FEEDBACK STABILIZERS FOR A CLASS OF IMPERFECTLY KNOWN HEREDITARY DESCRIPTOR SYSTEMS

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Abstract

A class of memoryless feedback controls is designed to asymptotically stabilize a class of imperfectly known, nonlinear, descriptor systems with time-delays. Each descriptor system, consisting of dynamic and static subsystems, contains discrete and distributed delays, and each dynamic subsystem is a time-delay system of the retarded type. A deterministic methodology based on Lyapunov theory and Lyapunov-Krasovskiĭ functionals is utilized and feedback controllers are synthesized that will ensure, under appropriate hypotheses and satisfaction of appropriate stability criteria, a uniform asymptotic stability property for the prescribed class of hereditary descriptor systems.

1 Introduction

There have been a few recent research investigations on descriptor systems using a deterministic approach, see, for example [3]-[7]. The work discussed in [4]-[5] and [7]-[8] only considered linear descriptor systems, and only [5] and [7]-[8] studied systems with uncertainty. The paper published in [3], investigated a nonlinear descriptor control system, but no uncertainty was considered and the systems were delay-free. Also, the class of descriptor systems investigated in [4] and [8] consisted of dynamic subsystems modelled as linear time-delay systems of the retarded type. To the authors' knowledge, there appears to be no studies on uncertain *nonlinear* descriptor systems with discrete and distributed time-delays.

The main objective of this paper is to design, using a deterministic approach, a class of robust memoryless feedback controls for uncertain nonlinear descriptor systems, subject to time-delays (discrete and distributed), in order to achieve a uniform asymptotic stability property. There are some advantages of using memoryless controls, namely past history of the states do not need to be stored. Therefore, for certain applications, memoryless controls may be more suitable. With respect to the imperfectly known systems, only uncertainty in the dynamic subsystem is considered in this investigation. Also, parametric uncertainty is not considered in this paper; instead, *a priori* bounding knowledge of the system uncertainty, in terms of growth conditions with respect to its arguments, is assumed. One feature of the controllers, employed to stabilize the class of uncertain systems, is that the gains depend explicitly on upper bounds of the uncertainty and, thus, robustness of the feedback controls is a consequence. Utilizing feedback controllers with memory, together with a deterministic methodology based on Lyapunov theory and Lyapunov-Krasovskiĭ functionals, a stability criterion is proposed that will ensure the desired stability property for the prescribed class of descriptor systems.

2 Class of descriptor systems

Consider a class of imperfectly known, hereditary, descriptor systems modelled by

$$\dot{x}(t) = a(t, x_t, y_t, u(t)) \tag{2.1}$$
$$0 = b(x_t, y_t, u(t)), \tag{2.2}$$

where

$$\begin{aligned} a(t, x_t, y_t, u, \mu) &= a_1(x_1, x_2, y_1, y_2) + I(t, \tau_1, a_2(x_1)) \\ &\quad + I(t, \tau_2, a_3(y_1)) + G_1(x_1)u \\ &\quad + h_1(t, x_1, x_2, x_3, y_1, y_2, y_3, u), \\ b(x_t, y_t, u, \mu) &= b_1(x_1, x_2, y_1, y_2) + I(t, \tau_1, b_2(x_1)) \\ &\quad + I(t, \tau_2, b_3(y_1)) + G_2(x_1)u, \end{aligned}$$

$$\begin{aligned} & x_1(t) = x(t), \quad x_2(t) = x(t-\rho_1), \\ & x_3(t) = I(t, \tau_1, x_1) := \int_{t-\tau_1}^{t} x_1(r)\,\mathrm{d}r, \\ & y_1(t) = y(t), \quad y_2(t) = y(t-\rho_2), \\ & y_3(t) = I(t, \tau_2, y_1) := \int_{t-\tau_2}^{t} y_1(r)\,\mathrm{d}r, \end{aligned}$$

$x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^l$, $u(t) \in \mathbb{R}^m$ is the control input, and $1 \le m \le \ell \le n$. The notation $x_t = x_t(r) := x(t+r)$ ($r \in [-\tau, 0]$, $\tau > 0$) is introduced to denote the restriction of $x(\cdot)$ to the interval $[t-\tau, t]$. let $\mathcal{Q}^p_{A,\tau} := \Big\{\psi \in W([-\tau, 0]; \mathbb{R}^p) : \|\psi\|_W < A, \text{ with } 0 < A < \infty\Big\}$, where $W([-\tau, 0]; \mathbb{R}^p)$ denotes the Banach Space of absolutely continuous functions with Square-integrate derivation with norm $\|\psi\|_W = \left\{ \|\psi(0)\|^2 + \int_{-\tau}^{0} \|\psi(\theta)\|^2 \,\mathrm{d}\theta \right\}^{\frac{1}{2}}$. It is assumed that the system (2.1-2.2) is subject to initial condition

$$x_{t_0}(\theta) = \psi_x(\theta), y_{t_0}(\theta) = \psi_y(\theta), \theta \in [-T, 0], \tag{2.3}$$

with $T = \max[\rho_1, \rho_2, \tau_1, \tau_2]$, and $\psi_x \in \mathcal{Q}^n_{A,T}$; $\psi_y \in \mathcal{Q}^l_{A,T}$. The discrete and distributed delays, represented by ρ_1, ρ_2 and τ_1, τ_2, respectively, are assumed to be bounded. It is assumed that the vector fields $a_1 \in C^1(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}^\ell; \mathbb{R}^n)$, $b_1 \in C^1(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}^\ell; \mathbb{R}^\ell)$, $a_2 \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, $b_2 \in C^1(\mathbb{R}^n; \mathbb{R}^\ell)$, $a_3 \in C^1(\mathbb{R}^\ell; \mathbb{R}^n)$, and $b_3 \in C^1(\mathbb{R}^\ell; \mathbb{R}^\ell)$ are known and satisfy $a_1(0,0,0,0) = 0$, $b_1(0,0,0,0) = 0$, $a_2(0) = a_3(0) = 0$, $b_2(0) = b_3(0) = 0$, $G_1(x_1) \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ and $G_2(x_1) \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^\ell)$ are known. The uncertainty in the system is represented by the nonlinear function $h_1 \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}^\ell \times \mathbb{R}^\ell \times \mathbb{R}^m \times \mathbb{R}^+_0; \mathbb{R}^n)$. For notational simplicity, let $\xi = [\xi_1\ \xi_2]^\mathrm{T} := [x_1\ y_1]^\mathrm{T}$ and $\eta = [\eta_1\ \eta_2]^\mathrm{T} := [x_2\ y_2]^\mathrm{T}$. It is well known that, for $(x_1, x_2, y_1, y_2) \mapsto f(x_1, x_2, y_1, y_2)$,

$$
\begin{aligned}
f(x_1, x_2, y_1, y_2) = &\int_0^1 (\partial_{x_1} f)(\alpha x_1, \alpha x_2, \alpha y_1, \alpha y_2)\, \mathrm{d}\alpha x_1 \\
&+ \int_0^1 (\partial_{x_2} f)(\alpha x_1, \alpha x_2, \alpha y_1, \alpha y_2)\, \mathrm{d}\alpha x_2 \\
&+ \int_0^1 (\partial_{y_1} f)(\alpha x_1, \alpha x_2, \alpha y_1, \alpha y_2)\, \mathrm{d}\alpha y_1 \\
&+ \int_0^1 (\partial_{y_2} f)(\alpha x_1, \alpha x_2, \alpha y_1, \alpha y_2)\, \mathrm{d}\alpha y_2.
\end{aligned}
$$

Thus, $a_1(x_1, x_2, y_1, y_2)$ can be expressed in the form

$$
\int_0^1 A_1^{(\xi)}(\alpha\xi, \alpha\eta)\, \mathrm{d}\alpha\xi + \int_0^1 A_1^{(\eta)}(\alpha\xi, \alpha\eta)\, \mathrm{d}\alpha\eta,
$$

where

$$
\begin{aligned}
A_1^{(\xi)}(\xi, \eta) &:= \left[\, (\partial_{\xi_1} a_1)(\xi_1, \eta_1, \xi_2, \eta_2)\ \ (\partial_{\xi_2} a_1)(\xi_1, \eta_1, \xi_2, \eta_2) \,\right], \\
A_1^{(\eta)}(\xi, \eta) &:= \left[\, (\partial_{\eta_1} a_1)(\xi_1, \eta_1, \xi_2, \eta_2)\ \ (\partial_{\eta_2} a_1)(\xi_1, \eta_1, \xi_2, \eta_2) \,\right].
\end{aligned}
$$

Also, in view of the continuity conditions on a_2, $I(t, \tau_1, f_2(x_1))$ can be expressed in the form

$$
\begin{aligned}
\int_{t-\tau_1}^t \int_0^1 &(\partial_{\xi_1} a_2)(\alpha\xi_1(r))\, \mathrm{d}\alpha\xi_1(r)\, \mathrm{d}r \\
&= \int_0^1 I(t, \tau_1, (\partial_{\xi_1} a_2)(\alpha\xi_1)\xi_1)\, \mathrm{d}\alpha
\end{aligned}
$$

and, in addition,

$$
I(t, \tau_2, a_3(\xi_2)) = \int_0^1 I(t, \tau_2, (\partial_{\xi_2} a_3)(\alpha\xi_2)\xi_2)\, \mathrm{d}\alpha.
$$

Similarly, $b_1(x_1, x_2, y_1, y_2)$ can be replaced by

$$
\int_0^1 B_1^{(\xi)}(\alpha\xi, \alpha\eta)\, \mathrm{d}\alpha\xi + \int_0^1 B_1^{(\eta)}(\alpha\xi, \alpha\eta)\, \mathrm{d}\alpha\eta,
$$

where

$$
\begin{aligned}
B_1^{(\xi)}(\xi, \eta) &:= \left[\, (\partial_{\xi_1} b_1)(\xi_1, \eta_1, \xi_2, \eta_2)\ \ (\partial_{\xi_2} b_1)(\xi_1, \eta_1, \xi_2, \eta_2) \,\right], \\
B_1^{(\eta)}(\xi, \eta) &:= \left[\, (\partial_{\eta_1} b_1)(\xi_1, \eta_1, \xi_2, \eta_2)\ \ (\partial_{\eta_2} b_1)(\xi_1, \eta_1, \xi_2, \eta_2) \,\right],
\end{aligned}
$$

and

$$
\begin{aligned}
I(t, \tau_1, b_2(\xi_1)) &= \int_0^1 I(t, \tau_1, (\partial_{\xi_1} b_2)(\alpha\xi_1)\xi_1)\, \mathrm{d}\alpha, \\
I(t, \tau_2, b_3(\xi_2)) &= \int_0^1 I(t, \tau_2 (\partial_{\xi_2} b_3)(\alpha\xi_2)\xi_2)\, \mathrm{d}\alpha.
\end{aligned}
$$

Hence, the descriptor system can be expressed as

$$
\begin{aligned}
\dot{\xi}_1(t) = &\int_0^1 A_1^{(\xi)}(\alpha\xi, \alpha\eta)\, \mathrm{d}\alpha\xi + \int_0^1 A_1^{(\eta)}(\alpha\xi, \alpha\eta)\, \mathrm{d}\alpha\eta \\
&+ \int_0^1 I(t, \tau_1, (\partial_{\xi_1} a_2)(\alpha\xi_1)\xi_1)\, \mathrm{d}\alpha \\
&+ \int_0^1 I(t, \tau_2, (\partial_{\xi_2} a_3)(\alpha\xi_2)\xi_2)\, \mathrm{d}\alpha \\
&+ G_1(\xi_1(t))u(t) \\
&+ h_1(t, \xi_1(t), \eta_1(t), \zeta_1(t), \xi_2(t), \eta_2(t), \zeta_2(t), u(t)), \\
0 = &\int_0^1 B_1^{(\xi)}(\alpha\xi, \alpha\eta)\, \mathrm{d}\alpha\xi + \int_0^1 B_1^{(\eta)}(\alpha\xi, \alpha\eta)\, \mathrm{d}\alpha\eta \\
&+ \int_0^1 I(t, \tau_1, (\partial_{\xi_1} b_2)(\alpha\xi_1)\xi_1)\, \mathrm{d}\alpha \\
&+ \int_0^1 I(t, \tau_2, (\partial_{\xi_2} b_3)(\alpha\xi_2)\xi_2)\, \mathrm{d}\alpha \\
&+ G_2(\xi_1(t))u(t),
\end{aligned}
$$

where $\zeta_1 := x_3$ and $\zeta_2 := y_3$.

3 Design objective and class of feedback control

First, some appropriate hypotheses are introduced. The Euclidean inner product (on $\mathbb{R}^n$ or $\mathbb{R}^\ell$ as appropriate) and the induced norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let $\chi_i^{(1)}(\xi_1) := \left\langle g_i^{(1)}(\xi_1), P_1\xi_1 \right\rangle$ and $\chi_i^{(2)}(\xi) := \left\langle g_i^{(2)}(\xi_1), P_2\xi_1 + P_3\xi_2 \right\rangle$, where $g_i^{(1)}, g_i^{(2)}$ denote the ith components of G_1 and G_2 respectively.

H1: There exist $p : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $q : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ such that

$$
h_1(t, \xi, \eta, \zeta, u) = q(t, \xi, \eta, \zeta, u) + G_1(\xi_1)p(t, \xi, \eta, \zeta, u),
$$

and there exist real constants $\bar{\nu}_i, \hat{\alpha}, \beta_i, \hat{\beta}, \gamma_i, \hat{\gamma}, \Gamma_i, \hat{\Gamma}, \delta_i, \hat{\delta}, \Delta_i, \hat{\Delta} \in \mathbb{R}^+_0$, such that

(i) $|p_i(t, \xi, \eta, \zeta, u)| \le \nu_i(t, \xi, \eta, \zeta) + \beta_i \|\xi\| + \gamma_i \|\eta_1\| + \Gamma_i \|\eta_2\| + \delta_i \|\zeta_1\| + \Delta_i \|\zeta_2\| + \kappa_i |u_i|$;

(ii) $\|q(t, \xi, \eta, \zeta, u)\| \le \sum_{i=1}^m \hat{\nu}_i(t, \xi, \eta, \zeta)|\chi_i^{(1)}(\xi_1)| + \hat{\alpha} + \hat{\beta} \|\xi\| + \hat{\gamma} \|\eta_1\| + \hat{\Gamma} \|\eta_2\| + \hat{\delta} \|\zeta_1\| + \hat{\Delta} \|\zeta_2\|$,

where $\nu_i : \mathbb{R} \times \mathbb{R}^{n+l} \times \mathbb{R}^{n+l} \times \mathbb{R}^{n+l} \to [0, \bar{\nu}_i]$, $\hat{\nu}_i : \mathbb{R} \times \mathbb{R}^{n+l} \times \mathbb{R}^{n+l} \times \mathbb{R}^{n+l} \to [0, \tilde{\nu}_i]$, $\kappa_i \in [0, 1)$ for all i, and u_i, p_i denote the ith components of u and p, respectively.

Remark: The vector fields p, q are said to represent the *matched* and *unmatched* components, respectively, of the uncertainty in the nonlinear dynamic time-delay subsystem.

H2: For all $\omega_i > 0$ and $0 \le \bar{Z}_i < 1$, the following inequality holds:
$|\chi_i^{(2)}(\xi)| \le \omega_i \|\xi\| + Z_i(\xi)|\chi_i^{(1)}(\xi_1)|$,
where $Z_i : \mathbb{R}^{n+l} \to [0, \bar{Z}_i]$, and
$Z_i \le \bar{Z}_i < 1 - \kappa_i$.

It is desired that a robust memoryless feedback control function, $x(t) \mapsto c(x, y)$, be designed so that the uncertain descriptor system has the property that a prescribed compact nonempty set, containing $0 \in \mathbb{R}^n$ is uniformly asymptotically stable, for definition, see [1].

The design of the feedback controls emulates the work in [1]. Here the class of feedback controls consists the functions $x \mapsto c(x, y) = [c_1(x, y) \dots c_m(x, y)]^{\mathrm{T}}$ with the following structure:

$$c_i(x, y) := -(1 - \kappa_i - \bar{Z}_i)^{-1}\Bigg[\mu_i(\gamma_i + \Gamma_i + \tau_1\delta_i + \tau_2\Delta_i) + \frac{{\pi_i}^2(\xi)}{\pi_i(\xi)|\chi_i^{(1)}(\xi_1)| + \lambda_i \|\xi\|^2}\Bigg]\chi_i^{(1)}(\xi_1), \tag{3.1}$$

where
$$x \mapsto \pi_i(\xi) := \bar{\nu}_i + \Bigg(\beta_i + \tilde{\nu}_i \|P_1\| + \frac{1}{2}(\gamma_i + \Gamma_i + \tau_1\delta_i + \tau_2\Delta_i)\Bigg)\|\xi\|,$$
and $\mu_i > 0, \lambda_i > 0, a_i \ge 0$ are design parameters.

4 Stabilization of the feedback controlled descriptor system

For this particular problem, the methodology is based on Lyapunov-Krasovskiĭ functionals and a Lyapunov analysis. Consider the Lyapunov-Krasovskiĭ functionals: $v_0(\phi(0)) := \langle\phi(0), EP\phi(0)\rangle$, and

$$v_1(\phi) := v_0(\phi(0)) + \int_{-\rho_1}^{0} \langle\phi_1(s), R_1\phi_1(s)\rangle \,\mathrm{d}s + \int_{-\rho_2}^{0} \langle\phi_2(s), R_2\phi_2(s)\rangle \,\mathrm{d}s,$$

where $\phi = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix}^{\mathrm{T}}$, $\phi_1, \phi_2 \in C([-T, 0]; \mathbb{R}^n)$, and $E = \begin{bmatrix} I_n & O \\ O & O_\ell \end{bmatrix}, P = \begin{bmatrix} P_1 & O \\ P_2 & P_3 \end{bmatrix}$, with I_n denoting the $n \times n$ identity matrix and O_ℓ denoting the $\ell \times \ell$ zero matrix. The particular structure of v_0 has previously been utilized for linear hereditary systems of the neutral type in [2]. Here, it is assumed that $R_1, P_1 \in \mathbb{R}^{n\times n}$ are symmetric and positive definite, and also, $R_2 \in \mathbb{R}^{\ell\times\ell}$ is symmetric and positive definite. Noting that $\langle\phi(0), EP\phi(0)\rangle = \langle\phi_1(0), P_1\phi_1(0)\rangle$, it follows that, along solutions to the descriptor system (2.1-2.2),

$$\dot{v}_1(\xi_t) = \Big\langle\dot{\xi}_1(t), P_1\xi_1(t)\Big\rangle + \Big\langle\xi_1(t), P_1\dot{\xi}_1(t)\Big\rangle + \langle\xi(t), R\xi(t)\rangle - \langle\eta(t), R\eta(t)\rangle ,$$

where $R = diag(R_1, R_2)$. Since P_1 is symmetric, then

$$\Big\langle\dot{\xi}_1, P_1\xi_1\Big\rangle + \Big\langle\xi_1, P_1\dot{\xi}_1\Big\rangle = \Big\langle\begin{bmatrix} \dot{\xi}_1 \\ 0 \end{bmatrix}, P\xi\Big\rangle + \Big\langle P\xi, \begin{bmatrix} \dot{\xi}_1 \\ 0 \end{bmatrix}\Big\rangle. \tag{4.1}$$

Hence, along solutions to (2.1-2.2) and utilizing (4.1), it is easily shown that

$$\begin{aligned} \dot{v}_1(\xi_t) = &\int_0^1 \Big\langle\begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix}, L(\alpha\xi(t), \alpha\eta(t))\begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix}\Big\rangle \,\mathrm{d}\alpha \\ &+ 2\int_0^1\int_{t-\tau_1}^{t} \langle M(\alpha\xi_1(r))\xi(r), P\xi(t)\rangle \,\mathrm{d}r\,\mathrm{d}\alpha \\ &+ 2\int_0^1\int_{t-\tau_2}^{t} \langle N(\alpha\xi_2(r))\xi(r), P\xi(t)\rangle \,\mathrm{d}r\,\mathrm{d}\alpha \\ &+ 2\langle G(\xi_1(t))u(t) \\ &+ h(t, \xi_1(t), \xi_2(t), \eta_1(t), \eta_2(t), \zeta_1(t), \zeta_2(t), u(t)), P\xi(t)\rangle, \end{aligned} \tag{4.2}$$

where

$$L(\xi, \eta) := \begin{bmatrix} L_1 + R_1 & L_2 & L_3 & L_4 \\ L_2^{\mathrm{T}} & L_5 + R_2 & L_6 & L_7 \\ L_3^{\mathrm{T}} & L_6^{\mathrm{T}} & -R_1 & 0 \\ L_4^{\mathrm{T}} & L_7^{\mathrm{T}} & 0 & -R_2 \end{bmatrix},$$

$$\begin{aligned} L_1 = &(\partial_{\xi_1}a_1)^{\mathrm{T}}(\xi, \eta)P_1 + P_1(\partial_{\xi_1}a_1)(\xi, \eta) \\ &+ (\partial_{\xi_1}b_1)^{\mathrm{T}}(\xi, \eta)P_2 + P_2^{\mathrm{T}}(\partial_{\xi_1}b_1)(\xi, \eta), \end{aligned}$$

$$L_2 = P_1(\partial_{\xi_2}a_1)(\xi, \eta) + P_2^{\mathrm{T}}(\partial_{\xi_2}b_1)(\xi, \eta) + (\partial_{\xi_1}b_1)^{\mathrm{T}}(\xi, \eta)P_3,$$

$$L_3 = P_1(\partial_{\eta_1}a_1)(\xi, \eta) + P_2^{\mathrm{T}}(\partial_{\eta_1}b_1)(\xi, \eta),$$

$$L_4 = P_1(\partial_{\eta_2}a_1)(\xi, \eta) + P_2^{\mathrm{T}}(\partial_{\eta_2}b_1)(\xi, \eta),$$

$$L_5 = (\partial_{\xi_2}b_1)^{\mathrm{T}}(\xi, \eta)P_3 + P_3^{\mathrm{T}}(\partial_{\xi_2}b_1)(\xi, \eta),$$

$$L_6 = P_3^{\mathrm{T}}(\partial_{\eta_1}b_1)(\xi, \eta), \quad L_7 = P_3^{\mathrm{T}}(\partial_{\eta_2}b_1)(\xi, \eta),$$

$$M(\alpha\xi_1) := \begin{bmatrix} (\partial_{\xi_1}a_2)(\alpha\xi_1) & 0 \\ (\partial_{\xi_1}b_2)(\alpha\xi_1) & 0 \end{bmatrix},$$

$$N(\alpha\xi_2) := \begin{bmatrix} 0 & (\partial_{\xi_2}a_3)(\alpha\xi_2) \\ 0 & (\partial_{\xi_2}b_3)(\alpha\xi_2) \end{bmatrix},$$

$$G(\xi_1) := \begin{bmatrix} G_1(\xi_1) \\ G_2(\xi_1) \end{bmatrix},$$

$$h(t, \xi_1, \eta_1, \zeta_1, \xi_2, \eta_2, \zeta_2, u) := \begin{bmatrix} h_1(t, \xi_1, \eta_1, \zeta_1, \xi_2, \eta_2, \zeta_2, u) \\ 0 \end{bmatrix}.$$

Let $J = \begin{bmatrix} J_1 & O \\ O & J_2 \end{bmatrix}, K = \begin{bmatrix} K_1 & O \\ O & K_2 \end{bmatrix}$ be symmetric matrices, with $J_1,\ J_2,\ K_1,\ K_2 > 0$, and consider the functional

$$v_2(\xi_t) := \int_0^1 \Bigg\{\int_0^{\tau_1}\int_{t-s}^{t} \langle\xi(r), X(\alpha\xi_1(r))\xi(r)\rangle \,\mathrm{d}r\,\mathrm{d}s + \int_0^{\tau_2}\int_{t-s}^{t} \langle\xi(r), Y(\alpha\xi_2(r))\xi(r)\rangle \,\mathrm{d}r\,\mathrm{d}s\Bigg\}\mathrm{d}\alpha,$$

where

$$
\begin{aligned}
X(\alpha\xi_1) &= M^{\mathrm{T}}(\alpha\xi_1)PJ^{-1}P^{\mathrm{T}}M(\alpha\xi_1) \\
&= \begin{bmatrix} X_1(\alpha\xi_1) & O \\ O & O \end{bmatrix}, \\
X_1(\alpha\xi_1) &:= \Big((\partial_{\xi_1} a_2)^{\mathrm{T}}(\alpha\xi_1)P_1 \\
&\quad + (\partial_{\xi_1} b_2)^{\mathrm{T}}(\alpha\xi_1)P_2 \Big) J_1^{-1} \Big((P_1(\partial_{\xi_1} a_2)(\alpha\xi_1) \\
&\quad + P_2^{\mathrm{T}}(\partial_{\xi_1} b_2)(\alpha\xi_1) \Big) \\
&\quad + (\partial_{\xi_1} b_2)^{\mathrm{T}}(\alpha\xi_1)P_3J_2^{-1}P_3^{\mathrm{T}}(\partial_{\xi_1} b_2)(\alpha\xi_1), \\
Y(\alpha\xi_2) &= N^{\mathrm{T}}(\alpha\xi_2)PK^{-1}P^{\mathrm{T}}N(\alpha\xi_2) \\
&= \begin{bmatrix} O & O \\ O & Y_2(\alpha\xi_2) \end{bmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
Y_2(\alpha\xi_2) &:= \Big((\partial_{\xi_2} a_3)^{\mathrm{T}}(\alpha\xi_2)P_1 \\
&\quad + (\partial_{\xi_2} b_3)^{\mathrm{T}}(\alpha\xi_2)P_2 \Big) K_1^{-1} \Big(P_1(\partial_{\xi_2} a_3)(\alpha\xi_2) \\
&\quad + P_2^{\mathrm{T}}(\partial_{\xi_2} b_3)(\alpha\xi_2) \Big) \\
&\quad + (\partial_{\xi_2} b_3)^{\mathrm{T}}(\alpha\xi_2)P_3K_2^{-1}P_3^{\mathrm{T}}(\partial_{\xi_2} b_3)(\alpha\xi_2).
\end{aligned}
$$

Then, clearly,

$$
\begin{aligned}
\dot{v}_2(\xi_t) &= \int_0^1 \Big\{ \langle \xi(t), (\tau_1 X(\alpha\xi_1(t)) + \tau_2 Y(\alpha\xi_2(t)))\xi(t) \rangle \\
&\quad - \int_{t-\tau_1}^{t} \langle \xi(r), X(\alpha\xi_1(r))\xi(r) \rangle \, \mathrm{d}r \\
&\quad - \int_{t-\tau_2}^{t} \langle \xi(r), Y(\alpha\xi_2(r))\xi(r) \rangle \, \mathrm{d}r \Big\} \, \mathrm{d}\alpha. \qquad (4.3)
\end{aligned}
$$

Define

$$
\begin{aligned}
v(\xi_t) &:= v_1(\xi_t) + v_2(\xi_t) \\
&\quad + A_{\rho_1} \int_{t-\rho_1}^{t} \|\xi(r)\|^2 \, \mathrm{d}r + A_{\rho_2} \int_{t-\rho_2}^{t} \|\xi(r)\|^2 \, \mathrm{d}r \\
&\quad + B_{\tau_1} \int_0^{\tau_1} \int_{t-s}^{t} \|\xi(r)\|^2 \, \mathrm{d}r \, \mathrm{d}s \\
&\quad + B_{\tau_2} \int_0^{\tau_2} \int_{t-s}^{t} \|\xi(r)\|^2 \, \mathrm{d}r \, \mathrm{d}s,
\end{aligned}
$$

where

$$
A_{\rho_1} := \hat{\gamma}\,\|P_1\| + \frac{1}{2}\sum_{i=1}^{m} \mu_i^{-1}\gamma_i,
$$

$$
A_{\rho_2} := \hat{\Gamma}\,\|P_1\| + \frac{1}{2}\sum_{i=1}^{m} \mu_i^{-1}\Gamma_i,
$$

$$
B_{\tau_1} := \hat{\delta}\,\|P_1\| + \frac{1}{2}\sum_{i=1}^{m} \mu_i^{-1}\delta_i,
$$

$$
B_{\tau_2} := \hat{\Delta}\,\|P_1\| + \frac{1}{2}\sum_{i=1}^{m} \mu_i^{-1}\Delta_i,
$$

then, using the inequality

$$
-\left\langle x, K^{-1}x \right\rangle + 2\left\langle x, y \right\rangle \le \left\langle y, Ky \right\rangle,
$$

for any symmetric $K > 0$ and for all x and y, and invoking (4.2-4.3), a straightforward analysis shows that, along solutions to (2.1-2.2),

$$
\begin{aligned}
\dot{v}(\xi_t) &\le \int_0^1 \Big\langle \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix}, \big(L(\alpha\xi(t), \alpha\eta(t)) + W(\alpha\xi(t))\big) \\
&\quad \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} \Big\rangle \, \mathrm{d}\alpha + 2\big\langle G(\xi_1(t))u(t) \\
&\quad + h(t, \xi_1(t), \xi_2(t), \eta_1(t), \eta_2(t), \zeta_1(t), \zeta_2(t), u(t)), P\xi(t) \big\rangle \\
&\quad + (A_{\rho_1} + A_{\rho_2} + \tau_1 B_{\tau_1} + \tau_2 B_{\tau_2}) \|\xi(t)\|^2 \\
&\quad - A_{\rho_1} \|\eta_1(t)\|^2 - A_{\rho_2} \|\eta_2(t)\|^2 \\
&\quad - B_{\tau_1} \int_{t-\tau_1}^{t} \|\xi(r)\|^2 \, \mathrm{d}r - B_{\tau_2} \int_{t-\tau_2}^{t} \|\xi(r)\|^2 \, \mathrm{d}r,
\end{aligned}
$$

where

$$
W(\xi) := \begin{bmatrix} W_1(\xi_1) & O & O & O \\ O & W_2(\xi_2) & O & O \\ O & O & O & O \\ O & O & O & O \end{bmatrix},
$$

$$
\begin{aligned}
W_1(\xi_1) &:= \tau_1(J_1 + X_1(\xi_1)) + \tau_2 K_1, \\
W_2(\xi_2) &:= \tau_2(K_2 + Y_2(\xi_2)) + \tau_1 J_2.
\end{aligned}
$$

It is supposed the following hypothesis holds.

H3: (i) There exist matrices P and R, with R_1, $R_2 > 0$, symmetric matrices J, K, with J_1, J_2, K_1, $K_2 > 0$, and a symmetric matrix-valued function $(\xi, \eta) \mapsto Q(\xi, \eta)$, satisfying $Q(\xi, \eta) > 0$ for all (ξ, η), such that

$$
L(\xi, \eta) + W(\xi) + Q(\xi, \eta) \le 0, \quad \forall\, (\xi, \eta).
$$

(ii) There exist a constant matrix $\tilde{Q} = \tilde{Q}^{\mathrm{T}} \in \mathbb{R}^{2n\times 2n}$ such that $Q(\xi, \eta) \ge \tilde{Q} > 0$.

In view of H1 and H3, it follows that

$$
\begin{aligned}
\dot{v}(\xi_t) &\le -\sigma_{\min}(\tilde{Q})\,\|\xi(t)\|^2 - \sigma_{\min}(\tilde{Q})\,\|\eta(t)\|^2 \\
&\quad + 2\sum_{i=1}^{m} \Big[\big(u_i(t, \xi, \eta, \zeta) + p_i(t, \xi, \eta, \zeta, u)\big)\chi_i^{(1)}(\xi_1) \\
&\quad + u_i(t, \xi, \eta, \zeta)\chi_i^{(2)}(\xi) \Big] + 2\left\langle q(t, \xi, \eta, \zeta, u), P_1\xi_1 \right\rangle \\
&\quad + (A_{\rho_1} + A_{\rho_2} + \tau_1 B_{\tau_1} + \tau_2 B_{\tau_2}) \|\xi(t)\|^2 \\
&\quad - A_{\rho_1} \|\eta_1(t)\|^2 - A_{\rho_2} \|\eta_2(t)\|^2 \\
&\quad - B_{\tau_1} \int_{t-\tau_1}^{t} \|\xi(r)\|^2 \, \mathrm{d}r - B_{\tau_2} \int_{t-\tau_2}^{t} \|\xi(r)\|^2 \, \mathrm{d}r, \qquad (4.4)
\end{aligned}
$$

then, invoking H1-H2, using feedback control (3.1), we have:

$$
\begin{aligned}
&2\sum_{i=1}^{m}\bigg[(u_i(t,\xi,\eta,\zeta)+p_i(t,\xi,\eta,\zeta,u))\chi_i^{(1)}(\xi_1)\\
&\quad+u_i(t,\xi,\eta,\zeta)\chi_i^{(2)}(\xi)\bigg]+2\left\langle q(t,\xi,\eta,\zeta,u),P_1\xi_1\right\rangle\\
&\le 2\bigg[\hat\beta\left\|P_1\right\|+\sum_{i=1}^{m}\Big(\lambda_i+\omega_i(1-\kappa_i-\bar Z_i)^{-1}\big(\beta_i+\tilde\nu_i\left\|P_1\right\|\big)\\
&+\omega_i^2(1-\kappa_i-\bar Z_i)^{-2}\mu_i\big(\gamma_i+\Gamma_i+\tau_1\delta_i+\tau_2\Delta_i\big)\Big)\bigg]\left\|\xi\right\|^2\\
&+\sum_{i=1}^{m}\frac{1}{2}\mu_i^{-1}(\gamma_i\left\|\eta_1\right\|^2+\Gamma_i\left\|\eta_2\right\|^2+\frac{\delta_i}{\tau_1}\left\|\zeta_1\right\|^2+\frac{\Delta_i}{\tau_2}\left\|\zeta_2\right\|^2)\\
&+2\left[\hat\alpha\left\|P_1\right\|+\sum_{i=1}^{m}\omega_i(1-\kappa_i-\bar Z_i)^{-1}\bar\nu_i\right]\left\|\xi\right\|\\
&+2\hat\gamma\left\|P_1\right\|\left\|\eta_1\right\|\left\|\xi\right\|+2\hat\Gamma\left\|P_1\right\|\left\|\eta_2\right\|\left\|\xi\right\|+2\hat\delta\left\|P_1\right\|\left\|\zeta_1\right\|\left\|\xi\right\|\\
&+2\hat\Delta\left\|P_1\right\|\left\|\zeta_2\right\|\left\|\xi\right\|. \qquad (4.5)
\end{aligned}
$$

Therefore, from (4.4) and (4.5), the following result holds for almost all t:

$$
\begin{aligned}
\dot v(\xi_t)\le&-\Big\{\sigma_{\min}(\tilde Q)-2\sum_{i=1}^{m}\lambda_i-k\Big\}\left\|\xi(t)\right\|^2-\sigma_{\min}(\tilde Q)\left\|\eta(t)\right\|^2\\
&+2\left[\hat\alpha\left\|P_1\right\|+\sum_{i=1}^{m}\omega_i(1-\kappa_i-\bar Z_i)^{-1}\bar\nu_i\right]\left\|\xi\right\|,
\end{aligned}
$$

where $k:=2(\hat\beta+\hat\gamma+\hat\Gamma+\tau_1\hat\delta+\tau_2\hat\Delta)\left\|P_1\right\|+\sum_{i=1}^{m}(\gamma_i+\Gamma_i+\tau_1\delta_i+\tau_2\Delta_i)\left(\frac{1}{2\mu_i}+2\omega_i^2(1-\kappa_i-\bar Z_i)^{-2}\mu_i\right)+2\sum_{i=1}^{m}\omega_i(1-\kappa_i-\bar Z_i)^{-1}(\beta_i+\tilde\nu_i\left\|P_1\right\|)$.

Minimizing k with the design parameter μ_i, one obtains

$$
\mu_i=\mu_i^*:=\frac{1-\kappa_i-\bar Z_i}{2\omega_i}
$$

and $k=2(\hat\beta+\hat\gamma+\hat\Gamma+\tau_1\hat\delta+\tau_2\hat\Delta)\left\|P_1\right\|+2\sum_{i=1}^{m}\omega_i(1-\kappa_i-\bar Z_i)^{-1}(\tilde\nu_i\left\|P_1\right\|+\beta_i+\gamma_i+\Gamma_i+\tau_1\delta_i+\tau_2\Delta_i)$.

Suppose

$$
\hat k:=\sigma_{\min}(\tilde Q)-k>0,
$$

then design parameters λ_i such that $\sum_{i=1}^{m}\lambda_i<\frac{1}{2}\hat k$, and

$\tilde k:=\hat k-2\sum_{i=1}^{m}\lambda_i>0$. Hence,

$$
\dot v(\xi_t)\le-\tilde k\left\|\xi(t)\right\|^2+2k_1\left\|\xi\right\|,
$$

where $k_1:=\hat\alpha\left\|P_1\right\|+\sum_{i=1}^{m}\omega_i(1-\kappa_i-\bar Z_i)^{-1}\bar\nu_i$.

Theorem 4.1 *Suppose Hypotheses* H1-H3 *hold, and choosing the design parameters to satisfy* $\hat k>0$, *then the compact set* $\mathcal{A}\supsetneqq\tilde{\mathcal{A}}:=\left\{x\in\mathbb{R}^n;\left\|x\right\|^2\le 2k_1\tilde k^{-1}\right\}$, *is a uniformly asymptotically stable set for the class of descriptor systems (2.1-2.2), subject to (2.3), using the feedback control* $u(t)=c(x,y)$, *where* $c_i(x,y)$ *are designed in (3.1).*

5 Example

It is assumed that a plant has a system model described by

$$
\begin{aligned}
\dot x_1(t)&=3x_1(t)+y(t)\\
\dot x_2(t)&=-4(x_2(t)-\tan^{-1}(x_2(t)))-\frac{1}{2}x_1(t-\rho_1)\\
&\quad-\int_{t-\tau_1}^{t}\sin(x_2(s))\,\mathrm{d}s+(x_1(t)+x_2(t))u(t)\\
&\quad+h_1\big(t,x(t),y(t),x(t-\rho_1),y(t-\rho_2),\\
&\qquad I(x,\tau_1),I(y,\tau_2),u(t)\big)\\
0&=-4y(t)-\frac{1}{2}\left(x_2(t)+\frac{x_2(t)}{1+x_2^2(t)}\right)-\frac{1}{2}y(t-\rho_2)\\
&\quad+\frac{1}{2}\int_{t-\tau_2}^{t}y(s))\,\mathrm{d}s,
\end{aligned}
$$

where
$x(t)=[x_1(t)\ \ x_2(t)]^T\in\mathbb{R}^2$, $y(t)\in\mathbb{R}^1$,
$h_1(t,x,y,\bar x,\bar y,\hat x,\hat y,u)=a(t)\dfrac{x_1^3(t)}{1+x_1^2(t)}$
$+\Big(b_1(t)\sin(\bar x_1^2)+b_2(t)\bar x_2\Big)\big(x_1+x_2\big)$ with $\bar x=[\bar x_1\ \ \bar x_2]^T$ represents uncertainty in the system and $t\mapsto a(t)$, $t\mapsto b_1(t)$, $t\mapsto b_2(t)$ are unknown functions with $|a(t)|\le\bar a$, $|b_1(t)|\le\bar b_1$ and $|b_2(t)|\le\bar b_2$ for all t. In this example, it is assumed that the discrete delays are $\rho_1=\rho_2=1$ and the distributed delays are $\tau_1=\frac{1}{2},\tau_2=1$.
Now $G_1(x)=\begin{bmatrix}0\\ x_1+x_2\end{bmatrix}$, Hypothesis 1 is satisfied with $\bar\nu=\bar b_1,\gamma_i=\bar b_2,\hat\beta_i=\bar a$ and $\hat\alpha=\tilde\nu=\beta_i=\hat\gamma=\Gamma_i=\hat\Gamma=\delta_i=\hat\delta=\Delta_i=\hat\Delta=\kappa_i=0$. Since $G_2(.)\equiv 0$, select $\omega_i=1$ and $\bar Z_i=0$, then Hypothesis 2 is always satisfied since $\chi_i^{(2)}(.)\equiv 0$.
Given $P_1=\begin{bmatrix}2&0\\0&1\end{bmatrix}$, $P_2=\begin{bmatrix}0&0\end{bmatrix}$, $P_3=\begin{bmatrix}1\end{bmatrix}$, $R_1=\begin{bmatrix}4&0\\0&1\end{bmatrix}$, $R_2=\begin{bmatrix}1\end{bmatrix}$ and $J_1=K_1=I_2$, $J_2=K_2=I_1$, it can be shown that Hypothesis 3 is satisfied with $\tilde Q=0.92$. Thus, with $\bar a=1/16,\bar b_1=1,\bar b_2=1/8$, and $\lambda_i=1/10$, Theorem 4.1 holds for this example.
The feedback control for the system is designed to be

$$
c_i(x,y)=-\left[\frac{1}{16}+\frac{\pi_i^2(x)}{\pi_i(x)|\chi_i^{(1)}(x)|+\frac{1}{10}(x_1^2+x_2^2)}\right]\chi_i^{(1)}(x),
$$

where $\pi_i(x)=1+\frac{1}{16}(x_1^2+x_2^2+y^2)^{\frac{1}{2}}$ and $\chi_i^{(1)}(x)=x_1x_2+x_2^2$.

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