THE APPLICATIONS AND A GENERAL SOLUTION OF A FUNDAMENTAL MATRIX EQUATION PAIR

Chia-Chi Tsui

743 Clove Road, NY 10310, USA,ctsui@ny.devry.edu

Keywords: output feedback, robust, fault isolation, eigenstructure assignment.

Abstract:
Equation TA - FT = LC (F is stable) is necessary and sufficient for the output of a feedback compensator (F,L,K_Z,K_y) to converge to a state feedback (SF) signal K**x**(t) for a constant K, where (A,B,C,0) is the open loop system and TB is the compensator gain to the open loop system input. Thus equation TB = 0 is the defining condition for this feedback compensator to be an output feedback compensator. Equation TB = 0 is also the necessary and sufficient condition to fully realize the critical loop transfer function and robust properties of SF control if K is systematically designed. Furthermore, because B is compatible to the open loop system gain to its unknown inputs and its input failure signals, TB = 0 is also necessary for unknown input observers and failure detection and isolation systems. Finally, this equation pair is the key condition of a really systematic and explicit design algorithm for eigestructure assignment by static output feedback control. This paper presents a general and exact solution which is uniquely direct, simple, and decoupled, to this matrix equation pair. An approximate solution which is general and simple, and which can be simply added to the exact solution to increase the row dimension of this solution, is also presented.

1 The Matrix Equation Pair and Its Applications

Consider the linear time-invariant irreducible system

$$\frac{d}{dt}\mathbf{x}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \quad (1.a)$$
$$\mathbf{y}(t) = C\mathbf{x}(t) \quad (1.b)$$

and its general linear feedback controller

$$\frac{d}{dt}\mathbf{z}(t) = F\mathbf{z}(t) + L\mathbf{y}(t) + TB\mathbf{u}(t) \quad (2.a)$$
$$\mathbf{w}(t) = -K_Z\mathbf{z}(t) + -K_y\mathbf{y}(t) \quad (2.b)$$

where **x**, **u**, **y**, and **z** are n, p, m, and r dimensional time signals, respectively.

It is well known that for $\mathbf{z}(t) \Rightarrow T\mathbf{x}(t)$ for a constant T,

$$TA - FT = LC \text{ (F is stable)} \quad (3)$$

is the necessary and sufficient condition [1]. From (1.b) and (2.b) it is obvious that (3) is also the necessary and sufficient condition for the controller output

$$\mathbf{w}(t) \Rightarrow -[K_Z:K_y][T':C']'\mathbf{x}(t) \equiv -\underline{K}\ \underline{C}\ \mathbf{x}(t) \equiv -K\mathbf{x}(t) \quad (4)$$

It is also proved in [2] that if (3) holds, then the poles of the feedback system of (1-2) are formed by the eigenvalues of F and A-BK.

The main problem which has limited the practical application of state space control theory is that the loop transfer function $L_{Kx}(s)$ $(=-K(sI-A)^{-1}B)$ and robustness properties of the state feedback (SF) control of (4) cannot be generally realized by the controller (2) [3, 4]. The necessary and sufficient condition to realize $L_{Kx}(s)$ by (2) is $K_Z(sI-F)^{-1}TB = 0\ \forall s$ [5]. Because K_Z and K_y of (4) must be free for any systematic design of K, and because $(sI-F)^{-1}$ is nonsingular, the necessary and sufficient condition to realize $L_{Kx}(s)$ (called "loop transfer recovery, LTR") is [6-8]

$$TB = 0 \quad (5.a)$$

It is obvious that (5.a) is the defining condition for the feedback controller (2) to be an output feedback compensator (OFC). Thus only OFC can realize fully the robustness property of SF control (if that OFC can satisfy (3) [15]).

In practice the plant system (1.a) is usually modeled with additional but undesirable input term (or terms) **d**(t), where **d**(t) is an unknown time function.

The controller (2) which estimates $\mathbf{x}(t)$ when $\mathbf{d}(t) \neq 0$ is called an "unknown input observer" (UIO) [9]. Failure detection and isolation (FDI) systems need to detect and isolate the non-zero occurrence of the failure signal $\mathbf{d}(t)$ among a number of different terms of $\mathbf{d}(t)$, and usually need a band of detectors of structure (2). It is required that each detector has its state $\mathbf{z}(t) \Rightarrow T\mathbf{x}(t)$ even though its designated term of $\mathbf{d}(t) \neq 0$ [10-11, 13]. Let us assume without lose of generality that B is also the gain of system (1.a) to $\mathbf{d}(t)$. Then it is obviously necessary in both UIO's and FDI systems that in addition to (3)

$$TB = 0. \qquad (5.b)$$

Finally, in the design of static output feedback control (SOF, $\mathbf{u}(t) = -K_y\mathbf{y}(t) = -K_yC\mathbf{x}(t)$) for eigenvalue/vector assignment, a really systematic design algorithm assigns n-m eigenvalues Λ_{n-m} and their left eigenvectors T_{n-m} first and then the remaining m eigenvalues Λ_m and their right eigenvectors V_m. More explicitly, equation pair

$$T_{n-m}A - \Lambda_{n-m}T_{n-m} = LC \text{ and } |[T_{n-m}':C']| \neq 0 \qquad (6.a)$$

is satisfied at the first step and then [8, 18]

$$AV_m - V_m\Lambda_m = BK \text{ and } T_{n-m}V_m = 0 \qquad (6.b)$$

is satisfied at the second step. The final answer is $K_y = K(CV_m)^{-1}$. The second equations of (6.a) and (6.b) together guarantee the existence of $(CV_m)^{-1}$ and that all n eigenvectors are linearly independent. While Equation (6.a) is trivial because it is the same requirement of existing state observers,(6.b) is the exact dual of equation pair (3) and (5) (if T_{n-m} of (6.b) is replaced by C).

To summarize, matrix equation pair (3) and (5) is the necessary and sufficient condition for an OFC to generate an SF control signal (4), the necessary and sufficient condition to realize the critical loop transfer function and robustness properties of SF control (4) of all systematic design (LTR), the necessary condition for an UIO and an FDI system, and the only non-trivial condition of the above eigenstructure assignment design. Therefore this equation pair is by far the most important and most fundamental equation in state space control design.

The purpose of this paper is to list together for the first time the above wide range of basic and important design applications under a single and simple mathematical requirement -- the matrix equation pair (3) and (5). Although a similar effort was made befor on equation (3) alone [20], the significance, necessity, and difficulty of adding (5) to (3) are obvious.

2 The Existing Solution

Almost all existing solutions of (3) and (5) also require $|\underline{C}|$ of (4) $\neq 0$. However as shown in the entire Section 1, $|\underline{C}| \neq 0$ is entirely unnecessary to all listed applications. Because in non-trivial cases p is much less n, a desirable K can usually be satisfied by $K = \underline{KC}$ without $|\underline{C}| \neq 0$. Requiring $|\underline{C}| \neq 0$ actually implies that the SF control (K) is designed regardless of $\underline{C}$ ($\equiv [T':C']'$), regardless of the information about the implementing controller (with key parameter T) and about the system output (with key parameter C), regardless of the information which is essential to the realization of the SF control when $\mathbf{x}(t)$ is not directly measurable, and regardless of the difference that $\mathbf{x}(t)$ is directly measurable or not. Hence although this requirement is prevalent in the past four decades and is essential to "separation principle" [4], it is not really rational and it causes a critical disadvantage that the corresponding solution of (3) and (5) cannot be valid for most open loop system conditions as shown below.

Because $|\underline{C}| \neq 0$ is also required, the existing solution of (3) and (5) requires the system (1) either has n-m stable transmission zeros or satisfies 1). minimum-phase (all transmission zeros are stable); 2). rank(CB) = p; and 3). m $\geq$ p [16, 14]. Because systems with m $\neq$ p do not have transmission zeros generically and systems with m = p and rank(CB) = p always have n-m transmission zeros [17], the second set of three conditions is more general than the first condition.

However, among this second set of three conditions, both minimum-phase and rank(CB) = p conditions are very restrictive. Because system (1) and its transmission zeros are supposed to be generally and randomly given, and because the stable and unstable regions are almost equally sized, the chance that not even a single transmission zero among the n - m transmission zeros (systems with m = p generically have this many transmission zeros [17]) is unstable (or minimum-phase) is very small for non-trivial systems where n >> m. Condition rank(CB) = p is not satisfied by many practical systems such as airborne systems either. To summarize, the existing solution of (3) and (5) (and $|\underline{C}| \neq 0$) does not exist for most systems.

Among the applications listed in Section 1, the LTR problem also has an asymptotically approximate solution [5]. This solution still is valid for minimum-phase systems (1) only, and requires asymptotic large gain L which is neither analytical nor acceptable in a robust control system [19].

For this reason, even though the SF control (4) which is designed on the condition $|\underline{C}| \neq 0$ can itself be optimal and ideal, its critical robustness properties cannot be sufficiently realized in most of the actual feedback systems.

The reason that condition $|\underline{C}| \neq 0$ has been added to (3) and (5) is that people did not study this equation pair directly -- they simply borrowed the existing solution of state observers (or (6.a), whose second part is $|\underline{C}| \neq 0$) for the solution of this equation pair. Another reason is that the decoupled solution T of (3) is not used to find the solution of (5). When T is not decoupled, its number of rows must be fixed (= n if $K_y=0$ or n-m if $K_y \neq 0$) and thus condition $|\underline{C}| \neq 0$ cannot be eliminated.

The only existing solution to this equation pair without condition $|\underline{C}| \neq 0$ seems to be in a very minor part (Section 4) of [14], and in [12] and [22], while none of these papers offered any approximate solution. The solution of [14] is valid for m > p only which is less general than our solution (see Theorem 1 and its proof). The solution of [12] and [22] is the state observer of some subsystems of a decoupled similarity transformation of (1) called "special coordinated basis" (s.c.b). Thus this solution is indirect and coupled. More critically, it is obvious and is accepted that the computation of s.c.b is very complicated and ill-conditioned [22].

3 A Direct, Simple, Exact, and Decoupled Solution

Before presenting a direct, simple, and exact solution of (3) and (5). Two important additional requirements on (3) and (5) should be mentioned.

The first is to maximize the row rank of matrix $\underline{C}$. Equation (4) shows that the higher this row rank, the less constrained the corresponding control of (4), the more the information T**x**(t) that is generated by controller (2), and the better the achievement of any of the applications of Section 1.

The second is to have a decoupled solution. Only for a decoupled solution can the number of rows of this solution be freely adjustable, and only then can an approximate solution be added to it and to incase rank($\underline{C}$) if the rank($\underline{C}$) of the exact solution is too low.

Therefore, we will set the initial number (r) of rows of T to its maximal possible value n - m. In case such a corresponding high row rank of $\underline{C}$ (= n) is not attainable (see section 2), the value r will be reduced because our solution is uniquely decoupled.

Our solution does not impose restrictions on the eigenvalues of matrix F. This feature enables the specified dynamic performance of the compensator (2). However, each stable transmission zero of system (1) should be matched by one of the eigenvalues of F. As will be proved in the next section, this requirement is necessary for the existence of the solution of (3) and (5) if $m \leq p$, and is also necessary to achieve the maximal possible row rank of $\underline{C}$.

Once the eigenvalues of F are selected, F will be set in Jordan form with real 2x2 Jordan block for complex conjugate eigenvalue pair and kxk Jordan block for eigenvalues of multiplicity k. Other forms of F of the same eigenvalues are just a similarity transformation away from this F. However, only the Jordan form enables the complete decoupling of the solution of (3) and (5), corresponding to each of the Jordan blocks [20].

For simplicity of presentation, we will also assume without loss of generality that in (1), $C = [C_1 : 0]$ ($|C_1| \neq 0$).

Let
$$T_i = \begin{bmatrix} \mathbf{t}_{i1} \\ \vdots \\ \mathbf{t}_{ik} \end{bmatrix} \qquad (7)$$
be the i-th block of rows of matrix T and corresponding to the i-th Jordan block F_i with dimension k, then the right n-m columns of (3) can be expressed as [20]

$$[\, \mathbf{t}_{i1} : \ldots : \mathbf{t}_{ik} \,] \left[I_k \otimes \left(A \begin{bmatrix} 0 \\ I_{n-m} \end{bmatrix}\right) - F_i' \otimes \begin{bmatrix} 0 \\ I_{n-m} \end{bmatrix} \right] = \mathbf{0} \qquad (8.a)$$

where $\otimes$ stands for the Kronecker product and I_k stands for a k-dimensional identity matrix. For example if k=1 and $F_i = \lambda_i$, then (8.a) becomes

$$\mathbf{t}_i \,(A - \lambda_i I) \begin{bmatrix} 0 \\ I_{n-m} \end{bmatrix} = \mathbf{0} \qquad (8.b)$$

Because the matrix of (8.a) has dimension kn x k(n-m) and because the form of C and the observability of (1) guarantee that this matrix is full column rank, there are km linear independent rows of $[\, \mathbf{t}_{i1} : \ldots : \mathbf{t}_{ik} \,]$ that can satisfy (8.a). These rows will form a kmxkn dimensional matrix D_i and

$$[\, \mathbf{t}_{i1} : \ldots : \mathbf{t}_{ik} \,] = \mathbf{c}_i D_i \equiv \mathbf{c}_i [D_{i1} : \ldots : D_{ik}] \qquad (9)$$

where $D_{ij} \in R^{kmxn}$ (j=1,...,k) and is the solution of the set of linear equations (8.a), and parameter $\mathbf{c}_i \in R^{1xkm}$ is completely free as long as (3) is concerned. Because (8.a) is completely decoupled $\forall i$, matrices D_i of (9) can be computed in complete parallel $\forall i$.

Now substitute (9) into (5) ($T_i B = 0$ $\forall i$), we have

$$[\mathbf{t}_{i1} B : \ldots : \mathbf{t}_{ik} B] = \mathbf{c}_i [D_{i1} B : \ldots : D_{ik} B] = \mathbf{0} \qquad (10)$$

which will be satisfied by the free parameter $\mathbf{c}_i$.

If the solution $\mathbf{c}_i$ of (10) is not unique, which is true when m > p+1, then the remaining freedom of $\mathbf{c}_i$ will be used to maximize the row rank of resulting matrix $\underline{C}$. This can be achieved using the existing general and systematic algorithms of [21] because (9) shares the same formulation of [21]

Once $\mathbf{c}_i$ and matrix T of (7) and (9) are determined, L equals

$$L = (TA - FT) \begin{bmatrix} I_m \\ 0 \end{bmatrix} C_1^{-1} \qquad (11)$$

4 Analysis of This Exact Solution and An Approximate Solution

<u>Theorem 1</u>: The non-zero solution of Section 3 satisfies (3) and (5), and is valid for any observable open loop system (1) if and only if it either has more outputs than inputs (m > p) or has at least one stable transmission zero.

<u>Proof</u>: It is obvious that based on the form of C, satisfying (7) to (9) implies that the right n-m columns of (3) are satisfied, and satisfying (11) implies that the left m columns of (3) are satisfied. This solution does not require any other conditions on (1) except its observability, and has its freedom explicitly expressed as parameter $\mathbf{c}_i$.

It is obvious that if m > p, then the dimension (km x kp) of matrix $[D_{i1}B : \ldots : D_{ik}B]$ of (10) implies that (10) and therefore (5) can always be satisfied by a nonzero $\mathbf{c}_i$.

If $m \leq p$ and if (and only if) system (1) has a stable transmission zero z_i, then there always exists a nonzero row vector say $[\mathbf{t}_i : \mathbf{l}_i]$ such that

$$[\mathbf{t}_i : -\mathbf{l}_i] \begin{bmatrix} A - z_i I & B \\ C & 0 \end{bmatrix} = \mathbf{0} \qquad (12)$$

Because z_i is matched by an eigenvalue λ_i of matrix F, the left n columns of (12) imply that $[\mathbf{t}_i : \mathbf{l}_i]$ is the respective i-th row of solution matrix T and L of (3) corresponding to λ_i. The right p columns of (12) imply that $\mathbf{t}_i$ satisfies the i-th row of (5). This argument also demonstrates that matching eigenvalues of F with the stable transmission zeros of (1) is necessary for the existence of the solution to (3) and (5) if $m \leq p$.

Because our solution to (3) and (5) is completely decoupled $\forall i$ (corresponding to each Jordan block F_i of F), the above existence of the i-th row of solution sufficiently implies the existence of the whole solution.

The condition of at least one stable transmission zero when $m \leq p$ is not mentioned in [14]. The necessity of this condition (at $m \leq p$) is incorrectly questioned by [12], and is not invalidated by the extremely rare situation of [12] (Example 1, where one row of the 2x2 transfer function matrix of (1) is zero -- so that every value is a transmission zero of (1)).

Theorem 1 implies that our solution of Section 3 does not have the strict restrictions of minimum-phase and rank(CB)=p at all (see Section 2). Because systems (1) with m = p generically have n - m transmission zeros [17], the chance that such systems have at least one stable transmission zero is very high (see Section 2). Hence the exact solution of this paper is general for all systems with m > p and almost all systems with m = p, and hence is general <u>for almost all systems.</u>

<u>Claim 1</u>: It is obvious that besides the eigenvalue selection freedom of F, the entire remaining freedom after (3) is represented by parameter $\mathbf{c}_i$ [20, 21]. It is also obvious that $\mathbf{c}_i$ is fully used to satisfy (5) in the form of (10), and is fully used to maximize the row rank of resulting matrix $\underline{C}$ as described by the paragraph just preceding equation (11).

<u>Theorem 2</u>: If the system (1) satisfies 1). minimum-phase, 2). rank(CB) = p, and 3). $m \geq p$, then the solution of Section 3 also satisfies $|\underline{C}| \neq 0$ (or rank($\underline{C}$) = maximal n).

<u>Proof</u>: The above three conditions imply the existence of unknown input observers which are identified by conditions (3), (5) and rank($\underline{C}$) = maximum n (see [16]). Because the eigenvalues of F of our solution are similarly selected as in the unknown input observer design [16], the proof follows directly from Claim 1. □

This theorem shows that the existing result of UIO and exact LTR state observers is only a special case of our solution of Section 3 (when rank($\underline{C}$) can reach its maximum value n).

<u>Claim 2</u>: Even if the system (1) does not satisfy either of the two conditions of Theorem 1, our solution still satisfies (3) (see the first part of the proof of Theorem 1). Our solution can also satisfy (5) approximately in least square sense because both (5) (and (10)) are in the form of a set of linear equations.

This claim implies that the approximate solution of (3) and (5) of this paper can be extended to all systems (unlike the existing asymptotical LTR result which is still limited to minimum-phase systems). In addition, the nature of the least square problem of a set of linear equations guarantees that this approximate solution is quite analytical and has finite gain only (unlike the existing asymptotic LTR result described in Section 2).

This approximate solution is very useful in case rank($\underline{C}$) of the exact solution is not high enough. As mentioned at the beginning of Section 3, a low rank($\underline{C}$) of (4) implies a weak (more constrained) SF control and less information T$\mathbf{x}$(t) which is generated by the observer or failure detector [13]. For example arbitrary pole assignment by this SF control is not possible if p x rank($\underline{C}$) $\leq$ n [23]. Because our approximate solution T satisfies (3) exactly and is decoupled, some of its rows (of T) can be simply added to the exact solution and to increase rank($\underline{C}$). This useful approximate solution is not offered at all by [12], [14], and [22] because of the indirect and coupled nature of their solution.

5 Conclusion

Equation pair (3) and (5) is necessary and sufficient for an output feedback compensator to generate a state feedback control signal K$\mathbf{x}$(t) of (4), and necessary and sufficient to realize (when $\mathbf{x}$(t) is not directly measurable) the critical loop transfer function and robustness properties of systematically designed state feedback control. This equation pair is also necessary for the basic result of unknown input observer, for a failure detection and isolation system, and for a really systematic eigenvalue/vector assignment design algorithm for static output feedback. Hence this matrix equation pair is fundamentally important in state space control systems design theory.

Almost all existing solutions of this equation pair attach a difficult and unnecessary additional condition $|\underline{C}| \neq 0$, and are invalid for most systems. Other existing solutions are indirect, coupled, and very unreliable in computation. However, Section 3 presented a simple, exact, direct, and decoupled solution to this equation pair which is general for most systems (Theorem 1). It is proved in Theorem 2 that our solution also satisfies $|\underline{C}| \neq 0$ whenever the solution exists. In addition, Claim 2 shows that the approximate version of our solution is much more general, analytical, and practical than the existing ones. This approximate solution also is, uniquely, decoupled and satisfying (3) exactly. These properties make our approximate solution uniquely useful because this solution can be simply added to the exact solution. The result of this paper is based uniquely on a decoupled solution of (3) [20].

References

[1] Luenberger, D.G., "An introduction to observers," IEEE Trans. **AC-16**, pp. 596-603, 1971.
[2] Tsui, C.C., "Unifying state feedback/LTR observer and constant output feedback design by dynamic output feedback," Proc. 1993 IFAC World congress, vol. 2, pp. 231-238.
[3] Doyle, J., "Guaranteed margins for LQG regulators," IEEE Trans. **AC-23**, pp. 756-757, 1978.
[4] Willems, J.C., "Book review of 'Mathematical System Theory: The Influence of R.E. Kalman,'," IEEE Trans. **AC-40**, 978-979, 1995.
[5] Doyle, J. and G. Stein, "Robustness with observers," IEEE Trans. **AC-24**, pp. 607-611, 1979.
[6] Tsui, C.C., "Unified output feedback design and loop transfer recovery," Proc. 1992 American Control Conf., pp. 3113-3118.
[7] Tsui, C.C., "A new design approach to unknown input observers,"IEEE Trans. **AC-41**, pp. 465-468, 1996.
[8] Tsui, C.C., Robust Control System Design -- Advanced State Space Techniques. marcel dekker,NY, 1996.
[9] Wang, S.H. E.J. Davison and P. Dorato, "Observing the states of systems with unmeasurable disturbances," IEEE Trans. **AC-20**, pp. 716-717.
[10] Ge, W. and C.F. Fang, "Detection of faulty components via robust observation," Int. J. Control, vol. 47, pp. 581-600, 1988.
[11] Tsui, C.C., "On the solution to the state failure detection problem," IEEE Trans. **AC-34**, pp. 691-697,1989
[12] Hou, M., A.C. Pugh and P.C. Muller, "Disturbance decoupled function observers," IEEE Trans. **AC-44**, 2, pp. 382-386, 1999.
[13] Tsui, C.C., "The design generalization and adjustment of a failure isolation and accommodation system," Int. J. Systems Science, vol. 28, no. 1, pp. 91-97, 1997.
[14] Barlow, J.B., M.M. Monahem and D.P. O'Leary, "Constrained matrix Sylvester equations," SIAM J. Matrix Anal. Appli., **13,**1,1-9,1992.
[15] Tsui, C.C., "The first general output feedback compensator that can implement state feedback control,"Int. J. Systems Science, **29**, 1, pp. 49-55, 1998.
[16] Kudva, P., N. Viswanadham and A. Ramakrishna, "Observers for linear systems with unknown inputs," IEEE Trans. **AC-25**, pp. 113-115, 1980.
[17] Davison, E.J. and S.H. Wang, "Properties and calculation of transmission zeros of linear multivariable systems," Automatica, vol. 10, pp. 643-658, 1974.
[18] Tsui, C.C., "A design algorithm of static output feedback control for eigenstructrue assignment," Proc. 1999 IFAC Congress, 2a-10-2.
[19] Shaked,U & E. Soroka, "On the stability robustness of the continuous-time LQG optimal control," IEEE Trans. **AC-30**, pp. 1039-1043, 1985.
[20] Tsui, C.C., "On the solution to the equation TA-FT=LC and its applications," SIAM J. Matrix Anal. Appl., **14**, 1, 33-44, 1993.
[21] Kautsky, J., N.K. Nichols and P. Van Dooren, "Robust pole assignment in linear state feedback," Int. J. Control, no. 5, pp. 1129-1155, 1985.
[22] Chu, D., "Disturbance decoupled observer design for linear time-invariant systems: A matrix pencil approach," IEEE Trans. **AC-45**, pp. 1569-1575, 2000.
[23] Wang, X.A., "Grassmannian, central projection, and output feedback pole assignment of linear systems," IEEE Trans. **AC-41**, pp. 786-794, 1996.