AIRCRAFT CONFLICT DETECTION: A METHOD FOR COMPUTING THE PROBABILITY OF CONFLICT BASED ON MARKOV CHAIN APPROXIMATION

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Abstract

We study the automated aircraft conflict detection problem. Specifically, we introduce a method for estimating the probability of conflict for two-aircraft encounters at a fixed altitude. The spatial correlation between the wind perturbations to the aircraft positions is taken into account. The procedure used to estimate the probability of conflict is based on the introduction of a Markov chain approximation of the stochastic process describing the relative position of the aircraft.

1 Introduction

In this paper, we deal with the problem of evaluating the criticality of an encounter situation where two aircraft are flying at the same altitude in a certain region of the airspace, each one trying to follow a prescribed flight plan. The encounter situation is said to generate a "conflict" if the two aircraft get closer than a minimum allowed distance, i.e., 5 nautical miles (nmi) in en-route airspace and 3 nmi in airspace close to the airports. The procedure used to prevent the occurrence of a conflict typically consists of two phases, namely, i) aircraft conflict detection, and ii) aircraft conflict resolution. Automated tools are currently being studied to support the air traffic controllers (ATCs) in performing these tasks. A comprehensive overview of the methods proposed in the literature can be found in [10]. In automated conflict detection, models for predicting the aircraft future positions are introduced and the possibility that a conflict would happen within a certain time horizon is evaluated based on these models ([13]-[14]). If a conflict is predicted, then a modification of the aircraft flight plans is determined in the conflict resolution phase. The objective is to avoid the predicted conflict from actually occurring, while taking into account the cost of the resolution action in terms of delays, fuel consumption, deviation from the previously planned itinerary, etc. ([2]-[6]).

In this paper we focus on the conflict detection issue and propose a probabilistic approach to this problem. We describe the aircraft motion as the solution to a stochastic differential equation and quantify the criticality of a two-aircraft encounter by means of the probability that a conflict occurs within a certain time horizon: the higher the probability of conflict, the more critical the two-aircraft encounter situation. The probabilistic methods proposed in the literature to compute the probability of conflict are generally based on a simple description of the aircraft future positions originally proposed in [13]. Specifically, the two aircraft positions are described as uncorrelated Gaussian random processes whose variance grows in time. However, the uncorrelation assumption does not seem realistic and could lead to erroneous estimations in this critical application. This is because wind is the main source of uncertainty and it causes a correlation between the aircraft positions that becomes stronger as the aircraft get closer.

Inspired by [11], we introduce a two-aircraft system model which takes into account the possible correlation between the aircraft positions due to the presence of wind. We assume that each aircraft is trying to follow a flight plan (i.e., to fly at constant speed along the straight lines between given timed waypoints), while its actual motion may deviate from it because of different sources of uncertainty. In particular, we assume that the errors in tracking the flight plan are mainly due to the wind that affects the aircraft velocity. This leads to a compromising model between the two conflicting objectives of being realistic and at the same time still simple enough for the problem to be computationally tractable. The inspiring model proposed in [11] is certainly interesting but too complicated as a basis to conceive a method for computing the probability of conflict.

The computation of the probability of conflict for the model proposed in this paper is based on the introduction of a Markov chain whose state space is obtained by gridding the region of the airspace where the encounter takes place. By appropriately choosing its transition matrix, the Markov chain can be guaranteed to converge weakly to the stochastic process modeling the aircraft relative position as the grid size approaches zero. The probability of conflict can then be approximated by the corresponding quantity associated with the Markov chain, which is much easier to compute. A procedure for computing the probability of conflict map (i.e., the probability of conflict as a function of the current relative position of the two aircraft) is described. This procedure is based on the assumption that the wind correlation structure is homogeneous and isotropic in space, i.e., it depends only on the aircraft distance. As expected, the computations show that the wind correlation effect cannot be neglected when estimating the probability of conflict.

The paper is organized as follows. In Section 2 we describe the two-aircraft model, and in Section 3 we define the problem of determining the probability of conflict based on this model. In Section 4, the Markov approximation scheme for estimating the probability of conflict is presented. An example is given in Section 5. Finally, some conclusions are drawn in Section 6.

2 Two-aircraft model

We consider an encounter where two aircraft are flying at the same altitude in a certain bounded open region U of the airspace. We refer to the two aircraft as "aircraft 1" and "aircraft 2". Since the aircraft are flying at the same altitude, the airspace can be identified with $\mathbb{R}^2$, hence $U \subset \mathbb{R}^2$.

We focus our attention on a certain time horizon $T = [0, t_f]$, where 0 denotes the current time instant and t_f is a positive real number (possibly infinity) representing the look-ahead time horizon. We assume that each aircraft tries to follow a certain flight plan during the time interval T starting from its current position, where the flight plan for aircraft i is defined through a velocity function $u_i : T \to \mathbb{R}^2$. So at any time $t \in T$, the plan of aircraft i is to fly at the constant speed $\|u_i(t)\|$ along a straight line with direction defined by $u_i(t)$. Typically, the velocity functions u_1 and u_2 are piecewise constant, modeling the fact that the aircraft are generally trying to follow piecewise linear motions specified by a series of timed way-points.

We start by describing the most general model. Due to the presence of the wind perturbation, the actual velocity of aircraft i at time $t \in T$ is the sum of $u_i(t)$ and an additional term representing the wind disturbance. The wind contribution can be further decomposed into two components: i) a deterministic term representing the nominal wind velocity, which is assumed to be available to ATC through measurements; and ii) a stochastic term representing the effect of air turbulence and errors in the wind speed measurements.

Denote by X_i the position of aircraft i. Then, X_1 and X_2 are governed by the stochastic differential equations:

$$\begin{aligned} dX_1(t) = &\quad f(X_1,t)dt + u_1(t)dt + g(X_1,t)dB(X_1,t), && (1)\\ dX_2(t) = &\quad f(X_2,t)dt + u_2(t)dt + g(X_2,t)dB(X_2,t), && (2) \end{aligned}$$

where $f : U \times T \to \mathbb{R}^2$ is a time-varying vector field on U such that $f(x,t)$ is the nominal wind velocity at position x at time t. $B(\cdot,\cdot)$ is a time varying random field on U modeling (the integral of) air turbulence perturbations on the aircraft velocity, and is specified by the following properties:

- for each $x \in U$, $B(x,t)$ is a standard two dimensional Brownian motion;
- $B(x,t)$ is time increment independent. This implies, in particular, that the collections of random variables $\{B(x,t_2) - B(x,t_1)\}_{x \in U}$ and $\{B(x,t_4) - B(x,t_3)\}_{x \in U}$ are independent for any $t_1 \le t_2 \le t_3 \le t_4$;
- for any $t_1 \le t_2$, $\{B(x,t_2) - B(x,t_1)\}_{x \in U}$ is an (uncountable) collection of Gaussian random variables with zero mean and covariance

  $$\begin{aligned} &E\big\{[B(x,t_2) - B(x,t_1)][B(y,t_2) - B(y,t_1)]^T\big\}\\ &= h(\|x-y\|)(t_2 - t_1)I_2,\ \forall x, y \in U, \end{aligned}$$

  where I_2 is the 2-by-2 identity matrix, and $h : \mathbb{R} \to \mathbb{R}$ is a continuous, decreasing function with $h(0) = 1$ and $h(\infty) = 0$. In addition, h has to be non-negative definite in the sense that the k-by-k matrix $[h(\|x_i - x_j\|)]_{i,j=1}^k$ is non-negative definite for arbitrary $x_1, \dots, x_k \in \mathbb{R}^2$ and positive integer k. See [1] for further details.

$B(\cdot,\cdot)$ is in fact Gaussian, stationary (its finite dimensional distributions remain unchanged when the origin of $U \times T$ is changed), and isotropic in space (its finite dimensional distributions are not affected by a change of orthonormal coordinates in U). Finally, $g : U \times T \to \mathbb{R}^{2\times 2}$ is the term introduced to modulate the variance of the air turbulence perturbations.

We assume that f, u_1 and u_2 are known to ATC throughout T. Also, we suppose for simplicity that $g \equiv \sigma I_2$ for some constant $\sigma > 0$, so that equations (1) and (2) can be rewritten as

$$\begin{aligned} dX_1(t) &= f(X_1,t)dt + u_1(t)dt + \sigma dB(X_1,t), && (3)\\ dX_2(t) &= f(X_2,t)dt + u_2(t)dt + \sigma dB(X_2,t). && (4) \end{aligned}$$

Note that the assumption that $g \equiv \sigma I_2$ jointly with the assumption that $B(\cdot,\cdot)$ is isotropic implies that the effect of the wind stochastic components on the aircraft position X_i remains the same under any change of orthonormal coordinates of U.

3 Computing the probability of conflict

In this section, we describe a procedure for estimating the probability of conflict, i.e., the probability that one of the aircraft enters the protection zone of radius r centered at the other aircraft, based on the model introduced in the previous section.

We assume for simplicity that the nominal wind velocity field f does not depend on the position $x \in U$, i.e., $f(\cdot,t) = f(t)$, $t \in T$, which means that at any given time the nominal wind field is uniform in the considered region U of the airspace. In this case, we can incorporate $f(t)$ into $u_i(t)$ and simply set $f(t) = 0$, $t \in T$. Thus Equations (3) and (4) become

$$\begin{aligned} dX_1(t) &= u_1(t)dt + \sigma dB(X_1,t), && (5)\\ dX_2(t) &= u_2(t)dt + \sigma dB(X_2,t). && (6) \end{aligned}$$

The equation governing the relative position $Y = X_2 - X_1$ of the two aircraft can be obtained by subtracting Equation (5) from Equation (6), and is given by

$$dY(t) = v(t)dt + \sigma[dB(X_2,t) - dB(X_1,t)], \qquad (7)$$

where we set $v \triangleq u_2 - u_1$.

Fix X_1, X_2 and define $Z(t) \triangleq B(X_2, t) - B(X_1, t)$. We claim that in terms of distribution,

$$Z(t) \sim \sqrt{2(1 - h(\|X_2 - X_1\|))}W(t), \tag{8}$$

where $W(t)$ is a standard two-dimensional Brownian motion. To show this, we notice first that $Z(0) = W(0) = 0$ and next verify that their variances are identical. In fact, for any $t_1 \leq t_2$, by the definition of $B(\cdot,\cdot)$ we have

$$\begin{aligned} &E\{[Z(t_2) - Z(t_1)][Z(t_2) - Z(t_1)]^T\} \\ =& [2 - 2h(\|X_2 - X_1\|)](t_2 - t_1)I_2 \\ =& E\{[W(t_2) - W(t_1)][W(t_2) - W(t_1)]^T\}. \end{aligned}$$

Since W and Z are both Gaussian processes with zero mean, $Z(t) \sim \sqrt{2(1 - h(\|X_2 - X_1\|))}W(t)$, for X_1, X_2 constant. In our case, X_1, X_2 are themselves stochastic processes whose outcomes depend on B, hence on Z, therefore this conclusion is in general not true. However, since the function h is usually very flat at $\|X_1 - X_2\| \geq r$, hence $h(\|X_1 - X_2\|)$ varies much more slowly than $W(t)$, we can think of $h(\|X_1 - X_2\|)$ as locally constant near each time instant. Therefore, Equation (8) still holds approximately. Given that h is a decreasing function with $h(0) = 1$ and $h(\infty) = 0$, Equation (8) says that the closer the aircraft get to each other, the more similar are the wind perturbations to their positions. When the aircraft are far away, the wind effect on each aircraft position becomes more and more uncorrelated. By making this approximation, we can replace Equation (7) with

$$dY(t) = v(t)dt + \sigma\sqrt{2[1 - h(\|Y\|)]}dW(t). \tag{9}$$

If we set $U_Y = \{y = x_2 - x_1 \in \mathbb{R}^2 : x_1, x_2 \in U\}$, and define D to be the subset of U_Y corresponding to an aircraft entering the protection zone around the other aircraft, then the problem of determining the probability of conflict over T becomes:

"Given the initial condition $Y(0) \in U_Y \setminus D$, compute $P\{Y(t) \in D \text{ for some } t \in T\}$, where Y is the solution to the ordinary stochastic differential equation (9) defined on the open set $U_Y \setminus D$ with initial condition $Y(0)$."

D is usually taken to be a closed disk of radius $r = 5$ nmi centered around the origin. To account for the possibility that $Y(t)$ hits the boundary of U_Y first than it hits D, we can choose U_Y to be large enough and declare the situation to be safe any time $Y(t)$ wanders outside of U_Y. Therefore, the quantity we are interested in is actually the probability of $Y(t)$ hitting D before $U_Y^c = \mathbb{R}^2 \setminus U_Y$ within the time interval T, namely,

$$\begin{aligned} P_c =& P\{\text{There exists some } t \in T \text{ such that } Y(t) \in D \\ & \text{and } Y(s) \in U_Y \text{ for all } 0 \leq s < t\}. \end{aligned} \tag{10}$$

Note that P_c is a function only of the initial relative position $Y(0)$. When we need to point this out, we write $P_c(Y(0))$.

4 Approximation using Markov chains

We now determine an approximation of the solution $Y(t)$ to Equation (9). The point is to discretize the state space U_Y into some grid points that constitute the state space of a Markov chain. By carefully choosing the transition probabilities, the solution to the Markov chain will converge weakly to that of the stochastic differential equation (9) as the grid size approaches zero. Therefore, if we choose a small grid size, a good estimate of P_c is provided by the corresponding quantity associated with the Markov chain, which is much easier to compute.

Fix a grid size δ. We next define a Markov chain $\{Q_{k\Delta t}, k \geq 0\}$, where $\Delta t > 0$ is a positive constant representing the time interval between successive jumps. We shall specify the value for Δt later. Denote by $\delta\mathbb{Z}^2$ the integer grids scaled by δ, namely, $\delta\mathbb{Z}^2 = \{(m\delta, n\delta)|\ m, n \in \mathbb{Z}\}$. Each grid point $q = (m\delta, n\delta)$ in $\delta\mathbb{Z}^2$ has four immediate neighbors:

$$\begin{aligned} q_l &= ((m-1)\delta, n\delta), & q_r &= ((m+1)\delta, n\delta) \\ q_d &= (m\delta, (n-1)\delta), & q_u &= (m\delta, (n+1)\delta). \end{aligned}$$

The state space of $Q_{k\Delta t}$ is $S = (U_Y \setminus D) \cap \delta\mathbb{Z}^2$, which consists of all of the grid points of $\delta\mathbb{Z}^2$ that lie inside U_Y but outside of D. The interior of S, denoted by S^0, consists of all those points in $\delta\mathbb{Z}^2$ that belong to S and such that their four immediate neighbors belong to S as well. The boundary of S is defined to be $\partial S = S \setminus S^0$, and is the union of two disjoint sets: $\partial S = \partial S_U \cup \partial S_D$, where points in ∂S_U have at least one neighbor outside of U_Y, and points in ∂S_D have at least one neighbor inside D. The transition probabilities of $Q_{k\Delta t}$ are such that each state in ∂S is an absorbing state, and starting from an arbitrary state $q = (m\delta, n\delta)$ in S^0, the transition probabilities are given by ($k \geq 0$):

$$P\{Q_{(k+1)\Delta t} = q' | Q_{k\Delta t} = q\} = \begin{cases} p_l^{k\Delta t}(q) = \exp(-\delta\xi_q^{k\Delta t})/C_q^{k\Delta t}, & q' = q_l; \\ p_r^{k\Delta t}(q) = \exp(\delta\xi_q^{k\Delta t})/C_q^{k\Delta t}, & q' = q_r; \\ p_d^{k\Delta t}(q) = \exp(-\delta\eta_q^{k\Delta t})/C_q^{k\Delta t}, & q' = q_d; \\ p_u^{k\Delta t}(q) = \exp(\delta\eta_q^{k\Delta t})/C_q^{k\Delta t}, & q' = q_u; \\ p_o^{k\Delta t}(q) = \chi_q^{k\Delta t}/C_q^{k\Delta t}, & q' = q. \end{cases} \tag{11}$$

The parameters in the above expression are chosen to be

$$\begin{aligned} \Delta t &= \lambda\delta^2, \\ \xi_q^{k\Delta t} &= \frac{v_1(k\Delta t)}{2\sigma^2[1 - h(\delta\sqrt{m^2 + n^2})]}, \\ \eta_q^{k\Delta t} &= \frac{v_2(k\Delta t)}{2\sigma^2[1 - h(\delta\sqrt{m^2 + n^2})]}, \\ \chi_q^{k\Delta t} &= \frac{1}{\lambda\sigma^2[1 - h(\delta\sqrt{m^2 + n^2})]} - 4, \\ C_q^{k\Delta t} &= \chi_q^{k\Delta t} + \exp(-\delta\xi_q^{k\Delta t}) + \exp(\delta\xi_q^{k\Delta t}) \\ &\quad + \exp(-\delta\eta_q^{k\Delta t}) + \exp(\delta\eta_q^{k\Delta t}), \end{aligned}$$

where $v_1(k\Delta t)$ and $v_2(k\Delta t)$ are the two components of the vector $v(k\Delta t)$. λ is a positive constant small enough such that

$\chi_q^{k\Delta t}$ defined above is positive for all m, n and all k. In particular, this is guaranteed if $\lambda < (4\sigma^2)^{-1}$.

$\{Q_{k\Delta t}, k \geq 0\}$ is a time-inhomogeneous Markov chain such that i) starting from a state in S^0 at time $k\Delta t$, $k \geq 0$, the chain jumps to one of its four neighbors or stays at the same state according to transition probabilities determined by its current location and the time $k\Delta t$, and ii) states in ∂S are absorbing.

Suppose that at some time instant $k\Delta t$ the chain is at state $q = (m\delta, n\delta) \in S^0$. Define

$$b_q^{k\Delta t} = \frac{1}{\Delta t} E\{Q_{(k+1)\Delta t} - Q_{k\Delta t} | Q_{k\Delta t} = q\}, \quad (12)$$

$$A_q^{k\Delta t} = \frac{1}{\Delta t} E\{(Q_{(k+1)\Delta t} - Q_{k\Delta t})(Q_{(k+1)\Delta t} - Q_{k\Delta t})^T | Q_{k\Delta t} = q\}. \quad (13)$$

Direct computation shows that

$$b_q^{k\Delta t} = \frac{2\delta}{C_q^{k\Delta t}\Delta t}\begin{bmatrix}\mathrm{sh}(\delta\xi_q^{k\Delta t})\\ \mathrm{sh}(\delta\eta_q^{k\Delta t})\end{bmatrix},$$

$$A_q^{k\Delta t} = \frac{2\delta^2}{C_q^{k\Delta t}\Delta t}\begin{bmatrix}\mathrm{csh}(\delta\xi_q^{k\Delta t}) & 0\\ 0 & \mathrm{csh}(\delta\eta_q^{k\Delta t})\end{bmatrix}.$$

If for each δ we choose m and n such that $(m\delta, n\delta)$ is closest to a fixed point $y \in U_Y$, then it can be verified that as $\delta \to 0$

$$b_{(m\delta,n\delta)}^{k\Delta t} \to v(k\Delta t), \quad A_{(m\delta,n\delta)}^{k\Delta t} \to 2\sigma^2[1 - h(\|y\|)]I_2.$$

By Theorem 8.7.1 in [3, pp.297] (see also [15]), we have

Proposition 1 *Suppose that the state of the Markov chain $\{Q_{k\Delta t}, k \geq 0\}$ is constant $Q_{k\Delta t}$ on each time interval $[k\Delta t, (k+1)\Delta t)$ between successive jumps. Then as $\delta \to 0$, the solution $\{Q_{k\Delta t}, k \geq 0\}$ converges weakly to the solution $\{Y(t), t \geq 0\}$ to the diffusion equation (9).*

Because of the weak convergence of $Q_{k\Delta t}$ to $Y(t)$, the probability (10) can be approximated by the probability

$$P_c^\delta \triangleq P\{Q_{k_f\Delta t} \in \partial S_D\} = P\{Q_{k\Delta t} \text{ hits } \partial S_D \text{ first than it hits } \partial S_U \text{ within } 0 \leq k \leq k_f\} \quad (14)$$

for small δ. Here $k_f \triangleq [t_f/\Delta t]$ denotes the largest integer not exceeding $t_f/\Delta t$ ($k_f = \infty$ if $t_f = \infty$), and the chain $\{Q_{k\Delta t}, k \geq 0\}$ is assumed to start from a point $(\bar{m}\delta, \bar{n}\delta) \in S$ closest to $Y(0)$.

For each $k = 0, \ldots, k_f$ and each $q = (m\delta, n\delta) \in S$, define

$$P_c^{(k)}(q) \triangleq P\{Q_{k_f\Delta t} \in \partial S_D | Q_{k\Delta t} = q\}.$$

Then $P_c^{(k)}$, $0 \leq k \leq k_f$, is a series of functions satisfying

$$P_c^{(k)}(m\delta, n\delta) = \begin{cases} p_o^{k\Delta t}(q)P_c^{(k+1)}(q) + p_l^{k\Delta t}(q)P_c^{(k+1)}(q_l) \\ \quad + p_r^{k\Delta t}(q)P_c^{(k+1)}(q_r) + p_d^{k\Delta t}(q)P_c^{(k+1)}(q_d) \\ \quad + p_u^{k\Delta t}(q)P_c^{(k+1)}(q_u), & \text{if } q \in S^0; \\ 1, & \text{if } q \in \partial S_D; \\ 0, & \text{if } q \in \partial S_U, \end{cases} \quad (15)$$

together with the initial condition

$$P_c^{(k_f)}(q) = \begin{cases} 1, & \text{if } q \in \partial S_D; \\ 0, & \text{otherwise.} \end{cases} \quad (16)$$

The desired quantity is thus $P_c^\delta = P_c^{(0)}(\bar{m}\delta, \bar{n}\delta)$.

Based on the preceding analysis, we can summarize the procedure to compute an approximation of P_c in (10) as follows:

Algorithm 1 *Given $Y(0)$ and $v(t) = u_2(t) - u_1(t)$, $t \in T$.*

1. *Fix $\delta > 0$. Define the Markov chain $\{Q_{k\Delta t}, k \geq 0\}$ with state space $S = (U \backslash D) \cap \delta\mathbb{Z}^2$, and transition probabilities given by* (11)*. Choose $\Delta t = \lambda\delta^2$ for some $\lambda \in (0, \frac{1}{4\sigma^2})$, and let $k_f = [t_f/\Delta t]$.*

2. *Initialize $P_c^{(k_f)}$ according to Equation (16).*

3. *For $k = k_f - 1, \ldots, 0$, compute $P_c^{(k)}$ from $P_c^{(k+1)}$ according to Equation (15).*

4. *Choose a point $(\bar{m}\delta, \bar{n}\delta)$ in S closest to $Y(0)$ and set $P_c^\delta = P_c^{(0)}(\bar{m}\delta, \bar{n}\delta)$.*

If δ is sufficiently small, then $P_c \simeq P_c^\delta$. In practice, Algorithm 1 works only for the finite-horizon case. If $t_f = \infty$, then the iterations will take infinitely many steps. Hence a different procedure should be adopted. A solution to the infinite horizon case is presented in the next section under the assumption that v is constant from some time instant on.

4.1 The infinite horizon case

We start by considering the case when the relative velocity v between the two aircraft is constant over T. We then study the case when v is constant from some time instant on.

If we arrange $P_c^{(k)}(m\delta, n\delta)$, $(m\delta, n\delta) \in S^0$, into a long column vector according to some fixed ordering of the points in S^0, and denote it by $\mathbf{P}_c^{(k)}$, then Equation (15) can be written as

$$\mathbf{P}_c^{(k)} = \mathbf{A}^{(k)}\mathbf{P}_c^{(k+1)} + \mathbf{b}^{(k)} \quad (17)$$

for suitably defined $\mathbf{A}^{(k)} \in \mathbb{R}^{|S^0|\times|S^0|}$ and $\mathbf{b}^{(k)} \in \mathbb{R}^{|S^0|}$, where $|S^0|$ denotes the cardinality of S^0.

Under the assumption that v is constant, $\mathbf{A}^{(k)} = \mathbf{A}$ and $\mathbf{b}^{(k)} = \mathbf{b}$, $k \geq 0$, and Equation (17) becomes

$$\mathbf{P}_c^{(k)} = \mathbf{A}\mathbf{P}_c^{(k+1)} + \mathbf{b}. \tag{18}$$

$\mathbf{A}$ is a sparse positive matrix with the property that the sum of its elements on each row is smaller than or equal to 1, where equality holds if and only if that row corresponds to a point in $(S^0)^0$, the interior of S^0, namely, a point in S^0 whose four neighbors all belong to S^0. On the other hand, $\mathbf{b}$ is a positive vector with nonzero elements on those rows corresponding to points on the boundary $\partial(S^0) = S^0 \setminus (S^0)^0$ of S^0.

Lemma 1 *The eigenvalues of $\mathbf{A}$ are all in the interior of the unit disk of $\mathbb{C}$.*

The proof of this lemma is straightforward, hence omitted here. As a result, we conclude that

Lemma 2 *Consider the discrete-time linear dynamic system that starts at some time $k = k_f < \infty$ and evolves backward in time according to Equation (18).*

- *There is a unique $\mathbf{P}^{(-\infty)} \in \mathbb{R}^{|S^0|}$ satisfying $\mathbf{P}^{(-\infty)} = \mathbf{A}\mathbf{P}^{(-\infty)} + \mathbf{b}$.*
- *Starting from any initial value $\mathbf{P}^{(k_f)}$, $\mathbf{P}^{(k)}$ converges to $\mathbf{P}^{(-\infty)}$ as $k \to -\infty$.*
- *If $\mathbf{P}^{(k_f)} \geq \mathbf{P}^{(-\infty)}$, then $\mathbf{P}^{(k)} \geq \mathbf{P}^{(-\infty)}$ for all $k \leq k_f$. Conversely, if $\mathbf{P}^{(k_f)} \leq \mathbf{P}^{(-\infty)}$, then $\mathbf{P}^{(k)} \leq \mathbf{P}^{(-\infty)}$ for all $k \leq k_f$.*

Proof: $\mathbf{P}^{(-\infty)} = (I - \mathbf{A})^{-1}\mathbf{b}$ since $I - \mathbf{A}$ is invertible by Lemma 1. Define $\mathbf{e}^{(k)} = \mathbf{P}^{(k)} - \mathbf{P}^{(-\infty)}$. Then $\mathbf{e}^{(k)} = \mathbf{A}\mathbf{e}^{(k+1)}$. So by Lemma 1, $\mathbf{e}^{(k)}$ converges to 0 as $k \to -\infty$. The last conclusion is a direct consequence of the fact that all components of the matrix $\mathbf{A}$ are nonnegative. ∎

If $(\bar{m}\delta, \bar{n}\delta) \in S$ is a point closest to $Y(0)$, then the element of $\mathbf{P}^{(-\infty)}$ corresponding to $(\bar{m}\delta, \bar{n}\delta)$ is the desired probabilistic quantity P_c^δ defined in Equation (14) for the infinite horizon case with constant velocity v. Furthermore, because of Lemma 2, we can estimate the speed of convergence of system (18) to $P^{(-\infty)}$ in the following way. Let the system start from two initial conditions that are one an upper bound of P_c^δ (one such example can be chosen to be 0 on ∂S_U and 1 on $\partial S_D \cup S^0$), and the other a lower bound of P_c^δ (for example, 1 on ∂S_D and 0 on $\partial S_U \cup S^0$). Then the iterated results for the two initial conditions will provides upper bounds and lower bounds for P_c^δ, respectively, which converge toward each other (hence to P_c^δ) as the number of iteration increases.

If the velocity $v(t)$, $t \in T$, is constant only starting from a certain time instant $\bar{t}$, one has to first determine $\mathbf{P}^{(-\infty)}$, and then execute Algorithm 1 replacing k_f with $[\bar{t}/\Delta t]$ and initializing $P_c^{(k_f)}$ over S based on $\mathbf{P}^{(-\infty)}$.

4.2 Extension to the case when the aircraft current position is uncertain

In Section 3, we formulated the problem of determining the probability of conflict P_c for a two-aircraft encounter under the assumption that the aircraft current positions are perfectly known. We then introduced a procedure to compute the conflict probability map, i.e., a function $P_c : U_Y \to [0,1]$ which, given the current relative position $Y(0)$ of the two aircraft, determines $P_c(Y(0))$. This procedure is extended here to address the case when the aircraft current positions $X_1(0)$ and $X_2(0)$ (hence $Y(0) = X_2(0) - X_1(0)$) are not known precisely. If $Y(0)$ can be described as a random variable with a certain distribution $p_Y(y)$, $y \in U_Y$, over U_Y, then the probability of conflict can be computed as $\int_{U_Y} P_c(y) dp_Y(y)$, which actually reduces to a finite summation if we adopt the approximation procedure for estimating $P_c(y)$.

5 An example

In this section we consider a two-aircraft encounter and compute the probability of conflict by using the procedure described in Section 4. The main objective of this section is to demonstrate through a numerical example that the correlation between the aircraft future positions cannot be generally neglected when computing the probability of conflict.

We consider the infinite horizon case and assume that the velocity function v is constant and given by $v(t) = [1\ 0]^T$, $t \geq 0$. we suppose that $\sigma = 1$, and the spatial correlation function is given by $h(d) = \exp(-cd)$, $\forall d \geq 0$, where $c > 0$ is some positive constant.

In Figure 1 we plot the level curves of the probability of conflict P_c^δ as a function of the initial relative position of the two aircraft for two different values of c ($\delta = 1$, $\lambda = 1/5$). We set $U_Y = [-200, 20] \times [-50, 50]$ and $D = \{y \in \mathbb{R}^2 : \|y\| \leq 3\}$.

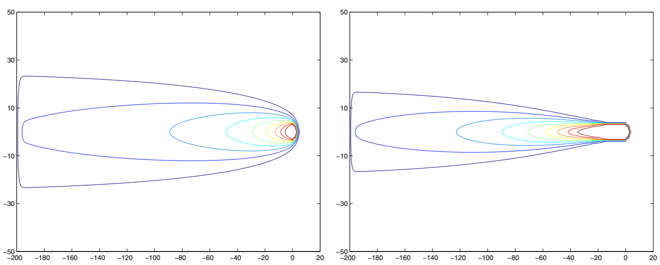

Figure 1: Map of the estimated probability of conflict for the correlation function $h(d) = exp(-cd)$, when the velocity function is constant (Left: $c = 1$; Right: $c = 0.01$).

Note that the shape and the extension of the level curves for given flight plans depends on the value of c, hence on the correlation structure of the wind perturbation.

In Figure 2, we set $c = 1$ and plot the probability of conflict for the case when v is piecewise constant given by $v(t) = [1, 0]^T$ for $t \in [0, 15]$, $v(t) = [0, 1]^T$ for $t \in [15, 30]$, and $v(t) = [1, 0]^T$ for $t \geq 30$. Note that the level curves of the probability

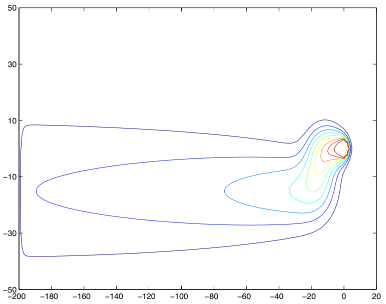

Figure 2: Map of the estimated probability of conflict for the correlation function $h(d) = exp(-d)$, when the velocity function is piecewise constant.

of conflict strongly depends on the aircraft flight plans.

6 Concluding remarks

In this paper we study the problem of determining the probability of conflict for a two-aircraft encounter in the level-flight case. We propose a simple kinematic model for the two-aircraft system. The distinguishing feature of our model with respect to the ones commonly adopted for developing conflict detection algorithms is that the correlation between the aircraft positions due to the wind perturbation effect is considered. We then introduce an approximation scheme for estimating the probability of conflict based on a simplified version of this model, where the nominal wind field is constant in space at each time, and the stochastic wind components have a spatial correlation structure that depends only on the distance of the aircraft. For an extension of this work to general nominal wind field case and the three dimensional case, see [7].

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