APPROXIMATE ROBUST RECEDING HORIZON CONTROL FOR PIECEWISE LINEAR SYSTEMS VIA ORTHOGONAL PARTITIONING

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Abstract

This paper considers an approximate robust receding horizon control algorithm for a class of hybrid systems by exploiting the equivalence between piecewise linear systems and mixed logical dynamical systems. The control algorithm consists of two control modes which are a state feedback mode and a receding horizon control mode. In the receding horizon control mode, the constrained positively invariant sets are used as the end set constraint and the control law is obtained as an explicit form off-line. To reduce the computations we propose an algorithm that will determine the approximate solution by using orthogonal partitioning.

1 Introduction

In recent years, model predictive control (receding horizon control) has attracted the attention of researchers [1]. Bemporad et al. have proposed the explicit controller for receding horizon control problems to reduce the computational complexity of on-line optimization [2]. This result is a breakthrough in the research of receding horizon control. The generalization of the problem is reported [3, 4] and the sub-optimal problem is developed [5]. Further it is well known that, in the practical applications, the control law is required to guarantee that the closed-loop system fulfills constraints. The robustness is important since when disturbances or model mismatch are present closed-loop performance can be poor with likely violations of the constraints and no convergence can be guaranteed. For the issue the terminal penalty and constraints play important role [6]. In [7] feedback min-max model predictive control for linear time invariant discrete-time systems is proposed and the control algorithm which guarantees a convergence to the invariant set with no constraint violation. On the other hand, hybrid systems arise in a large number of application areas, and are attracting increasing attention. The hybrid system framework allows to model a broad class of systems arising many applications and to address the cooperative control problems and reconfigure problems [8]. It is known that a class of hybrid models can be described by the piecewise linear systems.

We consider robust receding horizon control based on feedback min-max model predictive control [7] for piecewise linear systems as extending the class of system. However since in [7] the system is restricted to linear time invariant discrete-time systems, the control can not deal with hybrid systems directly and a method to construct the end set constraint is not given clearly. Then we proposed the robust receding horizon control for piecewise linear systems using the constraint positively invariant sets [9]. However, its on-line computation is demanding because of integer variables. Further, if we obtain the explicit representation of the solution as piecewise linear function the implementation of the algorithm is still computationally demanding.

In this paper, we propose a min-max approach for calculation of the min-max solution which moves the implementation off-line. Further, an approximate robust receding horizon control algorithm for piecewise linear systems is considered. We construct the algorithm by two control modes: the state feed back mode for keeping the state in a set and the receding horizon control mode for steering the state to the set. In the control algorithm we employ the equivalence [10] of the piecewise linear system form and the mixed logical dynamical (MLD) system form [11] to extend the system form. In receding horizon control mode we construct the end set constraint by using the intersection of the constrained positively invariant sets [12, 13]. This control algorithm guarantees convergence to the union of the constrained positively invariant sets and satisfying the constraints in spite of existence of disturbance. Further we employ the min-max approach for the control algorithm to reduce the on-line computation. We use the algorithm using vertices of the polyhedron for calculating the optimal input sequence and the worst disturbance and present the necessary and sufficient condition for the relation between the min-max solution and vertices of the polyhedron. By the algorithm we can obtain the optimal input sequence as the piecewise affine form with respect to the state. Exploiting approximate algorithm [14] we propose a partitioning algorithm for an approximate solution to reduce the complexity in terms of less regions.

2 Preliminary

2.1 Piecewise Linear Systems with Disturbance

In this paper, piecewise linear systems with disturbance are described by the following equation,

$$x(t+1)=A_ix(t)+B_iu(t)+B_ww(t) \text{ for } x(t)\in\mathcal{X}_i \quad (1)$$

where $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}^m$ is the input, $\mathcal{X}_i$ is the partition of the state set which satisfies $\mathcal{X}_i\cap\mathcal{X}_j=\emptyset$ and $\forall i\neq j$, $\cup_{i=1}^s\mathcal{X}_i=\mathbb{X}$ and we assume that (A_i,B_i) is controllable. The vector $w(t)\in\mathbb{W}\subset\mathbb{R}^l$ is an unknown bounded disturbance and the set $\mathbb{W}$ is convex and contains the origin. In addition the system is subject to constraints on either or both the states and the control inputs i.e. $x(t)\in\mathbb{X}$, $u(t)\in\mathbb{U}$, $\forall t\in\mathbb{N}$. We assume $\mathbb{X}$ and $\mathbb{U}$ are convex polyhedral. Consider the output to be constrained

$$y_c(t)=Cx(t)+Du(t)+D_ww(t). \quad (2)$$

By an appropriate choice of matrix C, D and a set $\mathbb{Y}$, all constraints mentioned can be summarized by

$$y_c(t)\in\mathbb{Y}. \quad (3)$$

Assume that the set $\mathbb{Y}$ is convex and contains the origin.

2.2 The Mixed Logical Dynamical Form of Piecewise Linear Systems

Here the mixed logical dynamical form [11] which is equivalent to piecewise linear systems is introduced. Consider the following general piecewise linear system

$$x(t+1)=A_ix(t)+B_iu(t) \text{ for } x(t)\in\mathcal{X}_i \quad (4)$$
$$y(t)=C_ix(t)+D_iu(t) \quad (5)$$

where $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}^m$ is the input, $\mathcal{X}_i$ is a partition of the state set which satisfies $\mathcal{X}_i\cap\mathcal{X}_j=\emptyset$ and $\forall i\neq j$, $\cup_{i=1}^s\mathcal{X}_i=\mathbb{X}$ and we assume that (A_i,B_i) is controllable.

The piecewise linear system (1) can be transformed mixed logical dynamical system formulation. The mixed logical dynamical system [11] form is

$$x(t+1)=Ax(t)+B_1u(t)+B_2\delta(t)+B_3z(t)+B_pw(t) \quad (6a)$$
$$E_2\delta(t)+E_3z(t)\leq E_1u(t)+E_4x(t)+E_5. \quad (6b)$$

where $x\in\mathbb{R}^n$ is the state of the system $u\in\mathbb{R}^m$ is the command input. $\delta\in\{0,1\}^{r_l}$ and $z\in\mathbb{R}^{r_c}$ represent respectively auxiliary logical and continuous variables. Assume that system (6) is completely well-posed [11], which in words means that for all x, u, w within a bounded set the variables δ, z are uniquely determined.

2.3 Constrained Positively Invariant Set

The constrained positively invariant set [12, 13] is explained in order to use it for an end set constraint of receding horizon control. Consider the control input $u=K_ix$ for the system (1) then the system can be rewritten as

$$x(t+1)=(A_i+B_iK_i)x(t)+B_ww(t) \quad (7)$$
$$y_c(t)=(C+DK_i)x(t)+D_ww(t). \quad (8)$$

For each closed-loop system, we define a state constraint set.

Definition 1 *[12, 13] State constraint set $X((C+DK_i),D_w,\mathbb{Y},\mathbb{W})$ is defined by*

$$X_i=\{x|(C+DK_i)x+D_ww\in\mathbb{Y},\forall w\in\mathbb{W}\}. \quad (9)$$

Remark 1 *Necessary and sufficient condition $y_c(t)\in\mathbb{Y}$ for possible disturbance $w(t)\in\mathbb{W}$ is $x(t)\in X_i$.*

Definition 2 *[12, 13] $\mathcal{O}_i\subset\mathbb{R}^n$ contains origin in its interior. $\mathcal{O}_i$ is a constrained positively invariant set, if it is a positively invariant set and is contained in $X_i((C+DK_i),D_w,\mathbb{Y},\mathbb{W})$.*

If a constrained positively invariant set exists, for any initial state $x(0)\in\mathcal{O}_i$ and $w(t)\in\mathbb{W}$, then $x(t)\in\mathcal{O}_i$ for all $t\in\mathbb{Z}^+$, where $\mathbb{Z}^+$ denotes the set of nonnegative integer.

Definition 3 *Maximal constrained positively invariant set is defined as follows*

$$\mathcal{O}_{\infty i}=\{x(0)|y_c(t|x(0),w)\in\mathbb{Y},\forall t\in\mathbb{Z}^+,\forall w\in\mathbb{W}\}. \quad (10)$$

Maximal constrained positively invariant set $\mathcal{O}_{\infty i}$ can be obtained by recursive process proposed in [12, 13]. Next we define a set which is used for an end set constraint of receding horizon control as

$$\mathcal{P}:=\cap_{i=1}^s\mathcal{O}_{\infty i}. \quad (11)$$

3 Robust Receding Horizon Control Problem

3.1 Robust Receding Horizon Control Law

Since the state can not be steered to the origin due to existing disturbance $w(t)$, the control objective is to drive the state of the system to the set which is constructed by invariant sets. In this paper, we propose 2 modes for the control law.

mode 1: The control law is the form $u=K_ix$.

mode 2: At time t, predictions for possible disturbance are represented by $\{w_{t+k}\}$, and $\{u_{t+k}\}$ denotes the input sequence for the disturbance realization. For the sake of simplicity we define $x_{t+k|t}:=x(t+k,x(t),u_0^{k-1},w_0^{k-1})$ and $\{z_{t+k|t}\}$, $\{\delta_{t+k|t}\}$ are similarly defined respectively.

The prediction for state $x_{t+k|t}$ is defined as follows

$$x_{t+k+1|t} = Ax_{t+k|t}+B_1u_{t+k}+B_2\delta_{t+k|t}+ B_3z_{t+k|t}+B_pw_{t+k} \quad (12a)$$
$$E_2\delta_{t+k|t}+E_3z_{t+k|t} \leq E_1u_{t+k}+E_4x_{t+k|t}+E_5 \quad (12b)$$

At current time t, let $x(t)$ be the current state. Consider the following min-max problem,

$$\min_{U} \max_{W} J(U, W, \Delta, Z, x(t)) \tag{13}$$

$$\text{subject to } \begin{cases} \text{the end set constraint} \\ (3),\ (6) \end{cases} \tag{14}$$

$$\begin{aligned} J(U, W, \Delta, Z, x(t)) := \|Px_{t+N|t}\|_\infty + \\ \sum_{k=0}^{N-1} \{ \|Q_1 x_{t+k|t}\|_\infty + \|Ru_{t+k}\|_\infty + \\ \|Q_2 \delta_{t+k|t}\|_\infty + \|Q_3 z_{t+k|t}\|_\infty \} \end{aligned} \tag{15}$$

where a notation $\|v\|_\infty$ denotes $\|v\| = \max_i |v_i|$ for $v = [v_1, v_2, \cdots, v_l]^T$ and $U := \{u_t, \ldots, u_{t+N-1}\}$, $W := \{w_t, \ldots, w_{t+N-1}\}$, $\Delta := \{\delta_{t|t}, \ldots, \delta_{t+N-1|t}\}$ $Z := \{z_{t|t}, \ldots, z_{t+N-1|t}\}$. N is predictive horizon and $P \in \mathbb{R}^{n\times n}$, $Q_1 \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m_c+m_l \times m_c+m_l}$, $Q_2 \in \mathbb{R}^{r_l\times r_l}$, $Q_3 \in \mathbb{R}^{r_c\times r_c}$ are nonsingular weighting matrices respectively.

The formulation (13), (14) can be written as a mixed integer linear programming by using following approach. First we introduce a vector V

$$\begin{aligned} V \ := \ \{\varepsilon_0^x, \ldots, \varepsilon_N^x, \varepsilon_0^\delta, \ldots, \varepsilon_{N-1}^\delta, \\ \varepsilon_0^z, \ldots, \varepsilon_{N-1}^z, \varepsilon_0^u, \ldots, \varepsilon_{N-1}^u\} \end{aligned} \tag{16}$$

where ε satisfies the following inequalities

$$\begin{cases} -\varepsilon_N^x \mathbf{1}_n \le \pm P x_{t+N|t}, \quad -\varepsilon_k^x \mathbf{1}_n \le \pm Q_1 x_{t+k|t} \\ -\varepsilon_k^u \mathbf{1}_m \le \pm R u_{t+k}, \quad -\varepsilon_k^\delta \mathbf{1}_{r_d} \le \pm Q_2 \delta_{t+k|t} \\ -\varepsilon_k^z \mathbf{1}_{r_c} \le \pm Q_3 z_{t+k|t} \end{cases} \tag{17}$$

$\mathbf{1}_k$ is a column vector of ones of length k, i.e. $\mathbf{1}_k := [1\ 1, \ldots, 1]^T \in \mathbb{R}^k$. Then the vector V represents an upper bound on $J(U, W, \Delta, Z, x(t))$ as

$$\tilde{J}(U, W, \Delta, Z, V, x(t)) := \sum_{i=0}^{N} \varepsilon_i^x + \sum_{i=0}^{N} (\varepsilon_i^u + \varepsilon_i^\delta + \varepsilon_i^z). \tag{18}$$

Concluding, the min-max problem (13), (14) can be denoted as

$$\min_{U} \max_{W} \tilde{J}(U, W, \Delta, Z, V, x(t)) \tag{19}$$

$$\text{s.t.} \begin{cases} x_{t|t} = x(t) \\ x_{t+N|t} \in \mathcal{P} \\ (3),\ (12),\ (17) \end{cases} \tag{20}$$

By plugging (12) into (19) and (20), and by defining the matrices G, S, F respectively the min-max problem (19) and (20) can be rewritten in the more simple form

$$\begin{aligned} &\min_{p_c, p_d} \max_{q} \hat{J}(p_c, p_d, q) \ \text{ s.t. } \ (p_c, p_d, q) \in \mathcal{S} \\ &\hat{J}(p_c, p_d, q) := f_c^T p_c + f_d^T p_d + g^T q \\ &\mathcal{S} := \{(p_c, p_d, q) : F_c p_c + F_d p_d + Gq \le Hx(t) + r\} \end{aligned} \tag{21}$$

where p_c denotes the continuous components of (U, V, Δ, Z) and p_d denotes discrete ones and the vector q represents a component of W.

By relaxing the conditions for the discrete component p_d as $0 \le p_d \le 1$, the min-max problem (21) can be rewritten as

$$\hat{J}^* = \min_{p} \max_{q} \hat{J}(p, q) \ \text{ s.t. } \ (p, q) \in \mathcal{S}' \tag{22}$$

$$\begin{aligned} \hat{J}(p, q) &:= [\, f_c^T \quad f_d^T \,] \begin{bmatrix} p_c \\ p_d \end{bmatrix} + g^T q \\ &= f^T p + g^T q \end{aligned} \tag{23}$$

$$\begin{aligned} \mathcal{S}' &:= \left\{ (p, q) : [\, F_c \quad F_d' \,] \begin{bmatrix} p_c \\ p_d \end{bmatrix} + Gq \le Hx(t) + r' \right\} \\ &= \{(p, q) : Fp + Gq \le Hx(t) + r'\}. \end{aligned} \tag{24}$$

where $f := [\, f_c^T \quad f_d^T \,]$, $p := \left[p_c^T \ p_d^T\right]^T$, $F := [\, F_c \quad F_d' \,]$.

3.2 The control algorithm

A robust receding horizon control algorithm for the piecewise linear systems (1) with disturbance $w(t) \in \mathbb{W}$ is presented as follows. Suppose u_t^* denotes the first element of optimal input sequence for the optimization problem (22).

Algorithm 1:
Data: $x(t)$
Algorithm: IF $x(t) \in \mathcal{P}$
THEN (mode 1) $u(t) = K_i x(t)$.
ELSE (mode 2) $u(t) = u_t^*$.

Theorem 1 *Suppose that* $\varepsilon_k^x = 0\ \forall x(k) \in \cup_{i=1}^{s} \mathcal{O}_{\infty i}$, $u(k) = K_i x(k)$. *Then the control law given by Algorithm 1 satisfies the constraints (3) and drives the state* $x(t)$ *to the union of constrained positively invariant sets* $\cup_{i=1}^{s} \mathcal{O}_{\infty i}$.

Proof: At time t, state $x(t)$, let $V_t^* := \tilde{J}(U_t^*, \Delta_t^*, Z_t^*, V_t^*, W_t^*, x(t))$ denotes the optimal cost which responds to the optimal input sequences U_t^*, Δ_t^*, Z_t^*, ${}_t^*V$ and the disturbance sequence W_t^*. At time t, the first element of the optimal sequence is applied, and disturbance takes a certain value $w(t)$.

At time $t+1$, consider an input sequence $U_{t+1} = \{u_{t+1}, u_{t+2}, \cdots, u_{N-1}, K_i x_{t+N|t}\}$ in which the last element might not be optimal. If the input sequence U_{t+1} is used we obtain the following inequality.

$$V_{t+1}^* \le \tilde{J}(U_{t+1}, \Delta_{t+1}^*, Z_{t+1}^*, V_{t+1}^*, W_{t+1}^*, x(t+1)) \tag{25}$$

The right hand side of inequality (25) leads

$$r.h.s = V_t^* - (\varepsilon_0^x + \varepsilon_0^u + \varepsilon_0^\delta + \varepsilon_0^u) + \varepsilon_{N+1}^x. \tag{26}$$

At time step $N+1$ we will obtain $x \in \mathcal{P}$ because of the terminal constraint $x_{t+N+1} \in \mathcal{P}$. Then the input must be $u(k) = K_i x_{t+N|t+1}$ from the Algorithm 1. We leads $\varepsilon_{N+1}^x = 0$, and

$$V_{t+1}^* \le V_t^* - (\varepsilon_0^x + \varepsilon_0^u + \varepsilon_0^\delta + \varepsilon_0^u). \tag{27}$$

Because $\varepsilon \geq 0$, the cost is monotonically nonincreasing. As it is bounded below by zero, it must consequently converge to a constant value, so that $V_t^* - V_{t+1}^* \to 0$ as $t \to \infty$. Then we have

$$\varepsilon_0^x + \varepsilon_0^u + \varepsilon_0^\delta + \varepsilon_0^u \leq V_t^* - V_{t+1}^*. \tag{28}$$

This leads $\varepsilon_0^x + \varepsilon_0^u + \varepsilon_0^\delta + \varepsilon_0^u \to 0$, $t \to \infty$. Hence by the definitions of ε the state converges to $\mathcal{P}$ which includes the origin. Further when the state in the set $\mathcal{P}$, the control law changes to $u = K_i x$. Consequently the control algorithm satisfies the constraints and drives the state in $\cup_{i=1}^s \mathcal{O}_i$ and in the control algorithmt the constraints is satisfied since the two control modes guarantee the constraints satisfaction. □

Theorem 1 guarantees that the state of the system can be steered to the set $\cup_{i=1}^s \mathcal{O}_{\infty i}$ with no constraint violation in spite of existence of disturbance. The set $\mathcal{O}_{\infty i}$ is depend on the design of feedback gain K_i, $\mathbb{X}$ and $\mathbb{U}$ and we can design the gain K_i. The control mode 1 is mode for keeping the state in the set $\mathcal{P}$ and mode 2 is mode for steering the state to the set $\mathcal{P}$. However, the computation of the algorithm is demanding since mode 2 solves the min-max optimization problem each time steps.

3.3 Piecewise Affine Controller

Here we consider the off-line computation of (21) to reduce the on line computation. We can obtain the min-max solution by calculating the vertices because the min-max problem (21) is linear [16]. The sequence is summarized algorithm 2.

Algorithm 2:

1) Obtain vertices of the polyhedron $\mathcal{S}'$ [17].

2) By the vertices obtained 1), define the vertices (p_i, q_i), $i = 1, 2, \ldots, l$ which satisfy

$$p_i = \arg\min_p \hat{J}(p, q), \text{ s.t. } (p, q_i) \in \mathcal{S}'. \tag{29}$$

3) The vertex which maximize $\hat{J}$ is a min-max solution.

Theorem 2 *Let the vector (p, q) be one of a vertex of the polyhedron $\mathcal{S}'$ for $x(t)$. For the vertex (p, q) let F_A, G_A, H_A, r_A represent the matrices corresponding to active constraints($_N$ denotes inactive constraints)*

$$[\, F_A \quad G_A \,] \begin{bmatrix} p \\ q \end{bmatrix} = H_A x(t) + r_A \tag{30}$$

$$[\, F_N \quad G_N \,] \begin{bmatrix} p \\ q \end{bmatrix} \leq H_N x(t) + r_N. \tag{31}$$

And matrices V_A and V_N are defined as $V_A := [\, F_A \quad G_A \,]$, $V_N := [\, F_N \quad G_N \,]$.

Then in the region $D(x)$

$$D(x) = \left\{ x(t) : \begin{array}{r} \{V_N (V_A^T V_A)^{-1} V_A^T H_A - H_N\} x(t) \\ < r_N - V_N (V_A^T V_A)^{-1} V_A^T r_A \end{array} \right\} \tag{32}$$

the state $x(t)$ and the vertex (p, q) are satisfy

$$\begin{bmatrix} p \\ q \end{bmatrix} = \psi(x(t))$$
$$\psi(x(t)) := (V_A^T V_A)^{-1} V_A^T (H_A x(t) + r_A). \tag{33}$$

Proof: When (p, q) is a vertex of the polyhedron $\mathcal{P}$, the matrix V_A is full rank. Hence $V_A^T V_A$ is nonsingular and we obtain the equation (33). By substituting the equation (33) into the equation (31) the region (32) is obtained. □

By theorem 2 the vertices of $\mathcal{S}'$ are piecewise affine with respect to the state $x(t)$ and the objective function $\hat{J}$ for the vertex (p, q) can be denoted as

$$\hat{J}^*(x(t)) := \left[f^T \; g^T \right] \psi(x(t)). \tag{34}$$

$\hat{J}^*$ is also piecewise affine with respect to the state $x(t)$. From these properties the following theorem can be obtained.

Theorem 3 *Let (p_1, q_1) be the min-max solution for the state $x(t)$ in the set of vertices of the polyhedron $\mathcal{S}'$ and $(p_i, q_i), i = 2, 3, \ldots, s$ denote other vertices. And let $(p_i, q_i), i = 2, 3, \ldots, k$ be the vertices which have intersecting points q_i for the set $\Pi_q(\mathcal{S}')$ and $D(x(t))$, $\psi(x(t))$ and $\hat{J}$ obtained by theorem 2 are denoted with appropriate subscripts.*

Then the conditions for the vector

$$\begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \psi_1(x(t)) \tag{35}$$

being the min-max solution for problem (21) are

(necessary condition) $x(t) \in D_1(x(t))$

(sufficient condition)

$$x(t) \in \left(\bigcap_{i=1}^k D_i(x(t)) \right) \cap \left\{ \begin{array}{r} x(t) : \hat{J}_1(x(t)) > \hat{J}_i(x(t)), \\ i = 2, 3, \ldots, k \end{array} \right\}$$

Proof: Necessary condition: Active constraints must be invariant when the min-max solution (p_1, q_1) is given by the equation (35). Hence by the theorem 2 the necessary condition is obtained.

Sufficient condition: If the vertex (p_1, q_1) is the min-max solution, then the object $\hat{J}_1$ must be larger than other objects for other vertices (p_i, q_i), $i = 2, 3, \ldots, k$. Hence the vertices (p_i, q_i) is represented by equation (33), and $\hat{J}_1 > \hat{J}_i$. □

By theorem 3 the linear min-max solution (p, q) is piecewise affine with respect to the state $x(t)$. Once we obtain the min-max solution for the state $x(t)$ the min-max solution for other state can be obtained by using the equation (35). Hence the min-max problem can be solved for each state by using theorem 2. The min-max model predictive control law $u(t)$ is the component of $\psi(x(t))$ which corresponds to u_t

$$u(t) = [0 \;\; \cdots \;\; 0 \;\; I \;\; 0 \;\; \cdots \;\; 0]\, \psi(x(t)). \tag{36}$$

Hence when we implement the control mode 2, only a piecewise affine function needs to be evaluated at each time step. Hence algorithm 1 can be modified as follows.

Algorithm 1':
Data: $x(t)$
Algorithm: IF $x(t) \in \mathcal{P}$
THEN (mode 1) $u(t) = K_i x(t)$.
ELSE (mode 2) $u(t) = [0 \ \cdots \ 0 \ I \ 0 \ \cdots \ 0]\,\psi(x(t))$.

The Algorithm 1' can be implemented using the pre-computed off-line explicit solution $\psi(x(t))$ and real time optimization can be avoided. However implementation of the Algorithm 1' may still require a significant amount of computations.

3.4 Approximate Algorithm

In this section, we consider an approximate algorithm for mode 2 control law by exploiting the approximate algorithm [14].

Error bounds: When constructing approximate min-max solutions it is necessary to compute the approximation error. The approximate solution $\hat{z}_0(x) = [p_0^T \ q_0^T]^T$ is defined on an arbitrary polyhedron $X_0 \subset \mathbb{R}^n$ and the corresponding cost is given by

$$\hat{J}_0 = f^T p_0 + g^T q_0. \tag{37}$$

We will compute bounds on the error $\hat{J}^* - \hat{J}_0$.

Let the polyhedron X_0 be represented by its vertices $\mathcal{V} = \{v_1, v_2, \ldots, v_M\}$. Define the linear function $\bar{J}(x) = \bar{L}_0 x + \bar{l}_0$ as the solution to the following LP:

$$\begin{aligned} &\min_{\bar{L}_0, \bar{l}_0} \left(\bar{L}_0 v + \bar{l}_0\right) \\ \text{s.t.} \quad &\bar{L}_0 v_i + \bar{l}_0 \ge \hat{J}(v_i), \ \ \forall i \in \{1, 2, \ldots, M\}. \end{aligned} \tag{38}$$

Similarly, define the linear function

$$\underline{J}(x) = \hat{J}^*(v) + \nabla^T \hat{J}^*(v)(x - v) = \underline{L}_0 x + \underline{l}_0 \tag{39}$$

where $v \in X_0$ is arbitrary. For the linear functions $\underline{J}$, $\bar{J}$ we obtain the following theorem.

Theorem 4 *For all $x \in X_0$ the following inequalities hold*

$$\underline{J}(x) \le J^*(x) \le \bar{J}(x). \tag{40}$$

Proof: The upper bound is obtained from the constaint in LP (38). The lower bound is derived as follows

$$\hat{J}^*(x) \ge \hat{J}^*(v) + \nabla^T \hat{J}^*(v)(x - v) \tag{41}$$

since $\hat{J}^*$ is convex. □

It follows that $\varepsilon_1 \le \hat{J}^* - \hat{J}_0 \le \varepsilon_2$ where

$$\varepsilon_2 = \max_{x \in X_0} (\bar{J} - \hat{J}_0), \ \ \varepsilon_1 = \max_{x \in X_0} (\hat{J}_0 - \underline{J}). \tag{42}$$

Then we introduce an value ε as the error bound.

$$\varepsilon = \max(-\varepsilon_1, \varepsilon_2) \tag{43}$$

Constraint Violation: Here we consider violation of the constraints by using the approximate solution. In the min-max problem the constraints is represented as

$$[F \quad G] \begin{bmatrix} p \\ q \end{bmatrix} \le Hx + r. \tag{44}$$

The vector $\tilde{\delta}(x)$ is defined by

$$\tilde{\delta}(x) = D([F\ G]\hat{z}_0(x) - Hx - r) \tag{45}$$

where D is a diagonal scaling matrix with positive elements. The maximum violation of each constraint within a polyhedron X_0 is computed by solving the following LP:

$$\delta_j = \max_{x \in X_0} \tilde{\delta}_j(x), \quad j = 1, 2, \ldots, q. \tag{46}$$

In order to reduce the computation we consider orthogonal partitioning (Figure 1). This partitioning method is porposed in [14]. We construct approximate solution $z_0(x)$ in polyhedron $X \subset \mathbb{R}^n$. First the algorithm will consider the whole region

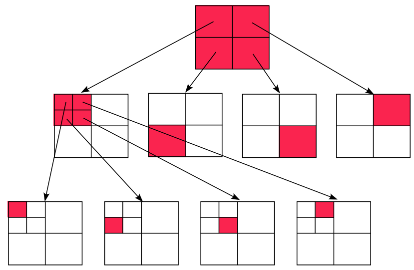

Figure 1: Partition of a rectangular region in a 2-dimensional sate space.

$X_0 = X$. The algorithm computes min-max solutions of the problrem (22) at the 2^n verteces of the hypercube X_0.

$$z_i^0(x) = K_i^0 x + g_i^0 \tag{47}$$

Then construct the approximate solution as

$$\hat{z}_0(x) = \frac{1}{2^n} \sum_{i=1}^{2^n} K_i^0 x + \frac{1}{2^n} \sum_{i=1}^{2^n} g_i^0. \tag{48}$$

Using the mesurements ε and δ_j the partitioning algorithm is obtained.

Algorithm 3(Partitioning Algorithm):

1) Initialize the partition to the whole hypercube, i.e. $\mathcal{X} = \{X\}$. Mark the hypercube X as unexplored.

2) Select any unexplored hypercube $X_0 \in \mathcal{X}$. If no such hypercube exists, terminate with the partition $\mathcal{X}$.

3) Compute the solution to the problem (21) at the 2^n vertices of the hypercube X_0.

4) From the min-max solutions at the vertices of the hypercube, compute a state feedback as an approximate solution for the region X_0.

5) Determine if the hypercube needs to be split in order to reduce the approximate error bound ε or the constraint violations bound δ. If $\varepsilon \leq \bar{\varepsilon}$, $\delta \leq \bar{\delta}$ holds then go to step 6. Otherwise, mark X_0 explored and to step 2.

6) Split the hypercube X_0 2^n cubes $X_1, X_2, \ldots, X_{2^n}$. Mark them all unexplored, remove X_0 form $\mathcal{X}$, add $X_1, X_2, \ldots, X_{2^n}$ to $\mathcal{X}$ and go to step 2.

The Algorithm 3 will not terminate before the cost and constraint errors respect their bounds in all hypercubes of the partition. In the Algorithm 3 $D > 0$, $\bar{\varepsilon}$ and $\bar{\delta} > 0$ can be considered as design parameters. For the min-max problem (22) the partitioning algorithm terminates after a finite number of steps with an approximate solution $\hat{z}_0$ and associate cost $\hat{J}_0$ that satisfies

$$\sup_{x \in X} \left| \hat{J}_0 - \hat{J}^* \right| \leq \bar{\varepsilon} \tag{49}$$

and constraints satisfying

$$\sup_{x \in X} D([F\ G]\hat{z}_0(x) - Hx - r) \leq \bar{\delta}. \tag{50}$$

4 Conclusion

In this paper, we propose an approximate robust receding horizon control algorithm for piecewise linear systems affected by additive bounded disturbances. It has two control modes based on feedback min-max model predictive control, in the receding horizon control mode we employ the equivalence of piecewise linear systems and MLD systems and propose the end set constraint which consists of constrained positively invariant sets. To reduce computations we propose an algorithm that will determine the approximate solution by using orthogonal partitioning.

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