The observer order reduction guaranteed by this bound is very significant even at the computer age.

I. Introduction

For a given linear time-invariant observable plant

$$\begin{aligned} d/dt\, \mathbf{x}(t) &= A\, \mathbf{x}(t) + B\, \mathbf{u}(t) \\ \mathbf{y}(t) &= C\, \mathbf{x}(t) \end{aligned} \qquad (1)$$

and its state feedback control $K\mathbf{x}(t)$ with $K \in R^{pxn}$ arbitrarily given, the function observer has the general state space model

$$d/dt\mathbf{z}(t) = F\mathbf{z}(t) + L\mathbf{y}(t) + TB\mathbf{u}(t) \qquad (2.a)$$
$$K\, \mathbf{x}(t) = K_z\mathbf{z}(t) + K_y\mathbf{y}(t) \qquad (2.b)$$

where $\mathbf{x}$, $\mathbf{u}$, $\mathbf{y}$, and $\mathbf{z}$ have dimensions n, p, m, and r, respectively. The stable poles (or the eigenvalues of F) of observer (2) are also arbitrarily given for a guaranteed rate of esitimation of $K\mathbf{x}(t)$.

The well known state observer is a special case of (2) in the sense that K = I. The observer (2) is strictly proper and proper if $K_y = 0$ or $\neq 0$, respectively. These differences are reflected by (2.b) <u>only</u> but not (2.a).

Because both $\mathbf{x}(t)$ and $\mathbf{y}(t) = C\mathbf{x}(t)$ are time varying signals, and because both K and C are constants, it is obvious that in order to generate $K\mathbf{x}(t)$ for a constant K, $\mathbf{z}(t)$ must converge to $T\mathbf{x}(t)$ for a constant T in (2.b). The necessary and sufficient condition for $\mathbf{z}(t) \rightarrow T\mathbf{x}(t)$ is [1]

$$TA - FT = LC, \text{ (F is stable)} \qquad (3)$$

Equation (3) concerns with the parameters of the dynamic part of observer (2.a) <u>only</u> but not of (2.b).

After (3) (or $\mathbf{z}(t) \rightarrow T\mathbf{x}(t)$) is satisfied, (2.b) becomes

$$K = K_zT + K_yC = [K_z:K_y][T':C']' \equiv \underline{KC} \qquad (4)$$

Equation (4) concerns with the parameters of the static output part of observer (2.b) <u>only</u> but not of (2.a). More important, because only (2.b) but not (2.a) reflects the difference between all types of observers, <u>only</u> (4) but not (3) can provide order difference and order reduction for different types of observers.

For example, because the number (p) of rows of K is usually much lower than n, the number (r) of rows of T needed to satisfy (4) can be much lower than n-m (or the number of rows of <u>C</u> can be much lower than n). This is the <u>only</u> reason that the function observer order r can be much lower than n-m.

Because high observer order has been

considered a major drawback that limits the practical application of state space control theory, the minimal order function observer design has been tried for years. This design can be divided into state space (of solving (3) and (4)) [2-6] and transfer function approaches [7-9]. The general upper bound of r from the transfer function approach is $\min\{n-m, p(v_1-1)\}$ [7] where v_1 is the highest observability index of plant (1), while the comparable general upper bound has not been guaranteed by the designs of [2-6]. For example the design of [3] is limited to the single-output plants only. In fact, the prevailing conclusion is that the really general and systematic minimal order function observer design procedure and the really general minimal function observer order have not been achieved [7, 9].

2 The Simplification To A Set Of Linear Equations Only

As analyzed in Section 1, only (4), which is a set of linear equations only, can provide the order reduction of function observers. To compute the solution of this equation with the minimal number of rows of T (or minimal observer order), really systematically, it is obvious that each row of T and each mode of the dynamic part (F,T,L) of observer (2.a) must be completely decoupled. It is also obvious that the remaining design freedom of T must be really fully usable in this computation. Because (F,T,L) must satisfy (3) first (see Section 1), the remaining freedom of T is also the remaining design freedom of (3) (and of the dynamic part of observer (2.a)).

Unfortunately, such a solution (F,T,L) of (3) has not been used in the existing designs of [2-9]. Consequently, the existing design must compute the solutions of (3) and (4) together and has not been able to compute the solution of (4) separately and therefore systematically. This is the simple and critical reason that the existing design of [2-6] cannot be really generally systematic and cannot guarantee a really general and really low upper bound of function observer order [10].

Such a solution is used in [11] to design the minimal order function observers. This solution is based on the Jordan form of matrix F and $C = [\,0 : C_1\,]$ ($|C_1| \neq 0$). This form of C can always be derived by similarity transformation. Then for distinct and real eigenvalues of F, the i-th row $\mathbf{t}_i$ of T can be expressed as

$$\mathbf{t}_i = \mathbf{c}_i D_i, \ \forall\, i \qquad (5)$$

where $\mathbf{c}_i \in R^{1\times m}$ is completely free and $D_i \in R^{m\times n}$ is formed by the basis vectors of $\mathbf{t}_i$ and can be fully determined from the equation

$$D_i(A - \lambda_i I)\begin{bmatrix} I_{n-m} \\ 0 \end{bmatrix} = 0, \ \forall\, i \qquad (6)$$

where λ_i is the i-th eigenvalue of F.

It is obvious that the left n-m columns of (3) can be satisfied by (5) and (6), and this result can be easily generalized to the general eigenvalue case of F [10]. The right and remaining m columns of (3) can always be satisfied by [10]

$$L = (TA - FT)\begin{bmatrix} 0 \\ I_m \end{bmatrix} C_1^{-1} \qquad (7)$$

Substitute (5) into (4), we have for $K_y = 0$ (strictly proper observer case):

$$K = K_z \begin{bmatrix} \mathbf{c}_1 & & \\ & \ddots & \\ & & \mathbf{c}_r \end{bmatrix} \begin{bmatrix} D_1 \\ \vdots \\ D_r \end{bmatrix} \qquad (8.a)$$

When $K_y \neq 0$ and matrix C is used in (4) (proper observer case), only the left n-m columns of (4) need to be satisfied (similar to the solution of (3)):

$$\underline{K} = K_z \begin{bmatrix} \mathbf{c}_1 & & \\ & \ddots & \\ & & \mathbf{c}_r \end{bmatrix} \begin{bmatrix} \underline{D}_1 \\ \vdots \\ \underline{D}_r \end{bmatrix} \qquad (8.b)$$

where $\underline{K}$ and $\underline{D}_i$ $\forall\, i$ are the left n-m columns of the corresponding matrices. The right and remaining m columns of (4) can always be satisfied by

$$K_y = (K - K_z T)\begin{bmatrix} 0 \\ I_m \end{bmatrix} C_1^{-1}$$

Inspection of (8) shows that it uniquely and truly unifies the strictly proper and proper observer cases, and is truly a set of linear equations with unknown parameters K_z (which fully represents the freedom of (2.b)) and $\mathbf{c}_i$'s (which fully represent the remaining freedom of (2.a) or (3)) on the same side of the equation.

It is obvious that the simplification to (8) is uniquely enabled by the solution (5-7) of (3).

3 The Unified, General, and Lowest Possible Observer Order Bound

As analyzed in Sections 1 and 2, only from (4) or (8) and only when (8) is computed independently, can the observer order reduction be determined really systematically and generally.

The algorithm of [11] uniquely and fully used the free parameters $\mathbf{c}_i$ to satisfy (8). As analyzed in [12], this algorithm guarantees that the observer order r satisfies

$$1 \leq r \leq \min\{n, v_1+...+v_p\}, \text{ if } K_z=0 \quad (9.a)$$

$$\text{and } 0 \leq r \leq \min\{n\text{-}m, (v_1\text{-}1)+...+(v_p\text{-}1)\}, \text{ if } K_z \neq 0 \quad (9.b)$$

where the v_i's are the plant observability indexes in descending order ($v_1+...+v_m = n$). This is lower than the existing general upper bound $\min\{n, pv_1\}$ and $\min\{n\text{-}m, p(v_1\text{-}1)\}$ [7] because the v_i's are in descending order.

We will show in the following and based on (8) alone, that (9) is also the completely unified and the lowest possible bounds of observer order r. As analyzed in Section 1, r is the lowest possible number of rows of T needed to satisfy (4).

From (8), 1 and 0 of (9) are indeed the lowest possible lower bound of r because K cannot be 0 while $\underline{K}$ can in (8.a) and (8.b), respectively. These two lower bounds are achievable whenever K is a linear combination of the rows of D_1 and of C in (8.a) and (4), respectively.

From (8), the observer order can reach n or n-m regardless of the values of D_i matrixes (or of T) because K can be I (state observer case). However, when the number of rows of K, p, is less than n, r may be lower than n or n-m and can be as low as its lower bound (1 or 0) as shown in Section 1 and in the previous paragraph. More important, when p is less than m, then r is guaranteed to be bounded by $v_1+...+v_p$ or $(v_1\text{-}1)+...+(v_p\text{-}1)$ which is always lower than n or n-m, respectively. Thus the function observer and its special state observer case, and the upper and lower bounds of r, are completely unified by (9).

From (8), the single-output plant case and the multi-output plant case are also completely unified. For single-output plant, which has m = 1 and v_1 = n, the two terms of the upper bounds of (9.a) and (9.b) become a unified n and n-1, respectively. As m is increased compared to p, or as the plant output observation information is increased and the number of state feedbacks to be estimated is decreased, the second term of the upper bound of (9.a) and (9.b), $v_1+...+v_p$ and $(v_1\text{-}1)+...+(v_p\text{-}1)$ respectively, become gradually lower than the respective first term n ($=v_1+...+v_m$) and n-m ($=(v_1\text{-}1)+...+(v_m\text{-}1)$). Hence not only the single and multiple output plant cases, but also the two terms of the upper bound of (9) are completely unified.

It should be noticed that the existing general observer order upper bound pv_1 and $p(v_1\text{-}1)$ cannot fit into this unification.

The complete unification of the function observer and state observer cases (for K arbitrary and K = I) and of the single-output and multi-output plant cases (for m = 1 and m > 1), as simply described in the previous two paragraphs, also clearly demonstrate that the upper bounds of (9) are the lowest possible.

4 The Significance of This Order Reduction

Section 3 shows that the design of [11], which is uniquely simplified to the solving of (8) only, can uniquely guarantee that the observer order be generally and systematically designed to reach its lower bound (1 or 0) when possible and be guaranteed to be limited by its upper bound of (9).

These bounds are the lowest possible.

This section will emphasize that the observer order upper bound (9) can be very significantly lower than the prevailing state observer order n or n-m. In addition, the practical significance of this analytical and general observer (or feedback controller) order reduction cannot be discounted by the newly developed computer numerical computation capability.

For the simplicity of presentation, we will consider the strictly proper observer case ($K_z = 0$) only.

As shown in Section 3, the upper bound $v_1+...+v_p$ of observer order r is always lower than n whenever m > p. For n » m » p and for evenly valued observability indexes v_i, which is very common in practice, it is obvious that this upper bound can be significantly lower than n

For example, in a circuit system with 100 capacitors, 10 voltage or current meters, and 2 controlled voltage or current inputs (n = 100, m = 10, and p = 2), and suppose $v_1 =...= v_{10} = 10$ ($v_1+...+v_m = n$), then the observer order (9) of the design of [11] can be guaranteed to be no higher than $v_1 + v_p = 20$, which is significantly lower than n = 100.

The controller order reduction from 100 to 20 can hardly be discounted, even by today's computer computational capability. It should be noted that practical problems are usually ill conditioned numerically. In such problems, even today's super-computer cannot compute accurately a 100-th order controller.

If the digital simulation of a 20-th order controller was formally impossible until now, then the significance of the above 100-to-20-th order reduction is feasible because of the new computer computation capability

In fact, the general and analytical design result such as (9) simply cannot be discounted by numerical computation capability, no matter how powerful this capability is. For example, the above 100-to-20-th order reduction can simply be a 1000-to-20-th order reduction, if the parameters n and m of that example are changed to 1000 and 100, respectively (v_i=10, i=1,...,100).
The fact is, high observer order has been considered as a major drawback of state space control theory for years, and the minimal order observer design has been tried by researchers for years. Hence the not fully successful past attempts of this task should not be a reason to discount the significance of this task, which is to design minimal order observer simply, generally and systematically and to achieve a generally guaranteed low observer order. This is especially true when such a task is already successfully accomplished by the design of [11].

5 Conclusion & Additional Significance

Even more significant than the claims of Sections 3 and 4, this paper also asserts the distinct design approach of [11]. That is to simplify the design problem to (4) or (8) only. The actual numerical methods for solving (8) for the lowest possible number of rows of T may have room for improvement (although the bounds (9) of that number are already the lowest possible). But this design approach is the only right approach to minimal order observer design. This is proved convincingly by the basic analysis, the design procedure, and the final results, of the first three sections of this paper.

This distinct design approach of observer/feedback controller has additional significance other than order reduction. From Section 1, the basic advantage of this design approach is at the full exploration of the common fact that the number (p) of state feedback controls is less than n in (4). This is the only significant fact for the improvement of observer design, and the only fact which makes the observer order reduction possible.

This fact is currently over looked -- the prevailing and existing state observer design always require the satisfaction of (3) and $|C|\neq 0$ together or the satisfaction of (3) and (4) for arbitrarily given K together. The consequence is the unsatisfactory design result to the additional and critical observer design requirements such as the failed the realization of robustness properties of state feedback control (LTR) [13, 14]. The existing LTR result, which is based on state observers, is invalid to most plants (nonminimum-phase or rank(CB)< p or m < p), while a new result which fully uses this distinct fact is valid for most plants (all plants with m > p and almost all plants with m = p) [13, 14].

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