STABILITY ANALYSIS AND CONTROL SYNTHESIS WITH D.C. RELAXATION OF PARAMETERIZED LMIS

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Abstract

Relaxation of an optimization problem under parametrized LMI constraint is discussed in this paper. Relaxation methods are based on convexfication using difference of convex and multiconvex techniques, thus the relaxed problems become numerically tractable. The d.c. relaxation is generalized and is imported into the multiconvex relaxation, then the difference of multiconvex relaxation is naturally defined. These two relaxation methods are applied to stability and state-feedback control synthesis of linear parameter varying systems. Numerical examples are illustrated for the applications to show the effectiveness of these methods.

1 Introduction

It is known that stability and performance analysis problems of linear time invariant systems are represented by feasibility problems or minimization problems under linear matrix inequality (LMI) constraint, that is, LMI problems [5]. LMI problems can be solved by efficient interior-point methods with worst-case polynomial complexity. Some LMI solvers [6] are available and are widely used. On the other hand, control problems, for example, stability analysis of linear parameter varying (LPV) systems are not LMI problems. These problems are represented by feasibility problems under parametrized LMI (PLMI) constraint, that is, PLMI problems [2]. PLMI includes a infinite number of LMIs, so that PLMI problem is NP-hard in general. Therefore, some relaxation methods reducing a PLMI constraint to a finite number of LMIs constraint have been proposed [1, 2]. Convexfication technique using difference of convex (d.c.) function [1] is efficiently used for relaxation. From global optimization point of view, d.c. structure [3] is a methodology for non-convex optimization problem. On the other hand, multiconvex (m.c.) function [2] is also used for relaxation. Multiconvexity is a result of non-convex quadratic optimization problems[4].

In this paper we propose two relaxation methods of parameterized LMI (PLMI) constraint on optimization problem. One is a generalized d.c. relaxation and the other is an unified method between the generalized d.c. and the m.c. relaxation, thus PLMI constraint becomes a finite number of LMIs. These two relaxation methods are applied to stability analysis and state-feedback control design of LPV systems.

The notation in this paper is as follows. $\mathbf{R}$ is the set of real scaler value. The set of $n \times m$ real matrices is denoted by $\mathbf{R}^{n\times m}$. $A \succ 0$ means that the matrix A is positive definite. $\mathbf{S}^k$ is the set of the k dimensional symmetric matrices and $\mathbf{S}_{++} = \{X \in \mathbf{S} : X \succ 0\}$. Ω is hyper-rectangle, then $\mathrm{vert}\,\Omega$ indicates the set of vertices of Ω. For $Z \in \mathbf{R}^{k\times k}$, $\mathrm{He}\{Z\}$ means $Z + Z^T$.

2 PLMI Problems

We consider an optimization problem to find decision variables such that PLMI constraint with quadratic parameter dependence. The problem is of the form

$$\begin{array}{ll}\text{find} & z\\ \text{such that} & M(z,\theta) \prec 0 \qquad \forall\theta\in\Omega,\end{array} \tag{1}$$

where c is a given vector, z is a vector of decision variables, $\theta = [\theta_1, \cdots, \theta_N]^T \in \Omega$ is a parameter vector,

$$M(z,\theta) = M_0(z) + \sum_{i=1}^{N} \theta_i M_i(z) + \sum_{i=1, i\le j}^{N} \theta_i\theta_j M_{ij}(z), \tag{2}$$

$M_i(z)$, $M_{ij}(z) \in \mathbf{S}^k$ are affine symmetric matrix-valued functions of z. In this paper, we call it PLMI problem.

As θ is the uncertain parameter vector, a PLMI constraint means an infinite number of LMIs constraint, so that PLMI problems (1) are hard to solve generally. If we relaxed a PLMI constraint to a finite number of LMIs, we can solve it tractably and would have an solution of the original PLMI problem (1). Therefore, our aim is to formulate a problem as follows.

Describe a relaxation problem of PLMI problem (1) defining a LMI problem as

$$\begin{array}{ll}\text{find} & \tilde z\\ \text{such that} & \tilde M(\tilde z,\theta) \prec 0 \qquad \forall\theta\in\Omega.\end{array} \tag{3}$$

Then $M(z,\theta) \preceq \tilde M(\tilde z,\theta) \prec 0$ for all $\theta \in \Omega$.

3 Relaxation methods

3.1 D.C. Relaxation

Definition 1 (D.C. Functions). *[3] A function $f(\theta)$ is d.c. if it can be expressed as the difference of two convex functions, i.e.,* *if $f(\theta) = f_1(\theta) - f_2(\theta)$, where f_1, f_2 are convex functions.*

Lemma 1 (Quadratic D.C. Functions). *$f_{zx}(\theta) = x^T M(z,\theta)x$ is quadratic d.c. function in the form $f_{zx}(\theta) = f_{1zx}(\theta) - f_{2zx}(\theta)$ if there exists $R_i \in \mathbf{S}^k_{++}$ $(i = 1,\dots,N)$ such that Hessian matrix $\nabla^2 f_{1zx}(\theta)$ is positive semi-definite, where*

$$f_{1zx}(\theta) = f_{zx}(\theta) + \sum_{i=1}^{N} \theta_i^2 x^T R_i x, \quad (4)$$

$$f_{2zx}(\theta) = \sum_{i=1}^{N} \theta_i^2 x^T R_i x, \quad (5)$$

and x is arbitrary nonzero vector.

A convex upper-bound of the concave function $-f_{2zx}(\theta)$ is given by the next lemma.

Lemma 2 (Affine Upper Bound). *For $\theta_i \in \mathbf{R}$, $R_i \in \mathbf{S}^k_{++}$ and $\mathcal{B}_i \in \mathbf{R}^{k\times k}$, we have*

$$-\theta_i^2 R_i \preceq -(\theta_i \mathcal{B}_i^T R_i + \theta_i R_i \mathcal{B}_i - \mathcal{B}_i^T R_i \mathcal{B}_i). \quad (6)$$

The lemma is shown from the quadratic form $(\theta_i I - \mathcal{B}_i)^T R_i (\theta I - \mathcal{B}_i) \geq 0$.

Now that the relaxed PLMI constraint is convex on parameter set, we can give the following relaxation of the PLMI problem (1).

Theorem 1 (Generalized D.C. Relaxation). *The following LMI problem is a relaxation of PLMI problem* (1).

$$\begin{array}{l} \text{find } z, R_i \in \mathbf{S}^k_{++}, Q_i \in \mathbf{R}^{k\times k} \\ \text{such that (7) and (8)} \quad \forall \theta \in \text{vert}\,\Omega, \end{array}$$

where

$$\begin{bmatrix} 2(M_{11}(z)+R_1) & M_{12}(z) & \cdots & M_{1N}(z) \\ M_{12}(z) & 2(M_{22}(z)+R_2) & & M_{2N}(z) \\ \vdots & & \ddots & \vdots \\ M_{1N}(z) & M_{2N}(z) & \cdots & 2(M_{NN}(z)+R_N) \end{bmatrix} \succeq 0, \quad (7)$$

$$\begin{bmatrix} M(z,\theta) & * & \cdots & * \\ Q_1^T - \theta_1 R_1 & -R_1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ Q_N^T - \theta_N R_N & 0 & 0 & -R_N \end{bmatrix} \prec 0 \quad (8)$$

for all $\theta \in \text{vert}\,\Omega$.

Proof. From (4), we have

$$f_{1zx}(\theta) = x^T M_0(z)x + \sum_{i=1}^{N} \theta_i x^T M_i(z) x + \sum_{i=1, i\leq j}^{N} \theta_i\theta_j x^T M_{ij}(z)x + \sum_{i=1}^{N} \theta_i^2 x^T R_i x. \quad (9)$$

If Hessian matrix $\nabla^2 f_{1zx}(\theta)$ is positive semi-definite, i.e.

$$\nabla^2 f_{1zx}(\theta) = \mathcal{A}^T(x)\mathcal{M}(z)\mathcal{A}(x) \succeq 0,$$

where $\mathcal{M}(z)$ and $\mathcal{A}(x)$ are matrices of dimension $kN \times kN$ and $kN \times k$ respectively, defined by

$$\mathcal{M}(z) = \begin{bmatrix} 2(M_{11}(z)+R_1) & M_{12}(z) & \cdots & M_{1N}(z) \\ M_{12}(z) & 2(M_{22}(z)+R_2) & & M_{2N}(z) \\ \vdots & & \ddots & \vdots \\ M_{1N}(z) & M_{2N}(z) & \cdots & 2(M_{NN}(z)+R_N) \end{bmatrix},$$

$$\mathcal{A}(x) = \begin{bmatrix} x & 0 & \cdots & 0 \\ 0 & x & & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & x \end{bmatrix},$$

then $f_{1zx}(\theta)$ is convex function. $\mathcal{M}(z) \succeq 0$ stands for (7). Then $f_{zx}(\theta)$ is quadratic d.c. function from Lemma 1. On the other hand, from (5) and Lemma 2, we have a convex upper bound of $-f_{2zx}(\theta)$ as

$$-f_{2zx}(\theta) \leq -\sum_{i=1}^{N} x^T(\theta_i \mathcal{B}_i^T R_i + \theta_i R_i \mathcal{B}_i - \mathcal{B}_i^T R_i \mathcal{B}_i)x.$$

Then the right hand side of the following inequality

$$f_{zx}(\theta) \leq f_{1zx}(\theta) - \sum_{i=1}^{N} x^T(\theta_i \mathcal{B}_i^T R_i + \theta_i R_i \mathcal{B}_i - \mathcal{B}_i^T R_i \mathcal{B}_i)x \quad (10)$$

is convex function of θ. If the right hand side of (10) is negative for all $\theta \in \text{vert}\ \Omega$, then $f_{zx}(\theta) < 0$ for all $\theta \in \Omega$. Now (10) means the following inequality

$$M(z,\theta) \prec M(z,\theta) + \sum_{i=1}^{N} (\theta_i I - \mathcal{B}_i)^T R_i (\theta_i I - \mathcal{B}_i) \prec 0. \quad (11)$$

Applying Shur complement [5] to the right hand side of (11), we have

$$\begin{bmatrix} M(z,\theta) & * & \cdots & * \\ R_1\mathcal{B}_1 - \theta_1 R_1 & -R_1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ R_N\mathcal{B}_N - \theta_N R_N & 0 & 0 & -R_N \end{bmatrix} \prec 0. \quad (12)$$

Replacing $\mathcal{B}_i^T R_i$ with matrices Q_i, we obtain (8). □

Remark 1. Although z, R_i, and B_i are decision variables, (11) is not affine on B_i, so that (11) is not LMI when $\theta \in \text{vert}\ \Omega$.

Remark 2. Relaxation variables R_i and $\mathcal{B}_i$ in Theorem 1 are correspond to scaler $r_i I$ and $0.5I$ in the article [1] respectively. Adopting matrix variables, we can earn a small relaxation term that is added to $M(z, \theta)$ on PLMI constraint, hence a tight relaxation can be achieved. In this sense, Theorem 1 generalizes the d.c. relaxation method [1] and gives a less conservative relaxation.

3.2 Difference of Multiconvex Relaxation

Definition 2 (M.C. Functions). *[2] A function $f(\theta)$ is multiconvex if $f(\theta_i)$ $(i = 1, \ldots, N)$ is convex on $\mathbf{R}$.*

Lemma 3 (Quadratic M.C. Functions). *[2] Assume that a quadratic function $f_{zx}(\theta)$ is multiconvex, that is*

$$\frac{\partial^2}{\partial \theta_i^2} f_{zx}(\theta) \geq 0 \quad (i = 1, \ldots, N). \tag{13}$$

Then, $f_{zx}(\theta) < 0$ for all $\theta \in \Omega$ is if and only if $f_{zx}(\theta) < 0$ for all $\theta \in \mathrm{vert}\,\Omega$.

Definition 3 (D.M.C. Functions). *A function $f(\theta)$ is difference of two multiconvex functions (d.m.c.) if $f(\theta) = f_1(\theta) - f_2(\theta)$, where f_1, f_2 are multiconvex functions.*

Lemma 4 (Difference of Quadratic M.C. Functions). *$f_{zx}(\theta)$ is difference of quadratic multiconvex function in the form $f_{zx}(\theta) = f_{1zx}(\theta) - f_{2zx}(\theta)$ if there exists positive definite symmetric matrix R_i $(i = 1, \ldots, N)$ such that* (13) *are satisfied.*

Theorem 2 (D.M.C. Relaxation). *The following LMI problem is a relaxation of PLMI problem* (1)*.*

$$\begin{array}{l} \text{find} \;\; z,\, R_i \in \mathbf{S}_{++}^k,\, Q_i \in \mathbf{R}^{k \times k} \\ \text{such that} \;\; (14) \;\; \text{and} \;\; (8) \quad \forall \theta \in \mathrm{vert}\,\Omega, \end{array}$$

where

$$M_{ii}(z) + R_i \succeq 0 \qquad (i = 1, \cdots, N). \tag{14}$$

Remark 3. Relaxation variables $\mathcal{B}_i$ in Theorem 2 are correspond to 0 in article [2]. For the same reason with Remark 2, Theorem 2 generalizes the m.c. relaxation method [2] and gives a less conservative relaxation.

Remark 4. As (14) is the block diagonal elements of (7), Theorem 2 gives a less conservative relaxation rather than Theorem 1.

4 Control Applications

We consider a LPV system as follows.

$$\Pi \qquad \dot{x}(t) = A(\theta)x(t) + B(\theta)u(t)$$

where, $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ and

$$\begin{bmatrix} A(\theta) & B(\theta) \end{bmatrix} = \begin{bmatrix} A_0 & B_0 \end{bmatrix} + \sum_{i=1}^{N} \theta_i \begin{bmatrix} A_i & B_i \end{bmatrix} + \sum_{i=1, i \leq j}^{N} \theta_i \theta_j \begin{bmatrix} A_{ij} & B_{ij} \end{bmatrix},$$

$$\begin{array}{l} \theta(t) = [\; \theta_1(t) \;\; , \ldots , \;\; \theta_N(t) \;]^T, \\ \Omega : \;\; \theta_i(t) \in [\; \underline{\theta}_i \;\; \bar{\theta}_i \;] \;\; \forall t \geq 0. \end{array}$$

For the system Π, stability analysis based on Lyapunov function $V(x) = x^T P^{-1} x$ such that $V > 0$ and $\dot{V} < 0$ along all admissible parameter trajectories and for all initial conditions are discussed in this section.

4.1 Stability Analysis

Lemma 5 (Stability). *The system Π is stable if there exists $P \in \mathbf{S}_{++}^n$ such that*

$$\mathrm{He}\,\{A(\theta)P\} \prec 0 \tag{15}$$

for all $\theta \in \Omega$.

As (18) is a parameterized LMI condition with non-convex or non-multiconvex on parameter space Ω generally, we could not solve it ready. Applying Theorem 1 to Lemma 5, we have a tractable sufficient stability condition.

Proposition 1 (Stability with D.C. Relaxation). *The system Π is stable if there exists z , $R_1, \ldots, R_N \in \mathbf{S}_{++}^n$, $Q_1, \ldots, Q_N \in \mathbf{R}^{n \times n}$ such that* (7) *and* (8) *for all $\theta \in \mathrm{vert}\,\Omega$ are satisfied, where $z = (P \in \mathbf{S}_{++}^n)$ and*

$$\begin{array}{rcll} M_0(z) & = & \mathrm{He}\,\{A_0 P\} & \\ M_i(z) & = & \mathrm{He}\,\{A_i P\} & (1 \leq i \leq N) \\ M_{ij}(z) & = & \mathrm{He}\,\{A_{ij} P\} & (1 \leq i < j \leq N) \\ M_{ii}(z) & = & \mathrm{He}\,\{A_{ii} P\} & (1 \leq i \leq N) \;. \end{array} \tag{16}$$

The proof is obtained by identifying the terms in (18) with $M(z, \theta)$. In the same way, applying Theorem 2, we have another relaxation.

Proposition 2 (Stability with D.M.C. Relaxation). *The system Π is stable if there exists z, $R_1, \ldots, R_N \in \mathbf{S}_{++}^n$, $Q_1, \ldots, Q_N \in \mathbf{R}^{n \times n}$ such that* (14) *and* (8) *for all $\theta \in \mathrm{vert}\,\Omega$ are satisfied, where $z = (P \in \mathbf{S}_{++}^n)$ and* (16) .

4.2 State-Feedback Control

For the system Π, we design a static state-feedback controller

$$u = Kx \;. \tag{17}$$

Then a controller with infinite number of LMI condition is given by the following lemma.

Lemma 6 (State-Feedback Stabilization). *The closed-loop system consisted of the system Π and* (17) *is stable if there exists $P \in \mathbf{S}_{++}^n$ and $S \in \mathbf{R}^{m \times n}$ such that*

$$\mathrm{He}\left\{ \begin{bmatrix} A(\theta) & B(\theta) \end{bmatrix} \begin{bmatrix} P \\ S \end{bmatrix} \right\} \prec 0 \;, \tag{18}$$

for all $\theta \in \Omega$. Then a state-feedback controller is

$$K = SP^{-1} \;. \tag{19}$$

We apply Theorem 2 to Lemma 6 and have tractable condition.

Proposition 3 (State-Feedback Stabilization with D.M.C. Relaxation). *The closed-loop system consisted of the system* Π *and* (17) *is stable if there exists* z , $R_1, \ldots, R_N \in \mathbf{S}^n_{++}$, $Q_1, \ldots, Q_N \in \mathbf{R}^{n\times n}$ *such that* (14) *and* (8) *for all* $\theta \in \mathrm{vert}\,\Omega$ *are satisfied, where* $z = (P \in \mathbf{S}^n_{++}, S \in \mathbf{R}^{q\times n})$ *and*

$$\begin{aligned} M_0(z) &= \mathrm{He}\left\{\begin{bmatrix} A_0 & B_0 \end{bmatrix}\begin{bmatrix} P \\ S \end{bmatrix}\right\} \\ M_i(z) &= \mathrm{He}\left\{\begin{bmatrix} A_i & B_i \end{bmatrix}\begin{bmatrix} P \\ S \end{bmatrix}\right\} \quad (1 \le i \le N) \\ M_{ij}(z) &= \mathrm{He}\left\{\begin{bmatrix} A_{ij} & B_{ij} \end{bmatrix}\begin{bmatrix} P \\ S \end{bmatrix}\right\} \quad (1 \le i < j \le N) \\ M_{ii}(z) &= \mathrm{He}\left\{\begin{bmatrix} A_{ii} & B_{ii} \end{bmatrix}\begin{bmatrix} P \\ S \end{bmatrix}\right\} \quad (1 \le i \le N) \end{aligned} \tag{20}$$

Then a state-feedback gain is given by (19).

5 Numerical Examples

We consider stability analysis and state-feedback control design with respect to the system Π. The following results are obtained by LMI Toolbox[6] on a PC with CPU Athlon MP1900+(Dual).

5.1 Stability Analysis

We consider the system Π with the following

$$\begin{aligned} &\left[\, A_0 \mid A_1 \mid A_2 \mid A_{11} \mid A_{22} \mid A_{12} \,\right] \\ = &\left[\begin{array}{cc|cc|cc} -1 & 1 & 1 & 0 & 0 & 0.2 \\ -1 & -5 & 0.1 & -0.2 & 0.1 & 0.3 \end{array}\right. \\ &\qquad \left.\begin{array}{cc|cc|cc} -0.3 & 0 & 0.3 & 0.5 & -2 & 0.4 \\ 0 & 0.2 & -0.3 & -0.1 & -0.2 & -1.2 \end{array}\right] . \end{aligned}$$

The region of the parameter is

$$\theta_1 \in [\ -2.0 \quad -0.5\ ] , \qquad \theta_2 \in [\ 0.5 \quad 0.5 + 9.5\alpha\ ]$$

where α takes $0 \le \alpha \le 1$. For this system, the following four relaxation methods are applied.

- d.c. relaxation (Tuan 1999) [1]
- generalized d.c. relaxation (Proposition 1)
- m.c. relaxation (Gahinet 1996) [2]
- d.m.c. relaxation (Proposition 2)

The maximum α is computed by dichotomy, then the maximum θ_2 is shown by Table 1.

5.2 State-Feedback Control

We consider the system Π with the following

$$\left[\, A_0 \mid A_1 \mid A_2 \mid A_{11} \mid A_{22} \mid A_{12} \,\right]$$

Table 1: Maximum value of θ_2: $\bar\theta_2$

Relaxation method	$\bar\theta_2$
d.c.	0.5557
generalized d.c.	1.0938
m.c.	0.7783
d.m.c.	1.1309

$$\begin{aligned} = &\left[\begin{array}{cc|cc|cc} 1 & 1 & 1 & 0 & 0 & 0.2 \\ -1 & 4 & 0.1 & -0.2 & 0.1 & 0.3 \end{array}\right. \\ &\qquad \left.\begin{array}{cc|cc|cc} -0.3 & 0 & 0.3 & 0.5 & -2 & 0.4 \\ 0 & 0.2 & -0.3 & -0.1 & -0.2 & -1.2 \end{array}\right] , \\ &\left[\, B_0 \mid B_1 \mid B_2 \mid B_{11} \mid B_{22} \mid B_{12} \,\right] \\ = &\left[\begin{array}{c|c|c|c|c|c} 0 & 0 & -0.2 & 0.5 & -0.1 & 1 \\ 1 & 0.1 & 0 & 0.1 & 0.4 & -0.2 \end{array}\right] . \end{aligned}$$

The region of the parameter is given by

$$\theta_1 \in [\ -0.3 \quad -0.1\ ] , \qquad \theta_2 \in [\ 0.1 \quad 0.3\ ] .$$

Then we have a state-feedback gain by Proposition 3.

$$K = [\ -13.0297 \quad -14.7593\ ]$$

6 Conclusion

We proposed a generalized d.c. relaxation and a difference of multiconvex relaxation of PLMI problems. Generalizing d.c. relaxation, we have a less conservative relaxation. Moreover, introducing d.c. structure into multiconvex relaxation, we obtain another less conservative relaxation. These relaxation methods are applied to stability analysis and state-feedback control synthesis of LPV systems. Numerical examples show the effectiveness of our relaxation methods.

References

[1] H.D.Tuan, P.Apkarian. "Relaxations of parameterized LMIs with Control Applications", *Int. J. Robust and Nonlinear Control*, **9**, pp. 59–84, (1999).

[2] P.Gahinet, P.Apkarian, M.Chilali. "Affine Parameter-Dependent Lyapunov Functions and Real Parametric Uncertainty", *IEEE Transactions on Automatic Control*, **41**, No.3, pp. 436–442, (1996).

[3] H.Tuy. "Convex Analysis and Global Optimization", *Kluwer Academic Publishers, Boston, Dordrecht*, (1998).

[4] R.Horst, P.M.Pardalos. "Handbook of Global Optimization", *Kluwer Academic Publishers, Boston, Dordrecht*, (1995).

[5] S.Boyd, L.El Ghaoui, E.Feron, V.Balakrishman. "Linear Matrix Inequalities in Systems and Control Theory", *SIAM Studies in Applied Mathematics, SIAM, Philadelphia*, (1994).

[6] P.Gahinet, A.Nemirovski, A.J.Laub, M.Chilali. "The LMI Control Toolbox", *The MathWorks Inc.*, (1995).