FREQUENCY DOMAIN DESIGN OF REDUCED ORDER H_∞ FILTERS FOR DISCRETE TIME SYSTEMS

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Abstract

In this contribution the frequency domain design of reduced order H_∞ filters of order n-κ is investigated for nth order discrete time systems with m measurements of which κ are undisturbed. Starting from the known time domain results, the polynomial matrices parameterizing reduced order *a priori* and *a posteriori* H_∞ filters are derived. A simple example demonstrates the proposed design procedure.

1 Introduction

The use of H_∞ filters, which estimate some linear combination of the system states in the H_∞ norm minimization sense, is appropriate when there is little knowledge of the statistics of the driving and of the measurement noise signals. Compared to minimum variance estimators (Kalman filters) they are less sensitive to uncertainty in the system parameters [10].

The H_∞ filtering problem was first considered in [3] and in [9] using a frequency domain approach. A solution of the H_∞ filtering problem in the framework of the Riccati equation approach is given in [14]. The corresponding theory has also been developed in the discrete time case (see e.g. [1], [13]).

This paper considers the frequency domain design of reduced order H_∞ filters for discrete time systems, where κ of the m measurements y_M of the nth order plant are not affected by disturbances. The resulting filter is of order n-κ, since it suffices to build an (n-κ)th order observer to reconstruct to whole system state. The H_∞ filter is characterized by polynomial matrices, parameterizing its discrete time transfer matrix. The H_∞ estimation problem can be solved under various patterns of information. In this contribution *a priori* and *a posteriori* H_∞ filtering are considered. The *a priori* H_∞ filter uses the measurements in a one step delay, whereas the *a posteriori* H_∞ filter uses the current measurements in order to generate the desired estimate. As a consequence, the filter channels related to the disturbed measurements are strictly proper in the case of *a priori* estimates and proper in the case of *a posteriori* estimates. When using such *a posteriori* H_∞ filters the H_∞ norm bound may be lower than the one obtained by *a priori* H_∞ filters.

After introducing the system descriptions in the time and in the frequency domain, the underlying H_∞ estimation problems are formulated in Section 2. This section also contains the reduced order H_∞ filter schemes in the time domain both for the *a priori* and the *a posteriori* estimation, forming the basis for the frequency domain solution derived in Section 3. A simple demonstrating example follows in Section 4.

2 Problem formulation and time domain results

Consider a time invariant, discrete time, linear system of order n with m_z unmeasurable outputs y_z, m measurements y_M, and q ≥ m disturbances w represented by

$$\begin{aligned} x(k+1) &= Ax(k)+Gw(k) \quad , \quad x(0)=0 \\ y_z(k) &= C_z x(k) \\ y_M(k) &= \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x(k) + \begin{bmatrix} D_1 \\ 0 \end{bmatrix} w(k) \\ &= C_M x(k) + Dw(k) \end{aligned} \tag{1}$$

with C_M having full row rank, $D_1 D_1^T > 0$, and A invertible. The output y_M is subdivided such that y_1 contains the m-κ disturbed measurements and y_2 the κ perfect ones with 0 ≤ κ ≤ m. It is assumed that the pair (C_M, A) is detectable. In the sequel, the composite matrix C will be used in different partitions

$$C = [C_z^T \;\; C_M^T]^T = [C_z^T \;\; C_1^T \;\; C_2^T]^T = [C_r^T \;\; C_2^T]^T \tag{2}$$

The frequency domain description of system (1) is

$$y(z) = \begin{bmatrix} y_z(z) \\ y_M(z) \end{bmatrix} = \left\{ C(zI-A)^{-1}G + \begin{bmatrix} 0 \\ D \end{bmatrix} \right\} w(z) \tag{3}$$

and it is assumed that the strictly proper part of this transfer matrix is represented in a left coprime matrix fraction description

$$C(zI-A)^{-1}G = \overline{D}^{-1}(z)\overline{N}(z) \tag{4}$$

with $\overline{D}(z)$ row reduced.

Given m measurements y_M find an H_∞ filter for the system (1), (3) that generates an estimate $\hat{y}_z(k)$ for the unmeasurable m_z linear combinations $y_z(k)$ of the state x(k) in the H_∞ norm minimization sense. With $l_2[0,\infty)$ denoting the set of real square summable functions on the interval $[0,\infty)$, define the (worst case) performance measure

$$M = \sup_{\substack{w \in l_2[0,\infty) \\ w \neq 0}} \frac{\|y_z - \hat{y}_z\|_2}{\|w\|_2} \tag{5}$$

when using the *a priori* estimate $\hat{y}_z(k)$, and in the case of an *a posteriori* estimate $\hat{y}_z{}^+(k)$ use

$$M^+ = \sup_{\substack{w \in l_2[0,\infty) \\ w \neq 0}} \frac{\|y_z - \hat{y}_z{}^+\|_2}{\|w\|_2} \tag{6}$$

Two (suboptimal) filtering problems are considered

1) A priori H_∞ filtering problem. Given a $\gamma > 0$, find a stable filter if it exists such that $M < \gamma$.

2) A posteriori H_∞ filtering problem. Given a $\gamma > 0$, find a stable filter if it exists such that $M^+ < \gamma$.

The time domain solutions to these problems are presented in [8]. In the sequel, however, other solutions are used, as they are better suited for the frequency domain design of the H_∞ filter.

Consider the system (1) with κ perfect measurements y_2 and a reduced order filter of order n-κ [2]

$$\begin{aligned} \xi(k+1) = {} & T(A - F_1C_1)\Theta\xi(k) + \\ & + T[F_1 \quad (A - F_1C_1)\Psi_2] \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} \end{aligned} \tag{7}$$

giving the state estimate

$$\hat{x} = \begin{bmatrix} C_2 \\ T \end{bmatrix}^{-1} \begin{bmatrix} y_2 \\ \xi \end{bmatrix} = [\Psi_2 \quad \Theta] \begin{bmatrix} y_2 \\ \xi \end{bmatrix} \tag{8}$$

Define the matrices

$$C_r = \begin{bmatrix} C_z \\ C_1 \end{bmatrix}; \quad R_{fr} = \begin{bmatrix} -\gamma^2 I_{m_z} & 0 \\ 0 & D_1D_1^T \end{bmatrix} \tag{9}$$

$$S_{fr} = [0 \quad GD_1^T]; \quad \tilde{R}_r = R_{fr} + C_r\tilde{P}C_r^T \tag{10}$$

$$\overline{P} = A\tilde{P}A^T + GG^T - L_r\tilde{R}_rL_r^T \tag{11}$$

$$X = C_2\overline{P}C_2^T \tag{12}$$

Then the optimal filter gain matrices result from

$$L_r = [\underset{n,m_z}{L_z}, \quad \underset{n,n-\kappa}{L_1}] = (A\tilde{P}C_r^T + S_{fr})\tilde{R}_r^{-1} \tag{13}$$

$$\Psi_2 = \overline{P}C_2^TX^{-1} \tag{14}$$

and $\tilde{P} = \tilde{P}^T \geq 0$ is a stabilizing solution of the algebraic Riccati equation (ARE)

$$\tilde{P} = A\tilde{P}A^T + GG^T - \Psi_2X\Psi_2^T - L_r\tilde{R}_rL_r^T \tag{15}$$

After solving $T\Psi_2 = 0$ with T having full row rank the matrix Θ is obtained from

$$[\Psi_2 \quad \Theta] = \begin{bmatrix} C_2 \\ T \end{bmatrix}^{-1} \tag{16}$$

The *a priori* estimate $\hat{y}_z(k)$ is obtained from the above results with $F_1 = L_1$ as

$$\hat{y}_z(k) = C_z\Psi_2y_2(k) + C_z\Theta\xi(k) \tag{17}$$

where

$$-\gamma^2I + C_z\tilde{P}C_z^T < 0 \tag{18}$$

must hold.

The *a posteriori* estimate $\hat{y}_z^+(k)$ is obtained from the above results with $F_1 = A\lambda_1$ where

$$\lambda_1 = (A - L_zC_z)^{-1}L_1 \tag{19}$$

as

$$\hat{y}_z{}^+(k) = \hat{y}_z(k) + C_z\lambda_1(y_1(k) - C_1\hat{x}(k)) \tag{20}$$

(see (8)) where

$$-\gamma^2I + C_z[\lambda_1D_1D_1^T\lambda_1^T + (I - \lambda_1C_1)\tilde{P}(I - C_1^T\lambda_1^T)]C_z^T < 0 \tag{21}$$

must hold.

The above presented optimal solution differs from the one presented in [8]. The matrix $\tilde{P}$ used here is related to the matrix $\overline{P}$ used in [8] by

$$\tilde{P} = (I - \Psi_2C_2)\overline{P}(I - C_2^T\Psi_2^T) \tag{22}$$

which has as a consequence

$$C_2\tilde{P} = 0 \tag{23}$$

For more details see [6].

3 Frequency domain design of discrete time H_∞ filters

3.1 The polynomial matrix equation for the fictitious H_∞ filter

The frequency domain design of the reduced order H_∞ filter is based on the left coprime factorization (4) of the system (1). As an intermediate result, the "fictitious" filter with

$$L_r = \begin{bmatrix} L_z & L_1 \end{bmatrix}$$

$$\begin{aligned} \xi_f(k+1) &= T(A - L_rC_r)\Theta\xi_f(k) + \\ &+ T[L_r \quad (A - L_rC_r)\Psi_2]\begin{bmatrix} y_z(k) \\ y_1(k) \\ y_2(k) \end{bmatrix} \end{aligned} \tag{24}$$

(with y_z as an input) is considered and the "realizable" filter (7) (case $F_1 = L_1$) results for $L_z = 0$.

Assumption 1. The factorization (4) is such that the highest row degree coefficient matrix $\Gamma_r\left[\overline{D}^\kappa(z)\right]$ has full rank, with

$$\overline{D}^\kappa(z) = \Pi\left\{\overline{D}(z)\begin{bmatrix} I & 0 \\ 0 & z^{-1}I_\kappa \end{bmatrix}\right\} \tag{25}$$

and $\Pi\{\cdot\}$ denoting taking the polynomial part. This can always be assured by appropriate left unimodular operations [12].

Assumption 2. The matrices $\tilde{R}_r$ and X (see (10) and (12)) have full rank. They can be computed from the frequency domain results (see below).

In [5] the relations between the time and the frequency domain parameterizations of reduced order observers have been presented. With $\tilde{\overline{D}}_f(z)$ parameterizing the fictitious filter (24) in the frequency domain, one has

$$\overline{D}^{-1}(z)\tilde{\overline{D}}_f(z) = C(zI - A)^{-1}\begin{bmatrix} L_r & \Psi_2 \end{bmatrix} + \begin{bmatrix} I_{m_z+m-\kappa} & 0 \\ 0 & 0_\kappa \end{bmatrix} \tag{26}$$

Theorem 1. With the system representation (4) such that Assumptions 1 and 2 hold, the polynomial matrix $\tilde{\overline{D}}_f(z)$ parameterizing the fictitious filter in the frequency domain solves the polynomial matrix equation

$$\begin{aligned} \tilde{\overline{D}}_f(z)\begin{bmatrix} \tilde{R}_r & 0 \\ 0 & X \end{bmatrix}\tilde{\overline{D}}_f^T(z^{-1}) &= \overline{D}(z)\begin{bmatrix} R_{fr} & 0 \\ 0 & 0 \end{bmatrix}\overline{D}^T(z^{-1}) + \\ &+ \overline{N}(z)\overline{N}^T(z^{-1}) + \overline{N}(z)\begin{bmatrix} 0 & D_1^T & 0 \end{bmatrix}\overline{D}^T(z^{-1}) + \\ &+ \overline{D}(z)\begin{bmatrix} 0 & D_1^T & 0 \end{bmatrix}^T\overline{N}^T(z^{-1}) \end{aligned} \tag{27}$$

Proof: Observing that

$$\begin{aligned} \tilde{P} - A\tilde{P}A^T &= (zI - A)\tilde{P}(z^{-1}I - A^T) + (zI - A)\tilde{P}A^T + \\ &+ A\tilde{P}(z^{-1}I - A^T) \end{aligned}$$

the ARE (15) can be written as

$$\begin{aligned} &(zI - A)\tilde{P}(z^{-1}I - A^T) + (zI - A)\tilde{P}A^T + A\tilde{P}(z^{-1}I - A^T) + \\ &+ \Psi_2X\Psi_2^T + L_r\tilde{R}_rL_r^T = GG^T \end{aligned}$$

Multiplying this with $C(zI - A)^{-1}$ from the left and with $(z^{-1}I - A^T)^{-1}C^T$ from the right (see (2)), then substituting (see (10) and (13))

$$C\tilde{P}C^T = \begin{bmatrix} \tilde{R}_r - R_{fr} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } A\tilde{P}C^T = \begin{bmatrix} L_r\tilde{R}_r - S_{fr} & 0 \end{bmatrix}$$

where we have used $C_2\Psi_2 = I$ following from (8) and (23), and reordering the result gives

$$\begin{aligned} &\left\{C(zI - A)^{-1}[L_r \;\; \Psi_2] + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right\}\begin{bmatrix} \tilde{R}_r & 0 \\ 0 & X \end{bmatrix} \\ &\left\{\begin{bmatrix} L_r^T \\ \Psi_2^T \end{bmatrix}(z^{-1}I - A^T)^{-1}C^T + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right\} = \begin{bmatrix} R_{fr} & 0 \\ 0 & 0 \end{bmatrix} + \\ &+ C(zI - A)^{-1}GG^T(z^{-1}I - A^T)^{-1}C^T + \\ &+ C(zI - A)^{-1}G\begin{bmatrix} 0 & D_1^T & 0 \end{bmatrix} + \left(C(zI - A)^{-1}G\begin{bmatrix} 0 & D_1^T & 0 \end{bmatrix}\right)^T \end{aligned} \tag{28}$$

which in view of (4) and (26) can be written as

$$\begin{aligned} \tilde{\overline{D}}_f(z)\begin{bmatrix} \tilde{R}_r & 0 \\ 0 & X \end{bmatrix}\tilde{\overline{D}}_f^T(z^{-1}) &= \overline{D}(z)\begin{bmatrix} R_{fr} & 0 \\ 0 & 0 \end{bmatrix}\overline{D}^T(z^{-1}) + \\ &+ \overline{N}(z)\overline{N}^T(z^{-1}) + \overline{N}(z)\begin{bmatrix} 0 & D_1^T & 0 \end{bmatrix}\overline{D}^T(z^{-1}) + \\ &+ \overline{D}(z)\begin{bmatrix} 0 & D_1^T & 0 \end{bmatrix}^T\overline{N}^T(z^{-1}) \end{aligned} \tag{29}$$

after the result has been multiplied by $\overline{D}(z)$ from the left and by $\overline{D}^T(z^{-1})$ from the right. ./.

Since the matrix $\begin{bmatrix} \tilde{R}_r & 0 \\ 0 & X \end{bmatrix}$ in (29) is indefinite, one must use J-spectral factorization [11] to obtain the (stable) spectral factor $\tilde{\tilde{D}}(z)$ of the known right hand side of (29), i.e.

$$
\begin{aligned}
\tilde{\tilde{D}}(z)\,J\,\tilde{\tilde{D}}^{T}(z^{-1}) &= \overline{D}(z)\begin{bmatrix} R_{fr} & 0 \\ 0 & 0 \end{bmatrix}\overline{D}^{T}(z^{-1}) + \\
&+ \overline{N}(z)\overline{N}^{T}(z^{-1}) + \overline{N}(z)\begin{bmatrix} 0 & D_1^{T} & 0 \end{bmatrix}\overline{D}^{T}(z^{-1}) + \\
&+ \overline{D}(z)\begin{bmatrix} 0 & D_1^{T} & 0 \end{bmatrix}^{T}\overline{N}^{T}(z^{-1})
\end{aligned} \tag{30}
$$

where J is a diagonal matrix with entries 1 and –1 on the main diagonal and $\det\tilde{\tilde{D}}(z)$ has its roots inside the unit circle. The factorization result for $\tilde{\tilde{D}}(z)$ may be such, that $\det\tilde{\tilde{D}}(z)$ contains $\mu > 0$ superfluous roots at $z = 0$ which is often the case for discrete time systems [7]. These must be extracted by right operations

$$\tilde{\tilde{D}}_{red}(z) = \tilde{\tilde{D}}(z)\,V_{ext}^{-1}(z) \tag{31}$$

with $\det V_{ext}(z) = z^{\mu}$ such that $V_{ext}^{-1}(z)JV_{ext}^{-T}(z^{-1}) = J$ (for an algorithm see [4]) to obtain a polynomial matrix $\tilde{\tilde{D}}_{red}(z)$ with $\deg[\det\tilde{\tilde{D}}_{red}(z)] = n - \kappa$.

Inspection of (26) shows, that (see [5])

$$\Gamma_r\left[\tilde{\overline{D}}_f(z)\right] = \Gamma_r\left[\overline{D}^{\kappa}(z)\right]\begin{bmatrix} I & 0 \\ C_2L_r & I_\kappa \end{bmatrix} \tag{32}$$

The time domain quantity C_2L_r can be obtained from the frequency domain parameters. Using (29), (30) and (31) gives

$$\tilde{\tilde{D}}_{red}(z)\,J\,\tilde{\tilde{D}}_{red}^{T}(z^{-1}) = \tilde{\overline{D}}_f(z)\begin{bmatrix} \tilde{R}_r & 0 \\ 0 & X \end{bmatrix}\tilde{\overline{D}}_f^{T}(z^{-1}) \tag{33}$$

If one knew the highest row degree coefficient matrix $\Gamma_r[\tilde{\overline{D}}_f(z)]$ one could compute $\tilde{\overline{D}}_f(z)$ as

$$\tilde{\overline{D}}_f(z) = \tilde{\tilde{D}}_{red}(z)\Gamma_r^{-1}[\tilde{\tilde{D}}_{red}(z)]\Gamma_r[\tilde{\overline{D}}_f(z)]$$

Substituting this and (32) in (33) one obtains

$$
\begin{aligned}
&\tilde{\tilde{D}}_{red}(z)J\tilde{\tilde{D}}_{red}^{T}(z^{-1}) = \tilde{\tilde{D}}_{red}(z)\Gamma_r^{-1}[\tilde{\tilde{D}}_{red}]\Gamma_r[\overline{D}^{\kappa}]\begin{bmatrix} I & 0 \\ C_2L_r & I \end{bmatrix} \\
&\begin{bmatrix} \tilde{R}_r & 0 \\ 0 & X \end{bmatrix}\begin{bmatrix} I & L_r^{T}C_2^{T} \\ 0 & I \end{bmatrix}\Gamma_r^{T}[\overline{D}^{\kappa}]\Gamma_r^{-T}[\tilde{\tilde{D}}_{red}]\tilde{\tilde{D}}_{red}^{T}(z^{-1})
\end{aligned}
$$

which after appropriate rearranging gives

$$
\begin{aligned}
&\Gamma_r^{-1}\left[\overline{D}^{\kappa}(z)\right]\Gamma_r\left[\tilde{\tilde{D}}_{red}(z)\right]\,J\,\Gamma_r^{T}\left[\tilde{\tilde{D}}_{red}(z)\right]\Gamma_r^{-T}\left[\overline{D}^{\kappa}(z)\right] = \\
&= \begin{bmatrix} \tilde{R}_r & \tilde{R}_rL_r^{T}C_2^{T} \\ C_2L_r\tilde{R}_r & C_2L_r\tilde{R}_rL_r^{T}C_2^{T} + X \end{bmatrix}
\end{aligned} \tag{34}
$$

At the left hand side of (34) are known frequency domain quantities, and from the right hand side the unknown quantity C_2L_r follows as

$$\begin{bmatrix} I \\ C_2L_r \end{bmatrix} = \begin{bmatrix} \tilde{R}_r & \tilde{R}_rL_r^{T}C_2^{T} \\ C_2L_r\tilde{R}_r & C_2L_r\tilde{R}_rL_r^{T}C_2^{T} + X \end{bmatrix}\begin{bmatrix} \tilde{R}_r^{-1} \\ 0 \end{bmatrix} \tag{35}$$

Consequently, Assumption 2 can be verified using frequency domain results. With C_2L_r from (35) the polynomial matrix $\tilde{\overline{D}}_f(z)$ can be computed from $\tilde{\tilde{D}}_{red}(z)$ via

$$\tilde{\overline{D}}_f(z) = \tilde{\tilde{D}}_{red}(z)\Gamma_r^{-1}\left[\tilde{\tilde{D}}_{red}(z)\right]\Gamma_r\left[\overline{D}^{\kappa}(z)\right]\begin{bmatrix} I & 0 \\ C_2L_r & I_\kappa \end{bmatrix} \tag{36}$$

3.2 The *a priori* filtering case

Assumption 3. With $\tilde{R}_r$ resulting from (34) $[I_{m_z}\ 0]\tilde{R}_r[I_{m_z}\ 0]^{T} < 0$ holds (this is the condition (18), assuring that the *a priori* filtering problem is solvable for the chosen γ).

In the realizable H_∞ filter (7), only the measurable outputs y_M are used. The polynomial matrix $\tilde{\overline{D}}(z)$ parameterizing the realizable reduced order H_∞ filter is related to the time domain parameters (case $F_1 = L_1$) by

$$\overline{D}^{-1}(z)\tilde{\overline{D}}(z) = C(zI - A)^{-1}\begin{bmatrix} 0 & F_1 & \Psi_2 \end{bmatrix} + \begin{bmatrix} I_{m_z+m-\kappa} & 0 \\ 0 & 0_\kappa \end{bmatrix} \tag{37}$$

[5]. A comparison of (26) and (37) shows that $\tilde{\overline{D}}(z)$ is given by

$$\tilde{\overline{D}}(z) = \overline{D}(z)\begin{bmatrix} I_{m_z} & 0 \\ 0 & 0_m \end{bmatrix} + \tilde{\overline{D}}_f(z)\begin{bmatrix} 0_{m_z} & 0 \\ 0 & I_m \end{bmatrix} \tag{38}$$

Using the results of [5] the quantity

$$\hat{y}_z(z) = \begin{bmatrix} I_{m_z} & 0_{m_z,m} \end{bmatrix}\tilde{\overline{D}}^{-1}(z)[\tilde{\overline{D}}(z) - \overline{D}(z)]\begin{bmatrix} 0 \\ y_M(z) \end{bmatrix} \tag{39}$$

can be shown to represent the transfer behaviour of the reduced order *a priori* H_∞ filter [6].

3.3 The *a posteriori* filtering case

Assumption 4. With $\tilde{R}_r$ resulting from (33) and $C_z\lambda_1$ computed below, the condition $[I_{m_z}\ C_z\lambda_1]\tilde{R}_r[I_{m_z}\ C_z\lambda_1]^{T} < 0$ holds (this is exactly (21), assuring that the *a posteriori* filtering problem is solvable for the chosen γ).

Solve again the J-spectral factorization problem (30), and compute $\tilde{\overline{D}}_f(z)$ (see (36)) and $\tilde{\overline{D}}(z)$ (see (38)). The polynomial matrix $\tilde{\overline{D}}^{+}(z)$ parameterizing the H_∞ filter in the *a posteriori* case is related to the time domain filter parameters by (37) for $F_1 = A\lambda_1$, namely

$$\overline{D}^{-1}(z)\tilde{\overline{D}}^{+}(z) = C(zI - A)^{-1}\left[0 \;\; A\lambda_1 \;\; \Psi_2\right] + \begin{bmatrix} I_{m_z+m-\kappa} & 0 \\ 0 & 0_\kappa \end{bmatrix} \tag{40}$$

To obtain the result (40) from (37), the gain matrix L_1 has to be substituted by

$$A\lambda_1 = (I - L_z C_z A^{-1})^{-1} L_1 \tag{41}$$

(see (19)). This results when adding the quantity

$$C(zI–A)^{-1}\left[0 \;\; L^{+}_{DIF} \;\; 0\right] \tag{42}$$

with

$$L^{+}_{DIF} = A\lambda_1 - L_1 = (I - L_z C_z A^{-1})^{-1} L_z C_z A^{-1} L_1 \tag{43}$$

to the factorization (37). Introducing the factorization

$$C(zI - A)^{-1} = \overline{D}^{-1}(z)\overline{N}_x(z) \tag{44}$$

the polynomial matrix $\tilde{\overline{D}}^{+}(z)$ of the reduced order H_∞ filter can be computed from

$$\tilde{\overline{D}}^{+}(z) = \tilde{\overline{D}}(z) + \overline{N}_x(z)\left[0 \quad L^{+}_{DIF} \quad 0\right] \tag{45}$$

In order to get the (time domain) quantity (43) from the frequency domain results consider the polynomial matrix

$$H_1(z) = \tilde{\overline{D}}_f(z) - \overline{D}(z)\begin{bmatrix} I & 0 \\ 0 & 0_\kappa \end{bmatrix} = \overline{N}_x(z)\left[L_r \quad \Psi_2\right] \tag{46}$$

If one knew the state space representation of the system, one could compute the factorization (44) and consequently also the gain matrices $\left[L_r \quad \Psi_2\right]$ from (46). This time domain characterization of the system is not known, but one can assume that there is an observable canonical realization of the system transfer matrix (4), giving rise to an (m_z+m)xn polynomial matrix $\overline{N}_x(z)$ in (44) of the form

$$\overline{N}_x(z) = \operatorname{diag}(\sigma_1^T, \ldots, \sigma_k^T) \tag{47}$$

where the σ_ν^T, $\nu = 1,2,\ldots,k$ are row vectors of the form $\left[z^{\delta_{r\nu}-1} \quad \cdots \quad z \quad 1\right]$ and the δ_{ri}, $i = 1,2,\ldots,m_z+m$ are the row degrees $\delta_{ri}\left[\overline{D}(z)\right]$. The σ_ν^T are only defined for such $i = \nu$, where $\delta_{ri}\left[\overline{D}(z)\right] \geq 1$. For all i where $\delta_{ri}\left[\overline{D}(z)\right] = 0$ the corresponding row of $\overline{N}_x(z)$ is a zero row.

With the above $\overline{N}_x(z)$, the entries in $\left[L_r \;\; \Psi_2\right] = [L_z \;\; L_1 \;\; \Psi_2]$ can be obtained from $H_1(z)$ by inspection. With this result

$$\left[L_z \quad 0 \quad 0\right]\overline{D}^{-1}(0)\overline{N}_x(0) = -L_z C_z A^{-1} \tag{48}$$

(see (44)) can be computed which gives L^{+}_{DIF} when substituted in (43) and finally $\tilde{\overline{D}}^{+}(z)$ when substituted in (45).

Using the results of [5] for $\tilde{\overline{D}}^{+}(z)$ the quantity

$$\begin{aligned} \hat{y}_z^{\,+}(z) = \left[I_{m_z} \quad 0_{m_z,m-\kappa} \quad 0_{m_z,\kappa}\right]\{(I - [0\;C\lambda_1\;\,0]) \\ \tilde{\overline{D}}^{+^{-1}}(z)[\tilde{\overline{D}}^{+}(z) - \overline{D}(z)] + [0\;C\lambda_1\;\,0]\}\begin{bmatrix} 0 \\ y_M(z) \end{bmatrix} \end{aligned} \tag{49}$$

can be shown to represent the transfer behaviour of the reduced order *a posteriori* H_∞ filter [6].

The quantity $C\lambda_1$ in the filter transfer behaviour of the reduced order a posteriori H_∞ filter can also be derived from the frequency domain results. Inspection of (40) shows that

$$[0\;C\lambda_1\;\,CA^{-1}\Psi_2] = -\overline{D}^{-1}(0)\tilde{\overline{D}}^{+}(0) + \begin{bmatrix} I_{m_z+m-\kappa} & 0 \\ 0 & 0_\kappa \end{bmatrix} \tag{50}$$

4 Example

Given a third order system with a (3,1) output vector $y^T = \left[y_z \quad y_1 \quad y_2\right]$ and a (3,1) input disturbance vector w. There is one disturbed (y_1) and one perfect measurement (y_2) (i.e. $m = 2$, $\kappa = 1$). Its frequency domain representation is

$$y(z) = \left\{\overline{D}^{-1}(z)\overline{N}(z) + \begin{bmatrix} 0 \\ D \end{bmatrix}\right\} w(z) \quad \text{with}$$

$$\overline{N}(z) = \begin{bmatrix} 0.7071 & -2.1213 & 0 \\ -1.4142 & -2.8284 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overline{D}(z) = \begin{bmatrix} 0.4714z - 0.4125 & 0.2357z - 1.0017 & 0.2357z + 0.5893 \\ -0.4714 & -0.9428 & 1.4142z + 0.4714 \\ 0.5774 & -0.5774 & -0.5774 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{i.e.} \quad D_1 = \left[0 \quad 0 \quad 1\right]$$

Since the measurement y_2 is not disturbed, the reduced order H_∞ filter is of order $n–\kappa = 1$.

To save space, we consider the *a priori* estimate only. The infimal value of γ is $\gamma_{opt} = \sqrt{35/6}$ [8]. For $\gamma = 2.41523$ and the above quantities, the right hand side of the polynomial equation (30) can be computed.

By J-spectral factorization (The first author thanks Polyx© for an α version of such a factorization program) one obtains

$$\overset{\approx}{D}(z)=\begin{bmatrix}-0.583z+0.112 & -0.908z-0.692 & -1.675z-0.122\\ -0.309z+0.134 & 0.570z-0.828 & -2.945z-0.146\\ 2.062z-0.215 & -0.171z+1.328 & -0.916z+0.234\end{bmatrix}$$

and

$$J=\mathrm{diag}(-1,1,1)$$

The determinant $\det\overset{\approx}{D}(z)$ contains two superfluous roots at z = 0 that can be extracted by

$$V_{ext}(z)=\begin{bmatrix}-1.0129z & 0.1586z & 0.0280z\\ 0.1611 & 0.9975 & 0.1760\\ 0 & 0.1738z & -0.9848z\end{bmatrix}$$

which meets $V_{ext}^{-1}(z)JV_{ext}^{-T}(z^{-1})=J$, giving (see (31))

$$\overset{\approx}{D}_{red}(z)=\begin{bmatrix}0.7817 & -1.2946z-0.6938 & 1.4913\\ 0.3054 & -0.8297 & 2.9996\\ -2.0357 & 1.3315 & 0.8718\end{bmatrix}$$

The determinant of this matrix has one root at z = 0.1429, (n-κ=1) which is the eigenvalue of the fictitious filter with input y. To get $\overset{\sim}{\overline{D}}_f(z)$ parameterizing this fictitious filter, one must compute C_2L_r via (34) and (35). With

$$\Gamma_r[\overline{D}^1(z)]=\begin{bmatrix}0.4714 & 0.2357 & 0\\ -0.4714 & -0.9428 & 1.4142\\ 0.5774 & -0.5774 & 0\end{bmatrix}\text{ and}$$

$$\Gamma_r[\overset{\approx}{D}_{red}(z)]=\begin{bmatrix}0 & -1.2946 & 0\\ 0.3054 & -0.8297 & 2.9996\\ -2.0357 & 1.3315 & 0.8718\end{bmatrix}$$

$C_2L_r=[-0.1714\quad 1]$ results giving (via (36))

$$\overset{\sim}{\overline{D}}_f(z)=\begin{bmatrix}0.4714z-0.6549 & 0.2357z+0.4125 & 0.8250\\ -0.7138 & 0.4714 & 1.4142\\ 0.5774 & -0.5774 & 0\end{bmatrix}$$

With $[I_{m_z}\ \ 0]\tilde{R}_r[I_{m_z}\ \ 0]^T=-2.7e-06$ the limit of condition (18) is nearly reached. The realizable filter with input y_M is parameterized by the polynomial matrix $\overset{\sim}{\overline{D}}(z)$ resulting from (38) as

$$\overset{\sim}{\overline{D}}(z)=\begin{bmatrix}0.4714z-0.4125 & 0.2357z+0.4125 & 0.8250\\ -0.4714 & 0.4714 & 1.4142\\ 0.5774 & -0.5774 & 0\end{bmatrix}$$

The root of $\det\overset{\sim}{\overline{D}}(z)$ is at z = 0, which is the eigenvalue of the optimal filter [8].

5 Conclusions

Based on the time domain results a frequency domain solution has been derived for the discrete time H_∞ estimation problem for nth order plants in the presence of κ perfect measurements. The H_∞ filter of order n-κ is parameterized by a polynomial matrix resulting from J-spectral factorization. Also *a posteriori* estimation can directly be handled in the frequency domain. The design results cover all cases between full-order and completely reduced order filters. A simple example demonstrated the design procedure.

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