PRESHAPING COMMAND INPUTS FOR EXPLICIT FRACTIONAL DERIVATIVE SYSTEMS: APPLICATION TO CRONE CONTROL

A. Poty, P. Melchior, F. Levron, B. Orsoni and A. Oustaloup

LAP - UMR 5131 CNRS
Université Bordeaux 1 - ENSEIRB
351, cours de la Libération, F33405 Talence cedex, France
Phone: +33 (0)540 006 607 - Fax: +33 (0)540 006 644
Email: melchior@lap.u-bordeaux.fr - URL:http:\\www.lap.u-bordeaux.fr

Keywords: Robotics, motion control, input-shaping, preshaping, path tracking, optimal control, fractional systems, Davidson-Cole filter

Abstract

Shaping command input or preshaping is used for reducing system vibration in motion control. Desired systems inputs are altered so that the system finishes the requested move without residual vibration. This technique, developed by N.C. Singer and W.P. Seering, is used for example in the aerospace field, in particular in flexible structure control. This paper presents the extension of this method for explicit fractional derivative systems (generalized derivative systems). A robustness study is then presented, and preshaping is applied to improve second generation CRONE control (explicit fractional derivative transfer function) response time. Shaper coefficients are calculated in real time using properties of explicit fractional derivative systems and results in simulation are given.

1 Introduction

Developing the control of a process that carries out a task results from two stages of design: *control synthesis* and *path planning*. Path planning is divided into two parts: *path generation* and *motion control*. By means of command inputs, path planning determines how to follow the path depending on the desired performances and physical constraints of the actuators. It is then necessary to determine an algorithm able to calculate command inputs for feedback control systems, while minimising the response time as well as the residual oscillations.

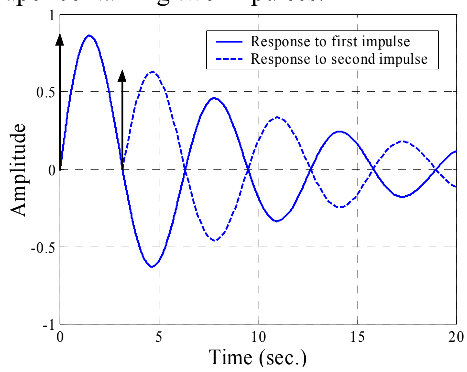
The CRONE team developed a solution based on an implicit fractional derivative filter [6][10]. A study compared this approach with traditional prefilters and Bang-Bang laws [7]. Another approach based on shaping command inputs was developed by N.C. Singer and W.P. Seering [12]. Shaping command input or preshaping is used to control the little-damped modes, and aims to eliminate residual vibration. It was applied to second order systems. Systems of order higher than 2 can always be separated into a cascade of order 2 cells [11]). The shaping command input technique consists in the convolution between step function and impulse sequence. The impulse amplitudes and their instant in time are calculated according to constraints.

This paper presents the extension of this method for explicit fractional derivative systems, robustness study and application to CRONE control. The final goal is to improve the second generation CRONE control (explicit fractional derivative transfer function) response time, applying these results [4]. Section 2 summarises Singer and Seering's preshaping method. In section 3, the shaper synthesis is applied to explicit fractional derivative systems. Section 4 presents the robustness study of the method and its application to second generation CRONE control. Shaper coefficients are calculated in real time using properties of explicit generalized derivative systems and results in simulation are given. Finally, section 5 presents main results of the method and some prospects.

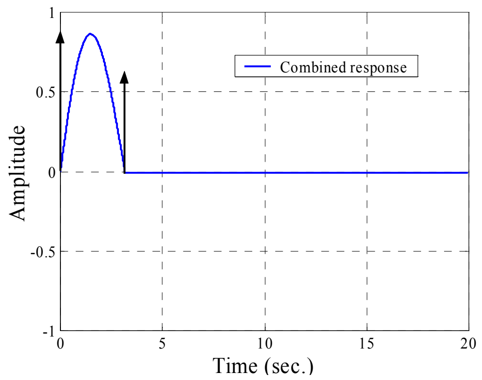
2 Shaping command input principle

Shaping input is obtained by convolving desired input with an impulse sequence. The amplitudes and instants of application constitute the shaper coefficients. The goal of the shaper synthesis is to calculate impulse amplitudes and instants of application, so that the shaping reference variable reduces or cancels the harmful effects of the mechanical system flexibility or the control law resonance.

Figure 1 represents the convolution between a step function and a shaper containing two impulses.



(a) Impulse response of a second-order system to an impulse (-), and to a delayed impulse (--)



(b) Combined impulse response

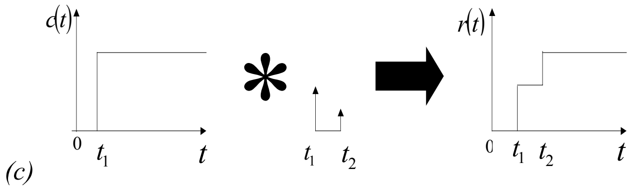


Fig. 1: Convolution between unit step function and a shaper with two impulses to produce a step command input

Shaper synthesis is obtained according constraint equations [12]. The system is:

$$\left\{\begin{array}{l}\displaystyle\sum_{j=1}^{N} A_j\left(t_j\right)^q e^{-\xi\omega\left(t_N-t_j\right)}\sin\left(t_j\omega\sqrt{1-\xi^2}\right)=0\\ \displaystyle\sum_{j=1}^{N} A_j\left(t_j\right)^q e^{-\xi\omega\left(t_N-t_j\right)}\cos\left(t_j\omega\sqrt{1-\xi^2}\right)=0\end{array}\right. . \qquad (1)$$

Resolution of (1) gives shaper called Zero Vibration(Derivative)^q or $\mathrm{ZV(D)}^q$ with $q \in R$. Shaper is thus the convolution between the step function and (q+2) impulses. However the disadvantage is: the higher the pulse number of the shaper, the more significant the response time. This is the robustness-speed compromise.

3 Shaping command input for explicit fractional derivative systems

3.1 Introduction

Singer and Seering's method for integer systems is now extended to explicit fractional derivative systems.

In the operational field, a fundamental system is called *explicit fractional derivative* [9] when its transfer function is described by:

$$F(s)=\frac{1}{1+(\tau s)^n} \text{ with } \quad \tau \in R \quad \text{and} \quad n \in C . \qquad (2)$$

The unit pulsed response given by (2) is:

$$y(t)=L^{-1}\left[\frac{1}{1+(\tau s)^n}\right] \text{with} \quad \tau \in R \quad \text{and} \quad n \in C . \qquad (3)$$

Calculation of the inverse transform of (3) is by integration of multiform functions by the residues [9]:

$$\begin{array}{l} y(t)=\left\{\dfrac{\tau^n \sin n\pi}{\pi}\displaystyle\int_0^{\infty}\frac{x^n e^{-xt}dx}{1+2(\tau x)^n\cos n\pi+(\tau x)^{2n}}\right\}u(t)\\ -\left\{\dfrac{2}{n}\tau^{-1}e^{t\tau^{-1}\cos\frac{\pi}{n}}\cos\left(t\tau^{-1}\sin\dfrac{\pi}{n}+\dfrac{\pi}{n}\right)\right\}u(t)\end{array} . \qquad (4)$$

Order n belongs to $\left]1,2\right[$. In this range, the transient is a positive damping, thus the system is stable. $y(t)$ in (4) results from two distinct response elements [8]:

- the first part is the stable aperiodic multimode
- the second part is a dominant oscillatory mode. It can be used to represent the control response. This is due to the explicit presence of two combined complex poles.

This oscillatory mode is robust because its damping ratio ξ is exclusively related to control order n, therefore independent of the transitional frequency.

Figure 2 represents the decomposition step response of an explicit fractional derivative system.

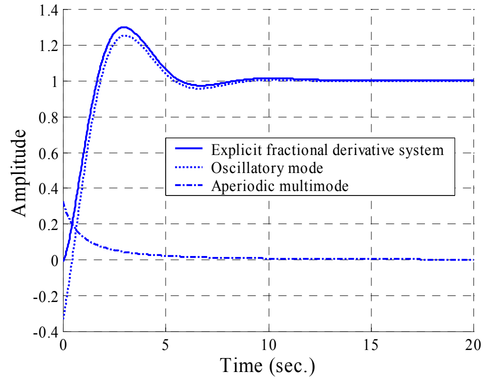


Fig. 2: Unit decomposition step response of an explicit fractional derivative system for n=1.5 and τ=1sec,(—)step response;(---) aperiodic multimode; (...) dominant oscillatory mode

Note: The greater n is, the quicker the aperiodic multimode will tend towards zero.

Considering that the peak on the oscillatory and global explicit generalized derivative curves do not coincide sufficiently well, two approaches are applied to Singer and Seering's method [12]:

- a temporal study on the oscillatory part
- a temporal study on the system's global response (4).

3.2 Shaper synthesis on the oscillatory part of the explicit fractional derivative system

This approach consists in determining the shaper synthesis on the oscillatory part (second order), then to apply the shaped command input thus obtained on the explicit fractional derivative system. This is a direct application of Singer and Seering's method.

According to the pulsed response expression (4), the oscillatory part is:

$$s_{osc}(t)=-\left\{\frac{2}{n}\tau^{-1}e^{t\tau^{-1}\cos\frac{\pi}{n}}\cos\left(t\tau^{-1}\sin\frac{\pi}{n}+\frac{\pi}{n}\right)\right\} . \qquad (5)$$

This is a direct application of Singer and Seering's method, and system of constraint equations is (with $q \in R$):

$$\left\{\begin{array}{l}\displaystyle\sum_{j=1}^{N} A_j(t_j)^q e^{\left(t_N-t_j\right)\tau^{-1}\cos\frac{\pi}{n}}\cos\left(t_j\tau^{-1}\sin\frac{\pi}{n}+\frac{\pi}{n}\right)=0\\ \displaystyle\sum_{j=1}^{N} A_j(t_j)^q e^{\left(t_N-t_j\right)\tau^{-1}\cos\frac{\pi}{n}}\sin\left(t_j\tau^{-1}\sin\frac{\pi}{n}+\frac{\pi}{n}\right)=0\\ \displaystyle\sum_{j=1}^{N} A_j=1\end{array}\right. . \qquad (6)$$

3.2.1 *ZV shaper*

By solving (6) for N=2, ZV shaper amplitudes and instants of application are:

$$\begin{cases} t_1 = 0 \quad , \quad A_1 = \dfrac{1}{1+e^{\pi\frac{\cos\pi/n}{\sin\pi/n}}} \\ t_2 = \dfrac{\pi}{\tau^{-1}\sin\dfrac{\pi}{n}} \quad , \quad A_2 = \dfrac{e^{\pi\frac{\cos\pi/n}{\sin\pi/n}}}{1+e^{\pi\frac{\cos\pi/n}{\sin\pi/n}}} \end{cases} . \qquad (7)$$

Note: A_i amplitudes are independent of τ .

Figure 3 represents the ZV shaper response hold for an explicit fractional derivative system for n=1.7 and τ=1sec.

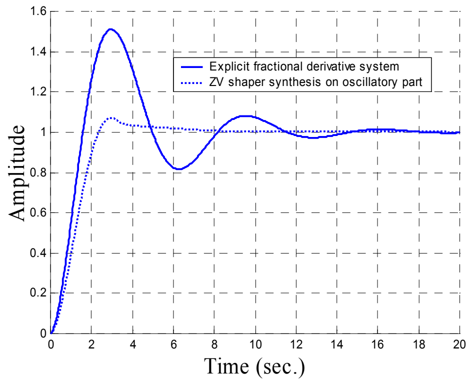


Fig. 3: Explicit fractional derivative system response to ZV shaper for n=1.7 and τ=1 sec

Contrary to the Singer and Seering's method applied to a perfectly known system, here a residual overshoot is always present. As calculations are carried out only on the system's oscillatory part, the oscillations due to (5) are perfectly cancelled following the application of the second impulse. However the aperiodic multimode, induced by the first then the second impulse, introduces a residual overshoot (the closer n is to 2, the smaller is the overshoot).To cancel the remaining overshoot, a prefilter could be introduced, for example Davidson-Cole [3]. Here the robustness of the ZVD has been chosen, along with the synthesis of the shaper starting from the oscillatory part of the explicit fractional derivative system.

3.2.2 *ZVD shaper*

The ZVD shaper is:

$$\begin{cases} t_1 = 0 \quad , \quad A_1 = \dfrac{1}{1+2K+K^2} \\ t_2 = \dfrac{\pi}{\tau^{-1}\sin\dfrac{\pi}{n}} \quad , \quad A_2 = \dfrac{K}{1+2K+K^2} \\ t_3 = \dfrac{2\pi}{\tau^{-1}\sin\dfrac{\pi}{n}} \quad , \quad A_3 = \dfrac{K^2}{1+2K+K^2} \\ \quad \text{with} \quad K = e^{\pi\frac{\cos\pi/n}{\sin\pi/n}} \end{cases} . \qquad (8)$$

Figure 4 represents the explicit fractional derivative system response to the ZVD shaper for n=1.7 and τ=1sec.

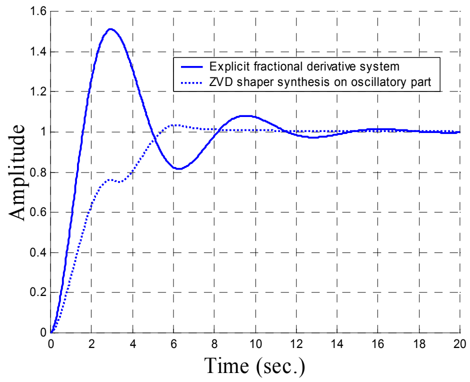


Fig. 4: Explicit fractional derivative system response to ZVD shaper with n=1.7 and τ=1 sec

As the shaper is not applied to the step response peak, but to the oscillatory curve peak, the robustness of the ZVD shaper to errors due to parametric variations is thus used, in order to cancel the residual overshoot. ZVD shaper synthesis on the oscillatory part, and its application on an explicit fractional derivative system, decreases the residual vibration. However it is not eliminated, and the response time is increased. The introduction of a ZVDD shaper would further reduce the residual vibration, but the response time would be increased.

3.3 Shaper synthesis on the global response of an explicit fractional derivative system

In this approach, the goal is to obtained the first impulse instant of the shaper more precisely. Thus calculation are done on the global response of an explicit fractional derivative system and not only on the oscillatory part. Taking into account the complexity of the equation (4), this unit pulsed response will be used [9]:

$$y_{imp}(t) = \sum_{i=0}^{\infty} \frac{(-1)^i}{\tau^{n(i+1)}} \frac{t^{n(i+1)-1}}{\Gamma(n(i+1))}.u(t) . \qquad (9)$$

Thus the step response is:

$$y_{ind}(t) = \sum_{i=0}^{\infty} \frac{(-1)^i}{\tau^{n(i+1)}} \frac{t^{n(i+1)}}{\Gamma(n(i+1)+1)}.u(t) . \qquad (10)$$

From (9), shaper amplitudes and instants of application can be calculated.

3.3.1 *Analytical expression*

- 1st approach:

From equation (9), the instants corresponding to the zeros of the pulsed response can be expressed. The first instant of application of the shaper is always taken equal to zero.

The first zero of the pulsed response is calculated using H.J. Hamilton's method [3]:

$$t_2 = -\left(\lim_{m\to\infty} \frac{a_0 R_{m-1}}{R_m}\right) \quad \text{with} \quad \begin{cases} R_m = \sum_{p=1}^{m} (-1)^{(p-1)} a_p a_0^{(p-1)} R_{m-p} \\ R_0 = 1 \end{cases} . \qquad (11)$$

$$\text{and} \quad \sum_{k=0}^{\infty} a_k x^k = 0 \quad \text{with} \quad a_k = \frac{(-1)^k}{\Gamma((k+1)n)}$$

Now, from this Hamilton method, and given that:

$$F(\tau s) \xrightarrow{\text{Laplace}^{-1}} f\left(\frac{t}{\tau}\right), \qquad (12)$$

calculations are carried out for τ=1sec, they will be divided by τ for the general case $(\forall\, \tau)$.

Applying H.J. Hamilton method with τ=1 sec, the result is given in appendix (7.1).

- 2nd approach:

The above result is not practical. A different approach consists in developing a_k in (11) to a series with order 2. The expression of the second impulse instant $(\forall n \in [1,2], \forall \tau)$ is written then according to (11) and (12):

$$t_2 = \frac{1}{\tau}\Big(-22.90 + 16.38 * n + 18.65 * (n-2)^2 + 13.88 * (n-2)^3 + 9.20 * (n-2)^4 + 4.36 * (n-2)^5 + 2.33 * (n-2)^6 + 0.62 * (n-2)^7 + 0.48 * (n-2)^8\Big)^{1/n}$$

This expression is much more practical, and is given with more significant digits in appendix (7.2).

3.3.2 Shaper amplitudes

The first overshoot can be formulated in two ways:

- by the reduced overshoot:

$$D = a.n^2 + bn + c \quad \text{with} \begin{cases} a = 79.195 \\ b = -138.507 \\ c = 59.528 \end{cases}, \qquad (14)$$

- or by introducing the instant of the zero of the impulse response (13) in the step response (10).

These two formulations are equivalent, as similar results are obtained in simulation. However thanks to its simplicity, the expression (14) is simpler and is thus preferred.

Figure 5 represents the explicit fractional derivative system response to a ZV shaper for n=1.7 and τ=1sec.

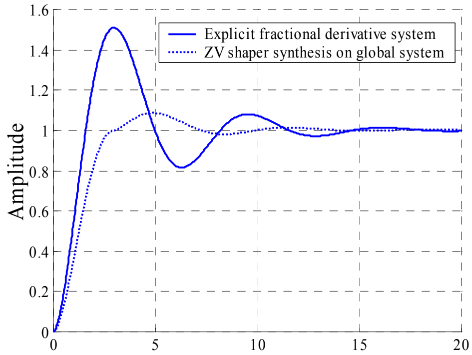


Fig. 5: Explicit fractional derivative system response to ZV shaper for n=1.7 and τ=1 sec

Using the shaper synthesis method, the new first overshoot is now smaller in accordance with the shaper principle on integer order systems. However the aperiodic multimode is still present, and the shaper no longer effects only the oscillatory part. Thus residual oscillations still appear after the second impulse. Thus a shaper with more impulses proves necessary.

The initial analytical calculation should thus be extended to include further pulsed response zeros. However for the moment this calculation remains impractical.

3.4 Comparison of the shaper synthesis methods

Figure 6 presents the explicit fractional derivative system response to the two shapers, ZV and ZVD. They are synthesised from the oscillatory part only, and also from the global response. The simulation uses: n=1.7 and τ=1 sec.

For the global response, the second approach, the exact analytic expression of the instant of the third impulse of the ZVD shaper is not known. It can be considered that $t_3 = 2 * t_2$ which is not exact, but a good approximation.

(13

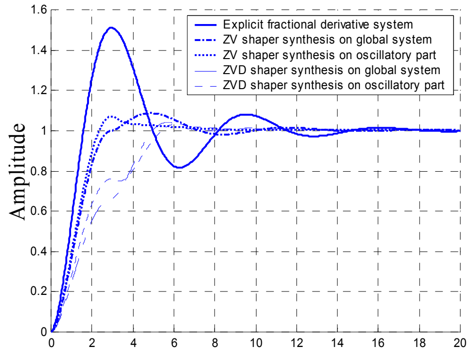


Fig. 6: Explicit fractional derivative system response to various shapers

A particular interest on the study of ZV shapers can be justified. Indeed, although ZV shapers oscillate more, they are faster. This is particularly the case on the global response where the desired command input at the second impulse is reached rapidly.

Also, ZVD shapers are more effective than ZV shapers to cancel residual vibration, but their drawback is the longer time response. It proves justified to simply assume that $t_3 = 2 * t_2$, because the oscillation achieved is reduced.

Lastly, the shapers used for the oscillatory part are easier to synthesise and program than those used for the global response. Also, not knowing even an approximation of the third zero expression of the pulsed response to an explicit fractional derivative system, a ZVDD shaper cannot be made. However it can be assumed that the inclusion of a fourth impulse in the shaper would further decrease residual oscillation and would increase response time.

According previous results, a robustness study is developed.

4 Robustness study of preshaper control applied to explicit fractional derivative systems

If natural frequency is uncertain, the shaper does not cancel the residual oscillation.

To increase the robustness of the input to variations of the system's natural frequency, shaper synthesis is calculated in real time using an explicit fractional derivative step response. Here, only ZV shaper synthesis is calculated.

4.1 Method

Three fundamental properties [8] of an explicit fractional derivative system are, for a fixed order n and variation of τ:

- first overshoot quasi-invariance (insensitivity of the damping ratio to parametric variations for a given range variation)
- similarity factor $(1/\tau^n)$ between the various step responses
- constant factor α between the instant of first inflection point (maximum of pulsed response) and of step response peak (first zero of pulsed response).

Figure 7 presents an explicit fractional derivative system step response for given n and a factor 9 for τ.

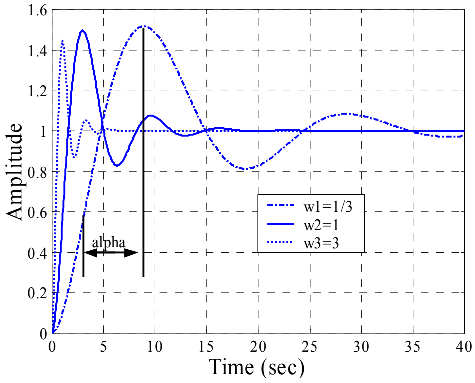


Fig. 7: Explicit fractional derivative system response for n=1.7 and τ=1sec (__), τ=3sec (--) and τ=1/3 sec(-.-)

Thus, for a such a system (n fixed) with variation of pulsation $(1/\tau)$, only the instants of application of the shaper can vary. Impulses amplitudes are always the same, according to the first property (expression (7) has amplitudes depending only on n). To synthesise a robust shaper in real time, it is thus necessary to calculate instants of application by exploiting the other two properties.

A particular instant must be expressed, for example the instant of the inflection point of the step response t_0. Next, as the analytical expression of factor α is known, it is easy to determine t_2, the instant of the step response peak.

To calculate factor α, two instants must be calculated:

- first t_0: the instant of the first point of inflection of the step response (maximum of the impulse response ⇔ first zero of the impulse response derivative)
- then t_2 (known): the instant of the step response peak (first zero of the impulse response).

Hamilton's method is used to determine the first instant:

$$\frac{ds_{imp}(t)}{dt} = \sum_{i=0}^{\infty} \frac{(-1)^i}{\tau^{n(i+1)}} \frac{(n(i+1)-1)t^{n(i+1)-2}}{\Gamma(n(i+1))}.u(t) \cdot \qquad (15)$$

Analytical expressions of the two instants are:

$$\begin{cases} t_0 = \frac{1}{\tau}\Big(-9.18+5.82*n+6.661070*(n-2)^2+5.57*(n-2)^3+3.57*(n-2)^4+ \\ \quad 1.92*(n-2)^5+0.88*(n-2)^6+0.35*(n-2)^7+0.13*(n-2)^8\Big)^{1/n} \\ t_2 = \frac{1}{\tau}\Big(-22.90+16.38*n+18.65*(n-2)^2+13.88*(n-2)^3+ \\ \quad 9.20*(n-2)^4+4.36*(n-2)^5+2.33*(n-2)^6+0.62*(n-2)^7+0.48*(n-2)^8\Big)^{1/n} \end{cases} . \qquad (16)$$

From (16), the analytical expression of α is:

$$\alpha = \frac{t_2}{t_0} = 2*\frac{\begin{array}{l}-0.25e^{19}*n+0.23e^{19}*n^4-0.39e^{19}*n^3-0.95e^{18}*n^5+0.41e^{19}*n^2 \\ -0.35e^{17}*n^7+0.24e^{18}*n^6+0.24e^{16}*n^8+0.71e^{18}\end{array}}{\begin{array}{l}-0.57e^{18}*n+0.83e^{18}*n^4-0.12e^{19}*n^3-0.37e^{18}*n^5+0.11e^{19}*n^2 \\ -0.17e^{17}*n^7+0.10e^{18}*n^6+0.13e^{16}*n^8+0.12e^{18}\end{array}} . \qquad (17)$$

Expression (17) shows a significant property: factor α is independent of τ; it is constant for a given value of n.

More exact expressions of t_0, t_2 and α are given in the appendix (7.2).

Neither τ nor t_2 need to be known in advance. Factor α and impulse amplitudes are calculated for a given n value. t_0 is detected, thus t_2 can be determined in real time using relation $t_2 = \alpha t_0$.

4.2 Simulation

The t_0 is detected using an algorithm in Matlab. The robustness of the response to parameter τ variations is then simulated.

The method is now applied to second generation CRONE control whose main property is a constant damping ratio directly related to non integer order n [8][9]. The transfer function is an explicit fractional derivative system with derivation order n between 1.3 and 1.5.

Figure 8 shows the step responses obtained for n=1.5 and τ varying by a factor of 9.

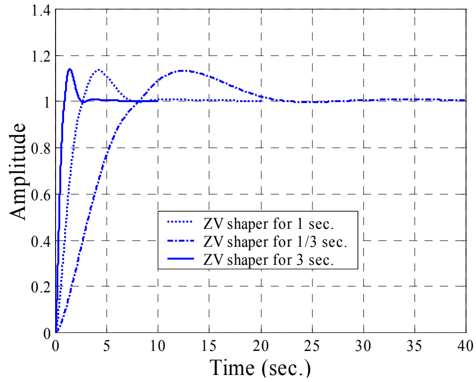


Fig. 8: CRONE control response to ZV shaper for τ varying by a factor of 9

These results are the same as in section 3. The method in real time is thus quite robust to variations of τ.

5 Conclusion

The overview of the Singer and Seering method (applicable only on second order systems) highlights its cancellation of the residual oscillations by using $ZV(D)^q$ shapers with $q \geq 0$, and amplitudes and instants of application are calculated from the impulse response. The disadvantage of this technique lies in the choice of q to counter parametric variations: the greater q is, the better the reduction in residual vibration, but the longer the response time.

The extension of this method for explicit fractional derivative systems is obtained through two approaches: one on the oscillatory part of the impulse response, and the other on the global impulse response of the system. The first approach, used on only the oscillatory part, is easier to synthesise thanks to simpler analytical expressions through direct application of Singer and Seering's method. The residual oscillation due to the oscillatory part are completely eliminated. However, because of the presence of aperiodic multimode, a residual overshoot persists. The second approach, used on the global impulse response, eliminates the first overshoot, in accordance with the shaper principle on integer order systems. Response time is also improved. However, because of the oscillations due to the oscillatory part, and because of the presence of the aperiodic multimode, a residual oscillation stay after the application of the second impulse.

Thanks to the three properties of explicit fractional derivative systems, a robust shaper is synthesised in real time. A constant factor α, for a given n, between the inflexion point and the step response peak is determined.

Simulations show an improvement of path tracking. Residual attenuation is decreased and response time is fast. Thus the preshaping method can be used efficiently with second generation CRONE control.

Shaping can thus uncouple the loop properties from tracking, for example by imposing the first overshoot of the step response. A single linear prefilter would be either non robust, or with a too long response time.

Thus a shaper including more impulses should now be designed. A method to determine a third instant of the shaper is by using a response to the original ZV shaper of an explicit fractional derivative system.

6 References

[1] M. Chossat. "Aide-mémoire de Mathématiques de l'ingénieur", *Editions Dunod*, , Paris, France, (1996).

[2] D.W. Davidson, R.H. Cole. "Dielectric relaxation in glycerol, propylen glycol and n-propanol", *J. Chem. Phys.*, **vol. 19**, (1951).

[3] H.J. Hamilton. "Roots of Equation by Fractional Iteration", *Duke math.*, J. 13, pp. 113-123, (1946).

[4] J. Kieffer, A.J. Cahill, M.R. James. "Robust and Accurate Time-Optimal Path-Tracking Control for Robot Manipulators", *IEEE Trans. On Robotics and Automation*, **vol. 13**, n°6, pp. 880-890, (1997).

[5] A. Le Mehaute, F. Heliodore, A. Oustaloup. "Cole-Cole relaxation and CRONE relaxation", *IMACS-IFAC symposium MCTS 91*, Lille, France, (1999).

[6] K.S. Miller, B. Ross. "An introduction to the fractional calculus and fractional differential equations", *Wiley-Interscience Publication*, (1993).

[7] B. Orsoni. "Dérivée généralisée en planification de trajectoire et génération de mouvement", *Thèse de doctorat, Université de Bordeaux 1*, (2002).

[8] A. Oustaloup. "La commande CRONE : Commande Robuste d'Ordre Non Entier", *Editions Hermès*, Paris, France, (1991).

[9] A. Oustaloup. "La dérivation non entière, théorie, synthèse et applications", *Editions Hermès*, Paris, France, (1995).

[10] S.G. Samko, A.A. Kilbas, O.I. Marichev. "Fractional Integrals and Derivatives", *Gordon and Breach Science Publishers*, (1993).

[11] N.C. Singer. "Residual Vibration Reduction in Computer Controlled Machines", *Thesis, Massachusetts Institute Of Technology*, January, (1989).

[12] N.C. Singer and W.P. Seering. "Preshaping Command Inputs to Reduce System Vibration", *Journal of Dynamic Systems, Measurement, and Control*, **Vol.112**, pp.76-82, (1990).

[13] W. Singhose, N. Singer, W. Seering. "Comparison of Command Shaping Methods for Reducing Residual Vibration", *3rd European Control Conference*, Rome, Italy, (1995).

7 Appendix

7.1 First zero analytical expression of the pulsed response to explicit fractional derivative systems

Application of the H.J. Hamilton method with τ=1sec gives the following result:

$$t_2 = \frac{\begin{array}{l}(10\Gamma(n)^4\Gamma(8n)\Gamma(7n)\Gamma(4n)^2\Gamma(3n)\Gamma(6n)\Gamma(2n)^4\Gamma(5n) - 4\Gamma(n)^3\Gamma(8n)\Gamma(7n)\Gamma(2n)^6\Gamma(4n)^2 \\ \Gamma(6n)\Gamma(5n) - 2\Gamma(2n)^7\Gamma(4n)^2\Gamma(n)\Gamma(7n)\Gamma(8n)\Gamma(5n)\Gamma(3n)^2 + 3\Gamma(2n)^7\Gamma(4n)\Gamma(n)^2\Gamma(7n) \\ \Gamma(8n)\Gamma(5n)\Gamma(6n)\Gamma(3n) - 2\Gamma(2n)^7\Gamma(n)\Gamma(6n)\Gamma(3n)^3\Gamma(7n)\Gamma(8n)\Gamma(4n) + \Gamma(2n)^7\Gamma(4n)^2 \\ \Gamma(5n)\Gamma(6n)\Gamma(3n)^3\Gamma(7n) + \Gamma(n)^6\Gamma(8n)\Gamma(7n)\Gamma(3n)^3\Gamma(4n)^2\Gamma(6n)\Gamma(5n) + 5\Gamma(n)^4\Gamma(8n) \\ \Gamma(7n)\Gamma(3n)^3\Gamma(4n)\Gamma(6n)\Gamma(2n)^3\Gamma(5n) - 4\Gamma(n)^3\Gamma(8n)\Gamma(7n)\Gamma(3n)^3\Gamma(4n)^2\Gamma(2n)^4\Gamma(6n) + \\ 3\Gamma(n)^2\Gamma(8n)\Gamma(7n)\Gamma(3n)^3\Gamma(2n)^5\Gamma(4n)^2\Gamma(5n) + 3\Gamma(n)^2\Gamma(8n)\Gamma(7n)\Gamma(3n)^3\Gamma(2n)^6\Gamma(5n) \\ \Gamma(6n) - 6\Gamma(n)^5\Gamma(8n)\Gamma(7n)\Gamma(3n)^2\Gamma(4n)^2\Gamma(6n)\Gamma(5n)\Gamma(2n)^2 - 12\Gamma(n)^3\Gamma(8n)\Gamma(7n)\Gamma(3n)^2 \\ \Gamma(4n)\Gamma(2n)^5\Gamma(5n)\Gamma(6n) - 2\Gamma(2n)^6\Gamma(4n)^2\Gamma(n)\Gamma(5n)\Gamma(6n)\Gamma(3n)^3\Gamma(8n) + 6\Gamma(n)^2\Gamma(8n) \\ \Gamma(7n)\Gamma(3n)^2\Gamma(2n)^6\Gamma(4n)^2\Gamma(6n))\Gamma(2n)\Gamma(5n)\Gamma(3n)\Gamma(9n)\end{array}}{\begin{array}{l}(2\Gamma(2n)^8\Gamma(n)\Gamma(5n)^2\Gamma(3n)^4\Gamma(7n)\Gamma(8n)\Gamma(9n)\Gamma(4n) + \Gamma(2n)^8\Gamma(4n)^2\Gamma(n)\Gamma(6n)\Gamma(3n)^4 \\ \Gamma(7n)\Gamma(8n)\Gamma(9n) + 15\Gamma(n)^5\Gamma(5n)^2\Gamma(3n)^2\Gamma(9n)\Gamma(8n)\Gamma(7n)\Gamma(4n)^2\Gamma(6n)\Gamma(2n)^4 - \\ 10\Gamma(n)^4\Gamma(5n)^2\Gamma(3n)\Gamma(9n)\Gamma(8n)\Gamma(7n)\Gamma(2n)^6\Gamma(4n)^2\Gamma(6n) - 3\Gamma(n)^2\Gamma(5n)^2\Gamma(3n)^4 \\ \Gamma(9n)\Gamma(8n)\Gamma(2n)^6\Gamma(4n)^2\Gamma(6n) - 3\Gamma(2n)^8\Gamma(n)^2\Gamma(5n)^2\Gamma(8n)\Gamma(9n)\Gamma(7n)\Gamma(3n)^3\Gamma(6n) - \\ 3\Gamma(2n)^8\Gamma(n)^2\Gamma(5n)\Gamma(8n)\Gamma(9n)\Gamma(7n)\Gamma(3n)^2\Gamma(4n)^2\Gamma(6n) + \Gamma(2n)^8\Gamma(n)^3\Gamma(5n)^2 \\ \Gamma(8n)\Gamma(9n)\Gamma(7n)\Gamma(4n)^2\Gamma(6n) + 2\Gamma(2n)^8\Gamma(n)\Gamma(5n)^2\Gamma(8n)\Gamma(9n)\Gamma(4n)^2\Gamma(3n)^3\Gamma(6n) + \\ \Gamma(n)^7\Gamma(5n)^2\Gamma(3n)^4\Gamma(9n)\Gamma(8n)\Gamma(7n)\Gamma(4n)^2\Gamma(6n) + 6\Gamma(n)^5\Gamma(5n)^2\Gamma(3n)^4\Gamma(9n)\Gamma(8n) \\ \Gamma(7n)\Gamma(4n)\Gamma(6n)\Gamma(2n)^3 - 5\Gamma(n)^4\Gamma(5n)\Gamma(3n)^4\Gamma(9n)\Gamma(8n)\Gamma(7n)\Gamma(4n)^2\Gamma(2n)^4\Gamma(6n) + \\ 6\Gamma(n)^3\Gamma(5n)^2\Gamma(3n)^4\Gamma(9n)\Gamma(8n)\Gamma(7n)\Gamma(2n)^6\Gamma(6)n + 4\Gamma(n)^3\Gamma(5n)^2\Gamma(3n)^4\Gamma(9n) \\ \Gamma(8n)\Gamma(7n)\Gamma(2n)^5\Gamma(4n)^2 - 6\Gamma(n)^2\Gamma(5n)\Gamma(3n)^4\Gamma(9n)\Gamma(8n)\Gamma(7n)\Gamma(2n)^7\Gamma(6n)\Gamma(4n) - \\ 6\Gamma(n)^2\Gamma(5n)^2\Gamma(3n)^3\Gamma(9n)\Gamma(8n)\Gamma(7n)\Gamma(2n)^7\Gamma(4n)^2 - 20\Gamma(n)^4\Gamma(5n)^2\Gamma(3n)^3\Gamma(9n) \\ \Gamma(8n)\Gamma(7n)\Gamma(4n)\Gamma(2n)^5\Gamma(6n) + 12\Gamma(n)^3\Gamma(5n)\Gamma(3n)^3\Gamma(9n)\Gamma(8n)\Gamma(7n)\Gamma(2n)^6\Gamma(4n)^2 \\ \Gamma(6n) - 7\Gamma(n)^6\Gamma(5n)^2\Gamma(3n)^3\Gamma(9n)\Gamma(8n)\Gamma(7n)\Gamma(4n)^2\Gamma(6n)\Gamma(2n)^2 + 2\Gamma(2n)^7\Gamma(4n)^2 \\ \Gamma(n)\Gamma(5n)^2\Gamma(6n)\Gamma(3n)^4\Gamma(7n)\Gamma(9n) - \Gamma(2n)^8\Gamma(4n)^2\Gamma(5n)^2\Gamma(6n)\Gamma(3n)^4\Gamma(7n)\Gamma(8n) + \\ 12\Gamma(n)^3\Gamma(5n)^2\Gamma(3n)^2\Gamma(9n)\Gamma(8n)\Gamma(7n)\Gamma(2n)^7\Gamma(4n)\Gamma(6n)\end{array}}$$

7.2 Analytical expression of pulsed response zeros

Instant t_0, the first zero of pulsed response derivative, is:

$$\begin{array}{l} t_0 = \left(-9.186185669387866 + 5.826793376339927 * n + 6.661070581812251 * (n-2)^2 + \right. \\ 5.573944136113842 * (n-2)^3 + 3.570509852818514 * (n-2)^4 + 1.922032225128115 * (n-2)^5 \\ + 0.8840609213371945 * (n-2)^6 + 0.3577272224636185 * (n-2)^7 \\ \left. + 0.1303944637606392 * (n-2)^8 \right)^{1/n} \end{array}$$

Instant t_2, the first step response peak, is:

$$\begin{array}{l} t_2 = \left(-22.90924356280700 + 16.38936760825690 * n + 18.65242688540431 * (n-2)^2 + \right. \\ 13.88732652470402 * (n-2)^3 + 9.202584058895127 * (n-2)^4 + 4.365735270691791 * (n-2)^5 \\ + 2.332088876877044 * (n-2)^6 + 0.6236486043071402 * (n-2)^7 + \\ \left. + 0.4883370091201022 * (n-2)^8 \right)^{1/n} \end{array}$$

Factor $\alpha = \frac{t_2}{t_0}$ is then:

$$\alpha = 2 * \frac{\begin{array}{l} -.2526099384179956e^{19} * n + .2388932024865575e^{19} * n^4 - .3920474286751685e^{19} * n^3 \\ -.9500391428791937e^{18} * n^5 + .4113205260973177e^{19} * n^2 - .3594871770807248e^{17} * n^7 \\ .2414737671901426e^{18} * n^6 + 2441685045600511e^{16} * n^8 + .7129030458246292e^{18} \end{array}}{\begin{array}{l} -.5742572872142102e^{18} * n + .8327202000387176e^{18} * n^4 - .1208982275746999e^{19} * n^3 \\ -.3705433190874063e^{18} * n^5 + .1105958709069648e^{19} * n^2 - .1728584197706609e^{17} * n^7 \\ +.1048005974803813e^{18} * n^6 + 1303944637606392e^{16} * n^8 + .1266146722290786e^{18} \end{array}}$$