COMPUTATION OF TIME-OPTIMAL SWITCHINGS FOR LINEAR SYSTEMS WITH COMPLEX POLES

F. Grognard*, **R. Sepulchre**†

* INRIA Sophia-Antipolis- COMORE Project, 2004 Av. des Lucioles, BP 93, 06902 Sophia-Antipolis Cedex, France,
frederic.grognard@inria.fr
† Institut Montefiore, B28, Université de Liège, B4000 Liège Sart-Tilman, Belgium,
r.sepulchre@ulg.ac.be

Keywords: Time-optimal, bang-bang control, MPC, satellite, algorithm.

Abstract

The minimum-time bounded control of linear systems is generically bang-bang and the number of switchings does not exceed the dimension of the system if the eigenvalues of the system matrix are real or if the initial condition is sufficiently close to the target. This paper extends the method of [8] for computing the switching times of time-optimal controllers to linear systems with complex poles and demonstrates its application on MPC schemes.

1 Introduction

In this paper, we will consider the problem of steering a solution from an initial condition z_0 to the origin for single-input linear systems

$$\dot{z} = Az + bv \tag{1.1}$$

subject to the input constraint

$$|v| \leq 1$$

where $z \in I\!R^n$, $v \in I\!R$, and the pair (A, b) is controllable.

The corresponding stabilization problem has long been recognized as a significant nonlinear control problem, so that many solutions have been proposed: anti-windup schemes, low-gain control laws or Model Predictive Control (MPC) schemes.

The anti-windup schemes are extensively used in industry but they are often ad-hoc and rarely propose stability proofs (see, for recent theoretical results [10] and [17]). Low-gain control laws provide proofs of semiglobal stability ([11, 12, 16]), but do so at the expense of performance. MPC schemes are also widely used in industry, but their application depends on the existence of fast algorithms for the computation of solutions of optimal control problems. In [2] and [4], this problem is avoided by giving an explicit form of the MPC controller which does not require the online computation. Such a controller cannot always be computed, so that one must rely on the online computation of the solution of optimal control problems. In this paper, we are interested in such an algorithm, where the cost to minimize is the total time.

A natural control method for linear systems with magnitude constraint is time-optimal control, which is well known to be bang-bang, with the switchings occuring on so called "switching curves" in the state space. The computation of those curves is equivalent to computing a feedback control law $v^*(z)$, and is untractable for large systems.

This practical limitation implies that the implementation of time-optimal control is best achieved through the computation of open-loop control. Also, due to the lack of robustness of open-loop control, it is suggested to close the loop by nesting this open-loop control in an MPC scheme: every τ units of time, a time-optimal control law $v^*(t)$ is numerically computed online with the current $z(k\tau)$ as initial condition, and this control law is applied during τ units of time; at time $(k+1)\tau$, the same control problem is recomputed,... It is therefore important to design algorithms that can rapidly solve online the optimal control problem that is posed every τ units of time. We focus on that problem in the special case of time-optimal control.

The challenge then consists in designing efficient iterative schemes to compute the time-optimal control law $v^*(t)$ for any given z_0. Several gradient-based iterative methods have been proposed. These gradient methods typically iterate on the adjoint initial or final state together with the time of response (see for instance [5, 6, 9, 13], and, for a summary of those methods, [14]). It is known that these methods are, in general, sensitive to the starting condition (initial guess) and have poor convergence properties.

In [8], we have presented an algorithm based on another approach: it uses the bang-bang property of the time-optimal controller. The algorithm is designed to operate when the number of switchings is less or equal to $n-1$. It sees the computation of the time-optimal control as the computation of the optimal sequence of switching times $0 = t_0 < t_1, \cdots < t_n = T$ or, equivalently, the optimal sequence of time intervals $\bar{x}_1 = t_1 - t_0, \bar{x}_2 = t_2 - t_1, \ldots, \bar{x}_n = t_n - t_{n-1}$. In this paper, we construct continuous time-systems $\dot{x} = f(x)$ which 'produce' the optimal sequence $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T$, in the sense that they possess an isolated equilibrium at $x = \bar{x}$ and that this equilibrium is asymptotically stable. The main result of [8] shows that, when the eigenvalues of A are real, the time-optimal controller presents $n-1$ switchings or less, and under proper time-scale decomposition, the semiglobal convergence of solutions to the desired equilibrium $x = \bar{x}$ can be enforced.

This paper will concentrate on the case where the eigenvalues of A are complex. In Section 2, we indicate a case where the number of switchings of the time-optimal controller is $n-1$ or less. The algorithm and the main convergence results are then given in Section 3. Finally we implement an MPC scheme for a change of orbit for a nonlinear model of a satellite in Section 4. Conclusions are given in Section 5.

2 Switchings in time-optimal controllers

The solution of the time optimal control problem

$$\begin{array}{lll} T^* = \min T & & \\ \quad \text{s.t.} & \dot{z} = Az + bv & \\ & z(0) = z_0 & (\mathcal{TO}) \\ & z(T) = 0 & \\ & |v(t)| \leq 1 & \end{array}$$

has long been characterized as a nice application of the Maximum Principle [15]. The time-optimal control is bang-bang and the switching times are the roots of $\eta^*(t)^T b$, where $\eta^*(t) = e^{-A^T t}\eta_0$ is the adjoint response of the system for a suitable vector η_0. Also, in the case of $\mathcal{TO}$, any bang-bang controller whose switching times correspond to the roots of some $\eta^*(t)^T b$ is time-optimal (the Maximum Principle is necessary and sufficient [1]). Theorem 1 employs this property and Proposition 1 to characterize a set of initial conditions that can be steered to the origin with a bang-bang control that involves at most $n-1$ switchings.

Notation 1 • *We will denote ω_{max} the maximum of the imaginary parts of the eigenvalues of A. When $\omega_{max} = 0$, $\frac{r}{\omega_{max}}$ denotes $+\infty$ (for $r > 0$).*

- *Let $T \in \mathbb{R}^+_*$. The set $\mathcal{C} \subset \mathbb{R}^n$ is the set of initial conditions z_0 that are null-controllable. The set $\mathcal{C}(T) \subset \mathcal{C}$ is the set of initial conditions z_0 that are null-controllable in time $t \leq T$.*

Proposition 1 *[18] Let $A \in \mathbb{R}^{n\times n}$, $b, \eta \in \mathbb{R}^n$ with the pair (A, b) controllable and $\eta \neq 0$. The number N of roots of the exponential polynomial $P(t) = \eta^T e^{-At} b$ inside the interval $[0, T]$ satisfies*

$$N \leq n - 1 + \frac{T\omega_{max}}{\pi} \tag{2.2}$$

Proposition 1 then results in the following theorem:

Theorem 1 *For any $z_0 \in \mathcal{C}(\frac{\pi}{\omega_{max}})$, there exists a unique bang-bang controller which steers $z(t)$ from z_0 to the origin with $n-1$ switchings or less and a total time inferior or equal to $\frac{\pi}{\omega_{max}}$. Moreover, this bang-bang controller is the solution of $\mathcal{TO}$.*

Proof: The fact that $z_0 \in \mathcal{C}(\frac{\pi}{\omega_{max}})$ implies that there exists a solution to $\mathcal{TO}$ with $T^* \leq \frac{\pi}{\omega_{max}}$. This controller is unique, bang-bang and we will show that it switches at most $n-1$ times. This results from the coincidence of the switching times of the optimal controller with the roots of $\eta^*(0)^T e^{-At} b$ and from the bound on the number of roots of an exponential polynomial given by Proposition 1:

(a) If $T^* < \frac{\pi}{\omega_{max}}$, Proposition 1 indicates that the number of roots of $P(t) = \eta^*(0)^T e^{-At} b$ inside the interval $[0, T^*]$ is inferior to a real number belonging to the interval $[n-1, n)$. Because the number of roots is an integer, the actual upper bound is equal to $n-1$, so that the number of switchings of $v^*(t)$ is inferior or equal to $n-1$.

(b) If $T^* = \frac{\pi}{\omega_{max}}$, the number of roots of $P(t)$ inside the interval $[0, T^*]$ is less or equal to n, according to (2.2). Two cases have to be considered: either this number of roots is inferior or equal to $n-1$, so that the number of switchings is also bounded by $n-1$, or the number of roots equals n. In this latter case, one root must be equal to 0 and one other equal to T^*. Otherwise, one could find a smaller interval containing n roots of $P(t)$, which is in contradiction with (2.2). This indicates that only $n-2$ roots lie in the interior of the interval $[0, T^*]$, so that only $n-2$ actual switchings take place.

We have now shown the existence of a bang-bang controller with $n-1$ switchings or less and $T \leq \frac{\pi}{\omega_{max}}$. Uniqueness is proven by showing that any such bang-bang controller is the unique solution of $\mathcal{TO}$: let $v(t)$ ($t \in [0, T]$) be a bang-bang control law that steers $z(t)$ from z_0 to the origin with $n-1$ switchings or less. Let $t = t_j$ ($j = 1, \cdots, N \leq n-1$) be the switching times.

(A) Let $T < \frac{\pi}{\omega_{max}}$. If $N < n-1$, then complement the list of t_j with $n-1-N$ (arbitrarily chosen) distinct values larger or equal to T (and smaller than $\frac{\pi}{\omega_{max}}$, if $\omega_{max} \neq 0$). One can find a non trivial η_0 such that $\eta_0^T e^{-At_j} b = 0$ ($j = 1, \cdots, n-1$). This means that the t_j are the $n-1$ roots of $\eta_0^T e^{-At} b = 0$ inside the interval $[0, t_{n-1}]$ of length inferior to $\frac{\pi}{\omega_{max}}$. From Proposition 1, we know that no other root can be found inside this interval $[0, t_{n-1}]$, so that $v(t)$ and $\operatorname{sign}(\eta_0^T e^{-At_j} b)$ have exactly the same switching times. It is then sufficient to pick $\eta(0) = \eta_0$ or $-\eta_0$ to ensure that $v(t)$ and $\operatorname{sign}(\eta(0)^T e^{-At_j} b)$ are identical. As a consequence, $v(t)$ is maximal, which is sufficient for $v(t)$ to be optimal in the case of $\mathcal{TO}$. Therefore $v(t)$ is equal to the unique $v^*(t)$.

(B) Let $T = \frac{\pi}{\omega_{max}}$. We will compare $v(t)$ and $v^*(t)$ (which produces the solution $z^*(t)$). Let t_1^* be the first switching time of $v^*(t)$ and $\tilde{t}_1 = \min(t_1, t_1^*)$. Two cases then arise: either $v(t) = v^*(t)$ or $v(t) = -v^*(t)$ in the interval $[0, \tilde{t}_1]$. If $v(t) = v^*(t)$ in the interval, then $z(\tilde{t}_1) = z^*(\tilde{t}_1)$. The control $v(t)(t \in [\tilde{t}_1, \frac{\pi}{\omega_{max}}])$ is then a bang-bang controller steering $z(t)$ from $z(\tilde{t}_1)$ to the origin in a time smaller than $\frac{\pi}{\omega_{max}}$, and with $n-1$ switchings or less. It is therefore optimal (see point **(A)**). By optimality of subtrajectories of an optimal solution, the same can be said of $v^*(t)$, so that $v(t) = v^*(t)$ for $t \in [\tilde{t}_1, \frac{\pi}{\omega_{max}}]$. Finally, $v(t) = v^*(t)$ for $t \in [0, \frac{\pi}{\omega_{max}}]$, so that $v(t)$ is solution of $\mathcal{TO}$. In the case where $v(t) = -v^*(t)$ in the interval $[0, \tilde{t}_1]$, it is clear that $T^* < \frac{\pi}{\omega_{max}}$ (in the case where $T^* = \frac{\pi}{\omega_{max}}$, $v^*(t)$ and $v(t)$ would be two different optimal solutions, which is impossible). The result of **(A)** implies that $v(t)$ ($t \in [\epsilon, \frac{\pi}{\omega_{max}}]$) is time-optimal from $z(\epsilon)$ with an optimal time $\frac{\pi}{\omega_{max}} - \epsilon$. As $\epsilon \to 0$, this optimal time tends to $\frac{\pi}{\omega_{max}}$ and $z(\epsilon)$ tends to z_0. By continuity of the optimal time with respect to the initial condition [1], the optimal time from z_0 should then be $\frac{\pi}{\omega_{max}}$, which is in contradiction with the observation that was made ($T^* < \frac{\pi}{\omega_{max}}$).

We have then shown that any bang-bang controller with $n-1$ switchings or less and $T \leq \frac{\pi}{\omega_{max}}$ that steers $z(t)$ from z_0 to the origin is the unique solution of $\mathcal{TO}$. Such a controller is therefore unique. □

Note that, for $z_0 \in \mathcal{C} \setminus \mathcal{C}(\frac{\pi}{\omega_{max}})$, this result is not valid anymore. This can be observed on the harmonic oscillator $\dot{z}_1 = z_2,\ \dot{z}_2 = -z_1 + v$; if we only consider the cases where $z_2(0) = 0$, we see that $z_0 \in \mathcal{C}(\frac{\pi}{\omega_{max}})$ if $|z_1(0)| \leq 2$. For any $0 < \epsilon < 2$, there exists a unique solution that steers $z(t)$ to the origin and that switch only once when $z_1(0) = -2 - \epsilon$ or $z_2(0) = 2 + \epsilon$; this solution is not time-optimal. The actual time-optimal solution should switch twice (see [1]). Also, for those initial conditions, there is an infinite number of solutions that steer $z(t)$ to the origin and switch twice.

As a consequence of this theorem, we will make the following assumption throughout this paper:

Assumption 1 *Suppose that $z_0 \in \mathcal{C}(\frac{\pi}{\omega_{max}})$.*

When $\omega_{max} > 0$, it is easily seen that $\mathcal{C}(\frac{\pi}{\omega_{max}})$ is a compact set with the origin in its interior, and whose border is the minimum isochrone corresponding to the time $T^* = \frac{\pi}{\omega_{max}}$. The set $\mathcal{C}(T)$ monotonically increases as a function of T and tends to $\mathcal{C}$ as T grows unbounded, which is also the case of $\mathcal{C}(\frac{\pi}{\omega_{max}})$ as ω_{max} goes to 0. In the limit, we recover the classical result that the time-optimal solution involves at most $n-1$ switchings when all the eigenvalues of A are real.

Theorem 1 justifies the approach that is taken in this paper: instead of looking for a time-optimal controller, or for the initial condition $\eta^*(0)$ of the adjoint system as previous algorithms did, we look for a controller that switches at most $n-1$ times. If the algorithm converges, Theorem 1 indicates that optimality can be tested as follows:

Optimality Test: *If $v(t)$ ($t \in [0, T]$) is a bang-bang controller that steers $z(t)$ from z_0 to 0 with $n-1$ switchings or less, and if $T \leq \frac{\pi}{\omega_{max}}$, then $v(t)$ is the time-optimal solution of $\mathcal{TO}$.*

3 An algorithm for the computation of bang-bang steering controls

Description of the algorithm

In the set $\mathcal{C}(\frac{\pi}{\omega_{max}})$, the search for the optimal control can be restricted to the steering controls that are defined by a sequence of n time intervals $x_i \triangleq t_i - t_{i-1}$ and the corresponding sequence of constant control values u_i. This class of piecewise constant controls is characterized by a pair of vectors (x, u), where x denotes the vector of time intervals and u denotes the vector of control values. The time-optimal solution is then defined by $(\bar{x}, \bar{u})$, with $|\bar{u}_i| = 1$.

From the solution of the linear system for $t \geq t_0 = 0$

$$z(t) = e^{At}\left(z(0) + \int_0^t e^{-A\tau} b v(\tau) d\tau\right),$$

it is seen that a control defined by the pair (x, u) will steer z_0 to $z = 0$ if it satisfies the 'steering equation'

$$\Phi(x)\, u = -z_0 \tag{3.3}$$

where the i-th column of the matrix Φ is

$$\Phi_{(:,i)}(x) \triangleq \int_{t_{i-1}}^{t_i} e^{-A\tau} b d\tau = \int_{\sum_{k=1}^{i-1} x_k}^{\sum_{k=1}^{i} x_k} e^{-A\tau} b d\tau$$

The equation $\Phi(x)\,\bar{u} = -z_0$ is the nonlinear equation to be solved to determine the optimal control. In contrast, (3.3) is linear in u and is easily solved for a given x. Denoting the open positive orthant $\mathcal{O}_n^+$, it can be seen that $\Phi(x)$ is regular inside the set $\mathcal{K} = \{x \in \mathcal{O}_n^+ | \sum_{i=1}^n x_i \leq \frac{\pi}{\omega_{max}}\}$, so that a unique solution $u(x) = -\Phi^{-1}(x) z_0$ of (3.3) exists for any x in $\mathcal{K}$. A natural class of iterative methods thus consists in updating the time intervals vector x such as to enforce convergence of the corresponding control vector $u(x)$ to a bang-bang sequence of magnitude $|u_i| = 1$.

The heuristics considered in [4] and [8] are the "decentralized" adaptation of the vector x: if $|u_i(x)|$ is larger than one, increase the length of the corresponding time interval x_i; if $|u_i(x)|$ is smaller than one, decrease the length of the corresponding time interval x_i.

In continuous-time, these heuristics yield the decentralized adaptation

$$\dot{x}_i = f_i(|u_i(x)| - 1) x_i,\ i = 1, \ldots, n \tag{3.4}$$

where f_i should be a (smooth) scalar function with its image in the first and third quadrant and should only vanish at zero. x_i multiplies f_i in order to guarantee the positive invariance of the open positive orthant.

Convergence

In [8], we have only considered the case where $\omega_{max} = 0$ (only real eigenvalues for A) and provided a global analysis of the continuous-time system (3.4) with the functions f_i selected as saturated linear functions, yielding the algorithm:

$$\epsilon_i \dot{x}_i = \text{sat}_M(|u_i(x)| - 1)x_i, \quad i \in \{1, \cdots, n\}, \; x_i(0) > 0 \tag{3.5}$$

where $\text{sat}_M(s) = \frac{Ms}{\max(|s|,M)}$ ($M > 0$). With $0 < \epsilon_n << \epsilon_{n-1} << \cdots << \epsilon_1$, a time-scale separation can be enforced between the different x_i dynamics, and the different control values $|u_i|$ successively converge to 1 (starting with $|u_n|$). Based on Theorem 1, the theorem of [8] can be generalized to the case where the eigenvalues of A are complex and $z_0 \in \mathcal{C}(\frac{\pi}{\omega_{max}})$.

Theorem 2 *If $z_0 \in \mathcal{C}(\frac{\pi}{\omega_{max}})$, then the equilibrium set Ω of*

$$\begin{cases} \dot{x}_1 & = & sat(|u_i(x)| - 1)x_i \\ & \vdots & \\ \epsilon_n \dot{x}_n & = & sat(|u_n(x)| - 1)x_n \end{cases} \tag{3.6}$$

inside $\mathcal{K} = \{x \in \mathcal{O}_n^+ | \sum_{i=1}^n x_i \leq \frac{\pi}{\omega_{max}}\}$ is non empty and is asymptotically stable. It is exponentially stable if Ω is a singleton.

Moreover, if A only has real non positive eigenvalues, the region of attraction of Ω in the positive orthant is enlarged at will in $\mathcal{O}_n^+$ by proper separation of the time-scales $\tau_n = \frac{t}{\epsilon_n}, \cdots, \tau_i = \frac{t}{\epsilon_i}$

In Theorem 2, numerical simulations suggest that the region of attraction of Ω includes the entire set $\mathcal{K}$. However, a theoretical characterization of the basin of attraction seems not immediate in the proof in [8]. Extension of the region of attraction of Ω beyond $\mathcal{K}$ is not feasible because of the possible singularity of $\Phi(x)$ and the possible existence of other equilibria outside $\mathcal{K}$.

A natural way of initializing the algorithm consists in taking all the elements of $x(0)$ very small. This almost ensures that $x(0)$ belongs to $\mathcal{K}$, and that convergence to the desired equilibrium takes place. However, convergence to the time-optimal solution can only be checked a posteriori by using the Optimality test of Section 2.

Implementation

We illustrate on Figure 1 the implementation of the algorithm on the controlled harmonic oscillator:

$$\begin{cases} \dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & -z_1 + v \qquad |v| \leq 1 \end{cases}$$

with $z_0 = (1 \;\; 1)^T$, an initial condition such that the time-optimal solution only presents one switching ($\bar{x} = (0.9305 \; 1.5709)^T$).

In order to implement the algorithm, we need to discretize it. The separation of the time-scales results in a very stiff set of differential equations, whose behavior can only be reproduced in discrete time by taking a very small discretization step. This results in slow convergence.

However, we have observed that the algorithm is robust to a reduction of the time-scales separation (see [7]). It tolerates that we take $\epsilon_i = 1$ for all i. As can be seen on Figure 1, this does not prevent the convergence from taking place, but the phase-plane is modified (compare the solid lines, where $\epsilon_1 = 1$ and $\epsilon_2 = 0.1$, with the dotted lines, where $\epsilon_1 = \epsilon_2 = 1$).

Without the time-scales separation, the differential equations are not stiff anymore, so that a simple large-step Euler discretization gives a good approximation of the behavior of the continuous system (compare the dotted and dash-dotted lines), and a very fast convergence (in the example, the equilibrium is reached in less than ten steps for the four initial conditions of Figure 1). The actual algorithm is then

$$x_i(k+1) = x_i(k) + \delta \; \text{sat}(|u_i(x(k))| - 1)x_i(k) \text{ for } i \in \{1, \cdots, n\}$$

where δ is the discretization step. We have shown in [7] that δ needs to be smaller than 1 to ensure invariance of the positive orthant. In the example, we have taken $\delta = 0.5$.

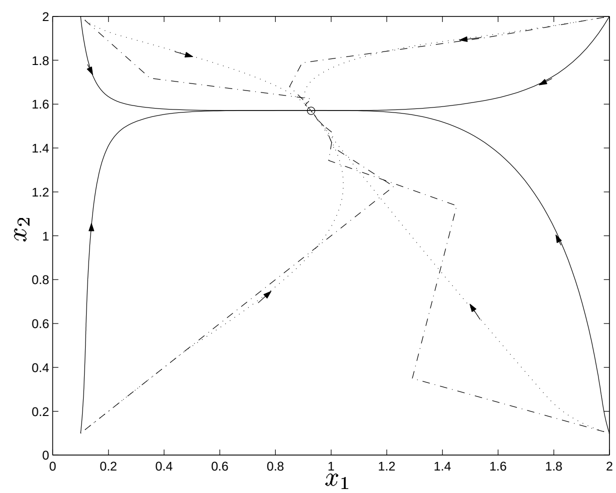

Figure 1: Phase plane of the evolution of the algorithm for the controlled harmonic oscillator with $z_0 = (1 \; 1)^T$. The continuous algorithm with time-scales separation (solid line), without time-scales separation (dotted-line), and the discrete algorithm without time-scales separation (dash-dotted line) are illustrated. The initial conditions for the algorithm which are illustrated are: $x_0 = (0.1 \;\; 0.1)^T$; $x_0 = (0.1 \;\; 2)^T$; $x_0 = (2 \; 0.1)^T$; $x_0 = (2 \; 2)^T$.

Heuristics for the utilization of the algorithm

From the comments on the initialization and the discretization of our algorithm, we suggest that $x(0)$ be picked close to the origin, and that a large-step Euler discretization be employed. After several steps of the algorithm, optimality of the solution is guaranteed if $T \leq \frac{\pi}{\omega_{max}}$ (see Theorem 1).

4 Time-optimal control in a receding horizon scheme

In this section, the application of receding horizon based on time-optimal control and saturated linear control applied to a nonlinear model of an orbiting satellite are compared.

Let us consider the orbital transfer problem for a satellite having a circular orbit around the earth. We consider that the target is a geostationary orbit. It evolves 36000 km above the earth, and its revolution takes 24 hours. The mass of the satellite is estimated at 2000 kg and the maximal thrust (in the direction of the tangent to the orbit) amounts to 2N. We suppose that the satellite starts its journey 400 km below the target geostationary orbit. The dynamics of this satellite are:

$$\left\{ \begin{array}{rcl} \ddot{r} & = & \omega^2 r - \frac{k}{r^2} \\ \dot{\omega} & = & -\frac{2\omega\dot{r}}{r} + \frac{v}{mr} \end{array} \right.$$

where r is the distance of the satellite to the center of the earth, ω is its angular velocity, and v is the tangential thrust [3]. The constant m is the mass of the satellite and $k = 3.9851.10^{14} m^3/s^2$ is known. The equilibrium of motion of a geostationary satellite satisfies $\bar{\omega} = \frac{2\pi}{86400} = 7.272\ 10^{-5} rad/s$ and $\bar{r} = 42238km$ (radius of the earth+36000 km). In order to apply time-optimal control, we compute the linearization of the system around the target equilibrium of motion and chose the variables like in [3]: $(z_1, z_2, z_3) = (r - \bar{r}, \dot{r}, (\omega - \bar{\omega})\bar{r})$. This results in the linearized system

$$\dot{z} = \begin{pmatrix} 0 & 1 & 0 \\ 3\bar{\omega}^2 & 0 & 2\bar{\omega} \\ 0 & -2\omega_0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{m} \end{pmatrix} v$$

which has its pole in 0 and $\pm\bar{\omega}i$. We have shown that a time-optimal solution that takes less than $T = \frac{\pi}{\bar{\omega}} = 43200s = 12h$ presents $n-1$ switchings or less (and any bang-bang solution presenting $n-1$ switchings or less with $T \leq 12h$ is time-optimal). Our algorithm can compute a bang-bang orbital transfer for the linear model if $T \leq 12h$: the control value $+2$ is applied during $x_1 = 13953$ seconds, followed by -2 during $x_2 = 14405$ seconds and $+2$ during $x_3 = 14475$ seconds. The transfer takes 42833 seconds, that is close to, but smaller than 12 hours. The Optimality test of Section 2 indicates that this bang-bang control is time-optimal for the linearized model. If we apply this strategy on the nonlinear model in open-loop, the nonlinearities prevent the transfer from being exactly achieved.

In order to compensate for the nonlinearities, a receding horizon scheme can be used: the time-optimal strategy (based on the linear model) is recomputed every ten minutes. However, the computed control law is not applied to the system as is. Indeed, once the first time-interval is elapsed, the solution x of the time-optimal control problem contains one value x_i, which is very small. Due to the nonlinearities, this value x_i is not exactly zero. Moreover, it can occur that $i = 1$, that is the solution of the time-optimal control problem starts with $u = +2$ for a very short time, and then switches to $u = -2$ for a long time. As this phenomenon can occur at each step of the Receding Horizon Scheme, the control law will present uselessly many switchings. We have eliminated this problem by ignoring the time intervals that are smaller than ten minutes, so that, if $x_1 < 600s$, the corresponding control is not applied. It is apparent on Figure 2 that this strategy leads to an exact transfer from one orbit to the other. This transfer takes 44400 seconds, that is a little bit more than twelve hours. It presents more than two switchings because the "errors" introduced by the nonlinearities need to be compensated for along the way. Basically, the control law is close to a strict bang-bang control with two switchings: the control value $+2$ is applied during 13800 seconds, followed by -2 during 16200 seconds and $+2$ during 14400 seconds. However, the compensation of the nonlinearities implies three occurrences of $u = +2$ during the second time interval, and one occurrence of $u = -2$ during the third interval.

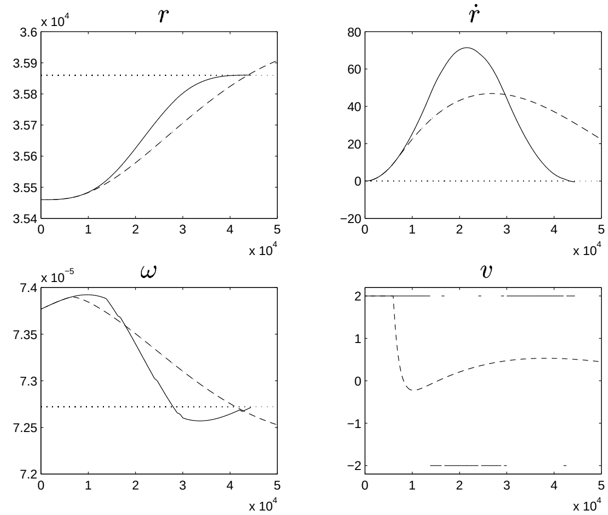

Figure 2: Orbital transfer using a receding horizon strategy (solid line) or a saturated linear controller (dash-dotted line)

A saturated linear controller is built for comparison. We choose to apply the design presented in [16]: a family of Riccati-based controllers is built, and a controller that does not saturate along the solution is chosen, so that convergence to the origin is not prevented by the saturation. In order to have a balanced convergence to the origin, we rescale the variables of the linear systems. Indeed, we have $z_1(0) = -400000$ and $z_3(0) = 44.1555$. Therefore, we define $w_1 = z_1/400000$, $w_3 = z_2/44.1555$, and $w_2 = z_3/60$ (based on the observation made on the time-optimal solution). Such an approach with Q, the identity matrix, as left hand side of the Riccati equation, yields the following controller, which does not saturate along the solution

$$u = -\,\text{sat}(2.1805\ 10^{-5} z_1 + 0.0474 z_2 + 0.1677 z_3) \quad (4.7)$$

By essence, this control design leads to controllers with infinite gain-margin. Therefore, we can replace (4.7) with

$$u = -\,\text{sat}(k(1.853\ 10^{-5} z_1 + 0.0341 z_2 + 0.1409 z_3)) \quad (4.8)$$

with $k > 1$. This will make better use of the available actuation, and still ensure stability in approximately the same region (we have taken $k = 10$). On Figure 2, it appears that the linear controller leads to a much slower convergence than the time-optimal one. It does not succeed in reproducing the two switchings. The first one is present (though early), but the second one is smoothed out.

Not surprisingly, the inclusion of the time-optimal controller inside an MPC loop yields improve performance with respect to what is obtained with a linear controller.

5 Conclusion

In this paper, we have proposed an algorithm that computes time-optimal switchings for linear systems with complex poles. The analysis extends previous results restricted to the case of real poles. Fast algorithms that compute bounded steering controls are of interest for the online calculation of bounded stabilizing feedbacks. The utilization of our algorithm in a receding horizon control implementation has been illustrated on a satellite example.

References

[1] Athans, M., Falb, P.L. (1966), *Optimal Control. An introduction to the theory and its applications*, Mc Graw-Hill, New York.

[2] Bemporad A., Morari M., Dua V., Pistikopoulos E.N.(2001), ”The Explicit Linear Quadratic Regulator for Constrained Systems”, Tech. Report AUT99-16, ETH, submitted to *Automatica*.

[3] Brockett R., “Finite dimensional linear systems,” John Wiley & Sons, 1970.

[4] De Doná, J.A. (2000), *Input Constrained Linear Control*, PhD. thesis, The University of Newcastle, Australia.

[5] Fadden, E.J., Gilbert E.G. (1964), “Computational aspects of the time optimal control problem,” *in* A.V. Balakrishnan & L.W. Neustadt, eds, “Computing Methods in Optimization Problems,” Academic Press, New York.

[6] Gilbert E.G. (1964), “The application of hybrid computers to the iterative solution of optimal control problems, *in* A.V. Balakrishnan & L.W. Neustadt, eds, “Computing Methods in Optimization Problems,” Academic Press, New York, pp. 261-284.

[7] Grognard F. (2001), “Control of constrained systems: closed-loop, open-loop and hybrid solutions,” PhD Thesis, Universit Catholique de Louvain, Belgium.

[8] Grognard F., Sepulchre R. (2001), ”Global stability of a continuous-time flow which computes time-optimal switchings”, 40th IEEE Conference on Decision and Control, Orlando (December 2001), pp. 3826-3831.

[9] Ho, Y.C. (1962), “A successive approximation technique for optimal control systems subject to input saturation,” *Journal of Basic Engineering, Transactions of ASME*, **84D**, 33-40.

[10] Kapoor N., Teel A., Daoutidis P. (1996), “On anti-integrator-windup and global asymptotic stability”, *Proceedings 13th IFAC World Congress*, San Francisco, **D**, pp. 67-72.

[11] Lin Z., Saberi A. (1993), “Semiglobal exponential stabilization of linear systems subject to input saturation via linear feedbacks”, *Systems & Control Letters*, **21**, no. 3, pp. 225-239.

[12] Megretski A. (1996), “L_2 BIBO output feedback stabilization with saturated control”, *Proceedings 13th IFAC World Congress*, San Francisco, **D**, pp. 435-440.

[13] Neustadt, L.W. (1960), “Synthesizing time-optimal control systems,” *Journal of Mathematical Analysis and Applications*, **1**, pp. 464-493.

[14] Plant, J.B. (1968), *Some iterative solutions in optimal control*, Research Monograph No 44, The M.I.T. Press, Cambridge, Massachussets.

[15] Pontryagin, L.S., Boltyanskii, V., Gamkrelidze, R., Mischenko, E. (1962), *The Mathematical Theory of Optimal Processes*, Interscience Publishers, Inc., New York.

[16] Saberi A., Lin Z., Teel A. (1996), “Control of Linear Systems with Saturating Actuators”, *IEEE Trans. on Automatic Control*, **41**, no. 3, pp. 368-378.

[17] Teel A. (1999), “Anti-windup for exponentially unstable linear systems,” *International Journal of Robust and Nonlinear Control*, **9**, no. 10, pp. 701-716.

[18] Voorhoeve, M. (1976), “On the oscillation of exponential polynomials,” *Mathematische Zeitschrift*, **151**, pp. 277-294.