INVESTIGATION OF OPTIMAL FILTERING AND SMOOTHING ALGORITHMS FOR ONE CLASS OF APPLIED PROBLEMS

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Abstract

The paper considers the optimal filtering and smoothing problems and peculiar features of their solutions when gravity anomaly is determined from an aircraft by using precision satellite measurements of coordinates and velocity.

1 Introduction

It is not infrequently that in practice the estimation accuracy of processes described by stationary equations can be increased by off-line processing, i.e., when it is possible to use not only previous, but also subsequent, relative to a current point, measurements. This situation occurs, for example, in surveying gravity anomalies (GA) from a vehicle, an aircraft, in particular. The possibilities to increase the accuracy of gravimetric surveys depend to a great extent on the possibility to obtain subcentimeter accuracies in determining coordinates and rather accurate velocity by using phase and Doppler data from a satellite navigation system (SNS) [1,2].

With the availability of precision coordinates and velocities provided by the SNS the problem of determining GA is reduced, as regards processing, to integration of all available data with the aim to derive the greatest possible GA accuracy. The difference between the second integral of the gravimeter readings and the altitude from the SNS, and the difference between the first integral of the gravimeter readings and the vertical velocity from the SNS are formed in order to exclude unknown altitude values h, vertical velocity V_z and acceleration $\dot{V}_z$. The differential measurements can be presented as follows [2]

$$z_h = \frac{\tilde{g}^{\text{gr}}}{p^2} - h^{\text{SNS}} = \frac{\tilde{g} + \delta g}{p^2} - \delta h \, ; \qquad (1)$$

$$z_{V_z} = \frac{\tilde{g}^{\text{gr}}}{p} - V_z^{\text{SNS}} = \frac{\tilde{g} + \delta g}{p^2} - \delta V_z \, . \qquad (2)$$

in operator form. Here $\tilde{g}$ - gravity anomaly; δg - gravimeter errors; $\tilde{g}^{\text{gr}}$ - gravimeter readings; $h^{\text{SNS}}, V_z^{\text{SNS}}$ - altitude and vertical velocity from SNS; $\delta h, \delta V_Z$ - the errors of SNS measurements.

The estimation problem consists in obtaining gravity anomalies, altitude and velocity using differential measurements (1), (2). Usually the processing is carried out with stationary low-frequency Butterworth filters for direct and reverse time whose parameters (order and cutoff frequency) are selected empirically in order to make gravity disturbance estimation most effective [1,4]. At the same time specifying stochastic models for $\tilde{g}$, $\delta h, \delta V_Z$ and δg the problem of gravity anomaly estimation can be formulated in the framework of the optimal filtering and smoothing theory [3]. However a number of questions arise. What models should be chosen to describe satellite measurement errors and GA? What are the procedures for optimal filtering and smoothing algorithms and what are their peculiarities? What is the increase in GA estimation accuracy in the smoothing mode as compared to that in the filtering mode? What is the benefit from integrated use of altitude and velocity?
It is the discussion of these questions that this paper is devoted to.

2 Statement of optimal filtering and smoothing problems and the algorithms for their solution

As a rule, time-invariant models are used in describing errors of satellite measurements and gravity anomalies [1-2]. Let an n - dimensional vector x and the corresponding m - dimensional measurements be specified as

$$\dot{x}(t) = Fx(t) + Gw(t) \, , \qquad (3)$$

$$y(t) = Hx(t) + v(t) \, , \qquad (4)$$

where F, G, H are time-independent $n \times n, n \times p, m \times n$ - dimensional matrices, respectively; $w(t)$, $v(t)$ - the p - and m - dimensional zero-mean white Gaussian noises independent of each other and $x(0)$. The intensities (power spectral density) for $v(t)$ and $w(t)$ are R>0 and E (unit matrix), respectively.

For the n - dimensional vector x or r - dimensional vector z related to it by the equation

$$z = Dx,$$

it is necessary for the steady-state mode to derive optimal (minimum variance) estimates and covariance matrices corresponding to them in the filtering problem with the use of the measurements $y(\tau)$ on the interval $\tau \in (-\infty, t)$ and in the smoothing problem with the use of the same measurements on the interval $\tau \in (-\infty, \infty)$. It is assumed that the matrices F, G, H are such that there exits a steady-state solution for these problems.

As is well known, the solution of the optimal filtering problem is determined by the Kalman filter [3]

$$\dot{\hat{x}}(t) = (F - PH^T R^{-1} H)\hat{x}(t)) + PH^T R^{-1} y(t). \quad (5)$$

Here P is the filtering error covariance matrix that characterizes the potential estimation accuracy in the real time mode. This matrix can be obtained from the solution of the following equation

$$PF^T + FP - PH^T R^{-1} HP + GG^T = 0. \quad (6)$$

As consideration is being given to the steady-state mode, the estimation algorithms of x and z can be described with the use of $n \times m$ and $r \times m$ transfer function matrices that can be written as

$$W_f^x(p) = (pE - F + PH^T R^{-1} H)^{-1} PH^T R^{-1}, \quad (7)$$

$$W_f^z(p) = D(pE - F + PH^T R^{-1} H)^{-1} PH^T R^{-1}. \quad (8)$$

Using the expression

$$-F + PH^T R^{-1} H = PF^T P^{-1} + GG^T P^{-1}$$

that results from (6), it is possible to present Eq. (7), (8) as

$$W_f^x(p) = P\left(pE + F^T + P^{-1} GG^T\right)^{-1} H^T R^{-1}, \quad (9)$$

$$W_f^z(p) = DP\left(pE + F^T + P^{-1} GG^T\right)^{-1} H^T R^{-1}. \quad (10)$$

The solution of the optimal smoothing problem can be represented as [3]

$$\dot{\hat{x}}_s(t) = (F + GG^T P^{-1})\hat{x}_s(t) - GG^T P^{-1} \hat{x}(t). \quad (11)$$

For the covariance matrix P^s, that characterizes the potential smoothing accuracy, i.e., estimation in the off-line mode, the following equation holds true

$$\left(F + GG^T (P)^{-1}\right)P^s + P^s\left(F + GG^T (P)^{-1}\right)^T = GG^T. \quad (12)$$

Writing the expression

$$\hat{x}_s(p) = -(pE - F - GG^T P^{-1})^{-1} GG^T P^{-1} \hat{x}(p)$$
$$= (-pE + F + GG^T P^{-1})^{-1} GG^T P^{-1} \hat{x}(p),$$

and taking into account (8), smoothing yields the following $n \times m$ and $r \times m$ matrices of transfer functions for the estimates x and z

$$W_s^x(p) = \left(-pE + F + GG^T P^{-1}\right)^{-1} \times$$
$$\times GG^T \left(pE + F^T + P^{-1} GG^T\right)^{-1} H^T R^{-1}, \quad (13)$$

$$W_s^z(p) = D\left(-pE + F + GG^T P^{-1}\right)^{-1} \times$$
$$\times GG^T \left(pE + F^T + P^{-1} GG^T\right)^{-1} H^T R^{-1}. \quad (14)$$

From (11) it follows that in the common case, in order to derive the optimal smoothing estimate it is necessary, first, to find the optimal estimate vector $\hat{x}(t)$, and then, using the vector $\hat{x}(t)$ as the n - dimensional measurement, process it with the use of the algorithm (11).

3 The efficiency of smoothing in determing gravity anomaly

In order to compare the efficiency of filtering and smoothing algorithms used in the processing of satellite measurements of altitude and vertical velocity in aerogravimetry, and also to study the efficiency of their integrated processing, it is necessary to specify some stochastic models for GA and errors of the satellite measurements. The previous investigations showed that the errors of the satellite measurements of altitude and vertical velocity can be described as the white noises with the intensities - R_{V_z} =(0.01m/s)^{2}s, and R_h=(0.005m)^{2}s, respectively [2], and the instrumental errors of the gravimeter used as the white noise with the intensity - (5mGal)^{2}s [8]. Gravity anomalies can be described by a process with the spectral density [5]

$$S_{\tilde{g}}(\omega) = 2\alpha^3 \sigma_{\tilde{g}}^2 \frac{5\omega^2 + \alpha^2}{(\omega^2 + \alpha^2)^3}, \quad (15)$$

where the $\sigma_{\tilde{g}}$ is the GA root mean square (RMS) value; α is the value reverse to the correlation interval. It is not difficult to show that this process can be described by using a third-order shaping filter with constant coefficients, with scalar forcing noise arriving at its input.

Under the assumptions made the problem reduces to estimation of the five-dimensional vector x with the components $x=(\delta h, \delta V_z, x_3, x_4, x_5)^T$. Here the first two equations have the form

$$\delta \dot{h} = \delta V_z, \quad \delta \dot{V}_z = \tilde{g} + q_g w_g, \qquad (16)$$

where w_g is the white noise that describes the gravimeter errors, while the other three equations correspond to the shaping filter for the equation $\tilde{g}$. The equations for the differential measurements of coordinates and velocity will have the form

$$y_i(t) = x_i(t) + v_i(t),\ i = \overline{1.2}. \qquad (17)$$

Any component of the vector x , can be estimated, but of the most interest is the anomaly value itself. Covariance matrices for filtering and smoothing problems were calculated for the models used. In calculations it was assumed that $\sigma_{\tilde{g}}$ =30mGal. In so doing, the values of the derivative for GA were assumed to be different - 3mGal/km, 5 and 10 mGal/km, with the corresponding correlation intervals $1/\alpha$ - 14 km., 8.5 and 4.3 km. The velocity was assumed to be equal to V =50m/s, which is typical of an aerogravimetric survey. Table 1 presents the calculation results as RMS errors in filtering and smoothing determined with the use of the diagonal elements of the covariance matrices.

Type of SNS measurements	Value of derivative, mGal/km		
	3	5	10
Velocity	7.7/2.2	11.2/3.5	18.8/6.7
Altitude	1.8/0.3	2.7/0.5	4.5/1.0
Velocity and altitude	1.8/0.3	2.7/0.5	4.5/1.0

T a b l e 1. RMS errors in GA filtering (numerator) and smoothing (denominator), mGal, for *V*=50m/s

The results presented in the Table allow the following conclusions:

- the accuracy of gravity estimation in the solution of the smoothing problem is 2–8 times higher than the accuracy achieved in the solution of the filtering problem;
- it is only with the use of altitude measurements for smoothing mode that the gravity estimation accuracy can be ensured at the state-of-the-art level of 1 mGal;
- integrated processing of coordinate and velocity measurements from the SNS does not sufficiently increase the accuracy of GA determination as compared to the case that only altitude is used.

It should also be noted that the calculation results were practically the same irrespective the white noise that describes instrumental errors of the gravimeter was present or not.

From the preceding it follows, that the investigation of filtering and smoothing may be further reduced to the solution of the problem for the case of scalar measurements of the altitude, neglecting the contribution from the instrumental errors of the gravimeter.

4 Simplification of the problem

It is clear that the potential accuracy derived, as well as the structure of the algorithms depend on the type of the models used for measurement errors and GA. The models describing measurement errors were derived from the analysis of the real data [2]. Model (15) is widely used in the problems that require stochastic description of GA [5]. At the same time it makes the algorithms more complicated in comparison with the case when a simpler model is used to describe GA. To find out if it is possible to use a simpler description for GA in the problem under consideration, it is reasonable to use the approach proposed in [7] for constructing algorithms that provide accuracy close potential. According to this approach the transfer function of the optimal algorithm is mainly determined by the crosspoint of spectral densities of the signal estimated and measurement errors and the inclination of these densities. The analysis of equation (15) shows that if $\omega >> \alpha$, the following approximation holds good for it

$$S_{\tilde{g}}(\omega) \approx q^2 / \omega^4. \qquad (18)$$

The latter corresponds to the description of GA in the form of the second integral of the white noise with the intensity similar to that of the forcing noise in the model (15), i.e., $q^2 = 10\alpha^3\sigma^2_{\tilde{g}}$.

It may be shown that for the assumption made (*V*=50m/s, R_{V_z} =(0.01м/s)^{2}s, R_h=(0.005m)^{2}s) the density (15) in the vicinity of its crosspoint with $\omega^2 R_{V_z}$, $\omega^4 R_h$ is approximated rather accurately by Eq. (15) (see Fig. 1).

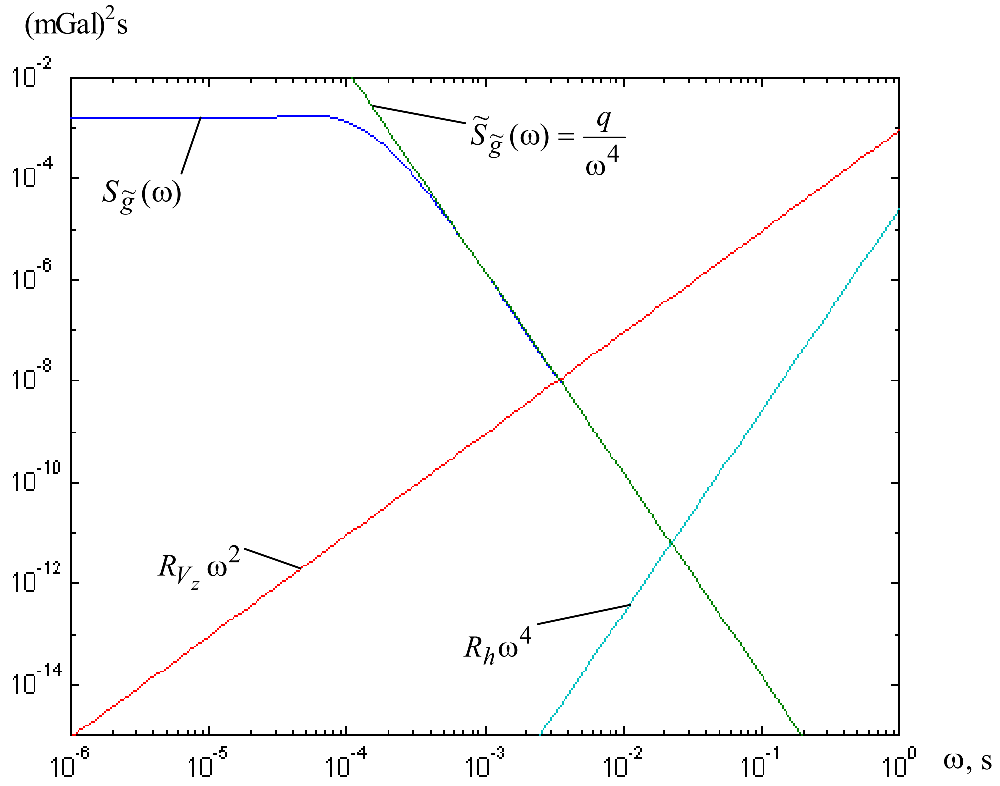

Fig 1. The spectral density of the signal and measurement errors in the problem of gravity anomaly estimation.

Taking into consideration the preceding, model (15) can be replaced for a simpler one in the form of (18). Thus the matrices F, G in (15) can be determined as

$$F=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}, G=\begin{bmatrix}0\\0\\0\\q\end{bmatrix}, \qquad (19)$$

and matrix $H=\begin{bmatrix}1&0&0&0\end{bmatrix}$ in (16), i.e., measurements are

$$y(t)=x_1(t)+v\ \ (t)\,. \qquad (20)$$

The investigation of efficiency of filtering and smoothing algorithms corresponding to (19), (20) proved to be highly efficient. It turned out, in particular, that their accuracy is close to potential in the conditions when the real model for GA is supposed to be more complicated and similar to model (15).

Using simplified models makes it possible to derive in explicit form analytic expressions for the covariance matrices P and P^S. That makes the analysis of the GA estimation potential accuracy for the steady-state mode of filtering and smoothing easier [2]. Besides the explicit expression can be obtained for transfer functions corresponding to these simplified models. Further let us see to what the filtering and smoothing procedures in these conditions are reduced and what their specific features are.

5 Analysis of the peculiarities of optimal filtering and smoothing algorithms in determining gravity anomalies

It can be shown that for model (19), (20) the transfer functions (9), (13) that define filtering and smoothing algorithms for steady-state mode have the following form

$$W_f^{x}(p)=W_B(p)\begin{bmatrix}\alpha\left(\frac{p}{\rho}\right)^3+\frac{\alpha^2}{2}\left(\frac{p}{\rho}\right)^2+\alpha\frac{p}{\rho}+1\\ p\left(\frac{\alpha^2}{2}\left(\frac{p}{\rho}\right)^2+\alpha\frac{p}{\rho}+1\right)\\ p^2\left(\alpha\frac{p}{\rho}+1\right)\\ p^3\end{bmatrix},$$

$$W_s^{x}(p)=W_B(p)\,W_B(-p)\begin{bmatrix}1\\p\\p^2\\p^3\end{bmatrix}, \qquad (21)$$

where

$$W_B(p) = \frac{\rho^4}{p^4 + \alpha p^3 \rho + \frac{\alpha^2}{2} p^2 \rho^2 + \alpha p \rho^3 + \rho^4} \quad (22)$$

4th-order Butterworth filter; and $\rho = \left(q^2/r^2\right)^{1/8}$, $\alpha = \sqrt{2(2+\sqrt{2})}$, $\beta = \sqrt{2}+1$, r^2 - intensity of the measurement noise.

Note the following peculiarity of the derived transfer functions. From (21) it follows that the realization of the optimal smoothing algorithm needs no filtering estimates for the whole state vector. As it can be concluded from (21), to realize the optimal smoothing algorithm it is sufficient to process the measurements (20) in direct time using the transfer function $W_B(p)$, and then to process the derived result in reverse time using the transfer functions, $j = 0,1,3$, depending on what state vector component is being estimated.

Let us discuss the cause for this peculiarity. For the class of the problems when H and D are rows, and G - the vector, expression (14) can be modified as

$$\begin{aligned} W_s^z(p) &= \left(\overline{W}_f^x(-p)\right)^T rP^{-1}GG^T P^{-1} r W_f^x(p) = \\ &= \overline{W}_*(-p) W_*(p), \end{aligned} \quad (23)$$

where

$$W_*(p) = G^T P^{-1} r W_f^x(p), \quad (24)$$

$$\overline{W}_*(-p) = G^T P^{-1} r \overline{W}_f^x(-p), \quad (25)$$

$$\overline{W}_f^x(p) = P\left(pE + F^T + P^{-1}Q\right)^{-1} D^T R^{-1}.$$

Here $W_*(p)$, $\overline{W}_*(p)$ represent the transfer function for single input single output systems, and $R = r^2$.

The Eq. (23) establishes the relation between the transfer functions corresponding to optimal smoothing and the transfer functions for optimal filtering for problem under consideration.

For the particular case that $D = H$,

$$W_s^z(p) = W_*(-p) W_*(p), \quad (26)$$

where

$$\overline{W}_*(p) = W_*(p) = G^T P^{-1} r W_f^x(p). \quad (27)$$

From (23) it follows that in order to derive an estimate for any component of the state vector, it is sufficient to process the initial measurements (20) in the filter with the transfer function $W_*(p)$, to store only one realization of the derived estimates regardless of the state vector x dimension, and then process this realization in reverse time in the filter with transfer function $\overline{W}_*(p)$. If in this case the state vector components that are being estimated and measured are the same ($D = H$), the transfer functions for direct and reverse time also identical. It is interesting to note that the transfer function (24) corresponds to the solution of the optimal filtering problem for the process $z(t) = rG^T P^{-1} x(t)$ using measurements $y(t) = Hx(t) + v(t)$. It is not difficult to illustrate this using (21) for problem (19), (20).

Also note that in the problem considered the state vector components are related to each other by a simple relation of the form $x_{j+1}(p) = p^j x_1(p)$, $j = 0,1,2$. As in smoothing the procedures of estimate calculation and such transformation of the initial process are commutative [6], then in the case under consideration it may be assumed that $D = H$ and, consequently, Eq. (26) holds true. In other words the problem can be reduced to derivation of a smoothed estimate of the first component of the state vector estimated and subsequent application of the operator p or p^2. It is possible to show that the equation obtained can be generalized for the cases that the models for GA are the third, fourth and so on integrals of the white noise.

An at last, note that the results derived give the possibility to establish the relations between the optimal smoothing algorithms for estimation GA and suboptimal algorithms using the Butterworth filters.

5 Conclusion

The problem of integrated processing of data from a satellite navigation system and gravimeter readings aimed at determining gravity anomalies has been formulated in the scope of the theory of optimal filtering and smoothing.

The efficiency of using optimal smoothing algorithms in comparison with the use of filtering algorithms has been analyzed for the models considered. It has been shown, in particular, that acceptable accuracies in determining anomalies can only be obtained with the use of satellite coordinate measurements for smoothing mode, while integration of measurements of coordinates and velocity proved to be ineffective.

The expression establishing the relation between the transfer functions corresponding to optimal smoothing and the transfer functions for optimal filtering have been derived for the class problems under consideration. It has been shown that, in order to derive an optimal estimate, it is sufficient to process initial scalar measurements in the filter with a specified transfer function, to store only one realization of the derived estimates regardless of the dimension of the state vector and then process this realization in reverse time with the similar transfer function. It has been shown that the first transfer function corresponds to the optimal filter that estimates a specified linear combination of state vector components that describes the estimated process.

The results derived give the possibility to establish the relations between the optimal smoothing algorithms for estimation GA and suboptimal algorithms using the Butterworth filters.

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