REFINED QUALITATIVE ANALYSIS FOR A CLASS OF NEURAL NETWORKS

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Keywords: Neural networks, nonlinear systems, sector nonlinearities, stability analysis, invariant sets.

Abstract

New results of qualitative analysis are presented for a class of neural networks (Hopfield-type), representing a refinement in the interpretation of their behaviour. The main instrument of this analysis consists in the individual monitoring of the state-trajectories by considering time-dependent rectangular sets that are forward invariant with respect to the dynamics of the investigated systems. Particular requirements for the rectangular sets approaching the equilibrium point allow a componentwise exploration of the stability properties, offering additional information with respect to the traditional framework (that expresses a global knowledge, built in terms of norms).

1. Introduction

The theory of *flow-invariant* (FI) sets [6] provides an extremely efficient instrument to refine the qualitative analysis of linear and nonlinear dynamical systems. By using FI rectangular sets with arbitrary time-dependence, this refinement allows a componentwise investigation of the system trajectories and, consequently, highlights two special types of asymptotic stability, called by Voicu, in his pioneering work on linear systems [10], *componentwise asymptotic stability* (CWAS) and *componentwise exponential asymptotic stability* (CWEAS). Unlike the classical concept of *asymptotic stability*, which gives collective characterizations of the state-trajectories expressed in terms of norms, CWAS and CWEAS permit an individual monitoring of each state variable approaching the equilibrium point. Later works [11], [9] broadened this research direction towards other types of dynamical systems. Recently, these two concepts were applied for the qualitative analysis of neural networks with delay [2], but their usage was not founded on a proper flow-invariance analysis and, therefore, the results delivered in [2] can be considerably improved.

The current paper applies the flow-invariance theory to study a class of dynamical neural networks described by the following differential equations:

$$\dot{x}_i(t) = -b_i x_i(t) + \sum_{j=1}^{n} w_{ij} f_j\left(x_j(t)\right) + u_i, \quad t \ge 0, \quad i = \overline{1,n}\,, \quad (1)$$

and provides a deeper insight into the asymptotic stability properties from the CWAS / CWEAS point of view. Dynamical systems belonging to class (1) are also known as *Hopfield-type networks* [9], with the following concrete meaning for the notations: x_i represents the neural voltage of the *i*-th neuron, $b_i > 0$ is a constant governing the changing rate of the *i*-th neuron, w_{ij} stands for the synaptic connection weight of the *j*-th neuron to the *i*-th neuron, f_i is the activation function of the *i*-th neuron and u_i is the constant external input.

For the activation functions in the model under study the following hypotheses were taken into consideration. For each $i = 1, 2, \ldots, n$,

(H1) f_i is continuous and

(H2) there exist an $\lambda_i > 0$ such that f_i satisfies the *slope condition* (that also guarantees f_i nondecreasing):

$$0 \le \frac{f_i(r) - f_i(s)}{r - s} \le \lambda_i, \quad \forall r, s \in \mathbb{R},\ r \ne s\,. \quad (2)$$

In our hypotheses, the activation functions are assumed to be neither differentiable nor bounded. Therefore, besides the commonly used bipolar sigmoid function, $f(s) = \left[1 - \exp(-\lambda s)\right] / \left[1 + \exp(-\lambda s)\right]$, $\lambda > 0$, there may also be utilized: the piecewise saturation function (characteristic for Cellular Neural Networks), $f(s) = \left[|\,\lambda s + 1\,| - |\,\lambda s - 1\,|\right]/2$, $\lambda > 0$, the piecewise linear one, $f(s) = \max\left(0, \lambda s\right)$, $\lambda > 0$, (encountered in optimization problems) and even the linear function $f(s) = \lambda s,\ \lambda > 0$. Moreover, hypothesis (H2) implies that if function f_i is continuously differentiable on $\mathbb{R}$, then its derivative satisfies $0 \le f_i'(s) \le \lambda_i$ for all values of s.

The dynamics of (1) has been extensively studied under various assumptions. The asymptotic stability of symmetrically connected networks ($w_{ij} = w_{ji},\ i, j = \overline{1,n}$) is now well ascertained (see for instance [5]). There is also a great number of papers which establish the (exponential) asymptotic stability for asymmetrically connected networks under various hypotheses on the activation functions: continuously differentiable and monotonically increasing [3], globally Lipschitz continuous [8], monotonically nondecreasing and partially Lipschitz continuous [4].

The case of Hopfield-type neural networks with delay has also been largely investigated. From our point of view, a special attention was given to [2] where some sufficient conditions were presented for CWAS (CWEAS) - therein named *guaranteed componentwise* (*exponential*) *convergence* - of the continuous-time model with delay

$$\dot{x}_i(t) = -b_i x_i(t) + \sum_{j=1}^{n} w_{ij} f_j\left(x_j(t)\right) + \sum_{j=1}^{n} d_{ij} f_j\left(x_j(t-\tau)\right) + u_i, \ t \ge 0, \ \tau > 0, \ i = \overline{1,n}, \quad (3)$$

with sigmoid-type (i.e. continuous, monotonically nondecreasing and bounded) activation functions that satisfy the slope condition (H2). System (3) is written with our own notations, which replace the original ones used in paper [2]. All over this text, our references and comparisons to the results in [2] take into discussion only the particular case of neural networks without time-delay resulting from model (3) for $d_{ij} = 0$, $i, j = \overline{1,n}$, i.e. omitting the third term and leading to model (1). A simple comparison of our (H1)-(H2) with the hypotheses in [2] shows that we have considered a broader class of activation functions than in [2] (implicitly requiring boundedness, unlike (H1)-(H2)).

In the first part of our paper, we assume that system (1) has a finite number of equilibrium points [5] and we study the behavior of the state-trajectories of (1) with respect to an equilibrium point, taking into account only hypotheses (H1) and (H2). In the second part, we introduce a supplementary hypothesis (H3) that ensures global CWEAS of system (1), and, subsequently, the uniqueness of the equilibrium point. This systematic analysis of CWAS and CWEAS properties, based on flow-invariance theory, leads us to more refined results than the ones presented in [2] and [3].

As special types of stability, CWAS and CWEAS are properties which characterize the state-space trajectories in the vicinity of a given equilibrium point. However, to simplify the formulation of our results, in the current paper, these two properties are directly associated with the system, since the overall context cannot create confusions.

For the sake of conciseness, the following notations are used all through this paper: $\boldsymbol{x} = [x_1, \dots, x_n]^T$, $\boldsymbol{u} = [u_1, \dots, u_n]^T$, $\boldsymbol{f}(\boldsymbol{x}) = [f_1(x_1), \dots, f_n(x_n)]^T$ (where T stands for the matrix transposition operator), $\boldsymbol{B} = \operatorname{diag}[b_1, \dots, b_n]$, $\boldsymbol{W} = [w_{ij}]$. With these notations, model (1) may be written in matrix form as:

$$\dot{\boldsymbol{x}}(t) = -\boldsymbol{B}\boldsymbol{x}(t) + \boldsymbol{W}\boldsymbol{f}\left(\boldsymbol{x}(t)\right) + \boldsymbol{u}, \quad t \ge 0. \quad (4)$$

Premises (H1) and (H2) ensure that for each initial condition

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0, \ t_0 \ge 0, \ \boldsymbol{x}_0 \in \mathbb{R}^n, \quad (5)$$

the Cauchy problem (4)-(5) has a unique solution $\boldsymbol{x}(t) = \boldsymbol{x}(t; t_0, \boldsymbol{x}_0)$ defined for all $t \ge t_0$.

Also, for two matrices having the same dimensions, $\boldsymbol{M} = [m_{ij}]$, $\boldsymbol{N} = [n_{ij}] \in \mathbb{R}^{m \times n}$, the matrix inequality $\boldsymbol{M} \ge \boldsymbol{N}$ ($\boldsymbol{M} > \boldsymbol{N}$), equivalent to $\boldsymbol{N} \le \boldsymbol{M}$ ($\boldsymbol{N} < \boldsymbol{M}$), is understood componentwise, i.e. $m_{ij} \ge n_{ij}$ ($m_{ij} > n_{ij}$) for all $i = \overline{1,m}$ and $j = \overline{1,n}$. This convention applies in the case of vectors or vector functions too. For a given matrix $\boldsymbol{M} = [m_{ij}]$, the matrix that has $|m_{ij}|$ as elements is denoted by $|\boldsymbol{M}|$. Furthermore, $\boldsymbol{O}$ and $\boldsymbol{0}$ stand for the null square matrix and the null vector, respectively, each of appropriate dimensions.

2. Flow-invariance of time-dependent rectangular sets

Let us assume that system (4) has a finite number of equilibrium points and let $\boldsymbol{x}_e$ be one of these points, i.e. $\boldsymbol{x}_e$ satisfies the equation $-\boldsymbol{B}\boldsymbol{x}_e + \boldsymbol{W}\boldsymbol{f}(\boldsymbol{x}_e) + \boldsymbol{u} = \boldsymbol{0}$. The dynamical behavior of the state-space trajectories of (4) may be analyzed by means of the deviations from the equilibrium point $\boldsymbol{x}_e$, denoted by $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{x}_e$, that satisfy

$$\dot{\boldsymbol{y}}(t) = -\boldsymbol{B}\boldsymbol{y}(t) + \boldsymbol{W}\boldsymbol{g}\left(\boldsymbol{y}(t)\right), \quad t \ge 0, \quad (6)$$

where $\boldsymbol{g}(\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{y} + \boldsymbol{x}_e) - \boldsymbol{f}(\boldsymbol{x}_e)$. Each component of $\boldsymbol{g}$ fulfills (H1)-(H2). Since $\boldsymbol{g}(\boldsymbol{0}) = \boldsymbol{0}$, relation (2) leads to

$$0 \le \frac{g_i(s)}{s} \le \lambda_i, \quad \forall s \in \mathbb{R}, \quad s \ne 0, \quad i = \overline{1,n}. \quad (7)$$

Obviously, $\boldsymbol{y}_e = \boldsymbol{0}$ is an equilibrium point of system (6).

The first problem addressed in this paper regards the study of flow-invariant sets *with respect to* (w.r.t.) system (6). Consider $\boldsymbol{p}, \boldsymbol{q} : \mathbb{R}_+ \to \mathbb{R}^n$ continuously differentiable vector functions, with positive components $p_i(t) > 0, q_i(t) > 0$, $\forall t \ge 0$, $i = \overline{1,n}$, and let us denote by

$$I_{[-\boldsymbol{p},\boldsymbol{q}]}(t) = [-p_1(t), q_1(t)] \times \dots \times [-p_n(t), q_n(t)], \quad (8)$$

the *time-dependent rectangular set* (TDRS) defined by $-\boldsymbol{p}$ and $\boldsymbol{q}$. In (8), $[\bullet]\times[\bullet]$ stands for the Cartesian product.

Definition 1. *TDRS* $I_{[-\boldsymbol{p},\boldsymbol{q}]}(t)$ *is flow-invariant* (*FI*) *w.r.t system* (6) *if for any* $t_0 \ge 0$ *and any initial condition* $\boldsymbol{y}(t_0) = \boldsymbol{y}_0 \in I_{[-\boldsymbol{p},\boldsymbol{q}]}(t_0)$, *the corresponding solution to* (6), $\boldsymbol{y}(t) = \boldsymbol{y}(t; t_0, \boldsymbol{y}_0)$, *remains inside* $I_{[-\boldsymbol{p},\boldsymbol{q}]}(t)$ *for all* $t \ge t_0$, *i.e.*

$$-\boldsymbol{p}(t_0) \le \boldsymbol{y}_0 \le \boldsymbol{q}(t_0) \ \Rightarrow \ -\boldsymbol{p}(t) \le \boldsymbol{y}(t) \le \boldsymbol{q}(t), \ \forall t \ge t_0. \blacksquare \quad (9)$$

In order to characterize the TDRSs that are flow invariant w.r.t. system (6), let us first introduce the following matrices

$$\begin{aligned}
&\boldsymbol{W}^+ = [w_{ij}^+] \in \mathbb{R}^{n \times n}, \quad \text{where} \quad w_{ij}^+ = \max\{w_{ij}, 0\}, \quad i, j = \overline{1,n}, \\
&\boldsymbol{W}^- = [w_{ij}^-] \in \mathbb{R}^{n \times n}, \quad \text{where} \quad w_{ij}^- = \max\{-w_{ij}, 0\}, \quad i, j = \overline{1,n}, \\
&\boldsymbol{W}^d = [w_{ij}^d] \in \mathbb{R}^{n \times n}, \quad \text{where} \quad w_{ij}^d = \begin{cases} 0, & \text{if } i \ne j, \\ w_{ii}, & \text{if } i = j, \end{cases} \\
&\widetilde{\boldsymbol{W}}^+ = [\tilde{w}_{ij}^+] \in \mathbb{R}^{n \times n}, \quad \text{where} \quad \tilde{w}_{ij}^+ = \begin{cases} w_{ij}^+, & \text{if } i \ne j, \\ 0, & \text{if } i = j, \end{cases} \\
&\widetilde{\boldsymbol{W}}^- = [\tilde{w}_{ij}^-] \in \mathbb{R}^{n \times n}, \quad \text{where} \quad \tilde{w}_{ij}^- = \begin{cases} w_{ij}^-, & \text{if } i \ne j, \\ 0, & \text{if } i = j, \end{cases}
\end{aligned} \quad (10)$$

defined by means of the elements of weight matrix $\boldsymbol{W}$, that satisfies the identities: $\boldsymbol{W} = \boldsymbol{W}^d + \widetilde{\boldsymbol{W}}^+ - \widetilde{\boldsymbol{W}}^- = \boldsymbol{W}^+ - \boldsymbol{W}^-$ and $|\boldsymbol{W}| = |\boldsymbol{W}^d| + \widetilde{\boldsymbol{W}}^+ + \widetilde{\boldsymbol{W}}^- = \boldsymbol{W}^+ + \boldsymbol{W}^-$.

Theorem 1. *The TDRS* $I_{[-\boldsymbol{p},\boldsymbol{q}]}(t)$ *defined by* (8) *is flow-invariant w.r.t. system* (6)*-if and only if*

$$\begin{bmatrix}\dot{\boldsymbol{p}}(t)\\ \dot{\boldsymbol{q}}(t)\end{bmatrix} \geq -\begin{bmatrix}\boldsymbol{B} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{B}\end{bmatrix}\begin{bmatrix}\boldsymbol{p}(t)\\ \boldsymbol{q}(t)\end{bmatrix} + \\ +\begin{bmatrix}\boldsymbol{W}^d+\widetilde{\boldsymbol{W}}^+ & \widetilde{\boldsymbol{W}}^-\\ \widetilde{\boldsymbol{W}}^- & \boldsymbol{W}^d+\widetilde{\boldsymbol{W}}^+\end{bmatrix}\begin{bmatrix}-\boldsymbol{g}(-\boldsymbol{p}(t))\\ \boldsymbol{g}(\boldsymbol{q}(t))\end{bmatrix},\ \forall t \geq 0. \quad (11)$$

Proof. By applying Lemma 4.2 from [6], $I_{[-\boldsymbol{p},\boldsymbol{q}]}(t)$ is FI w.r.t. system (6) if and only if

$$\begin{aligned} -\dot{p}_i(t) &\leq b_i p_i(t) + w_{ii} g_i\left(-p_i(t)\right) + \sum_{\substack{j=1\\ j\neq i}}^{n} w_{ij} g_j(y_j),\\ \dot{q}_i(t) &\geq -b_i q_i(t) + w_{ii} g_i\left(q_i(t)\right) + \sum_{\substack{j=1\\ j\neq i}}^{n} w_{ij} g_j(y_j), \end{aligned} \quad (12)$$

for all $t \geq 0$, $i=\overline{1,n}$, and all $y_j \in \left[-p_j(t), q_j(t)\right]$, $j=\overline{1,n}$. For an arbitrary $t\geq 0$ each g_j is continuous and nondecreasing on the compact interval $\left[-p_j(t), q_j(t)\right]$. The following inequalities are fulfilled

$$\begin{aligned} w_{ij}^{+} g_j\left(-p_j(t)\right) - w_{ij}^{-} g_j\left(q_j(t)\right) \leq w_{ij} g_j(y_j) \leq \\ \leq w_{ij}^{+} g_j\left(q_j(t)\right) - w_{ij}^{-} g_j\left(-p_j(t)\right) \end{aligned} \quad (13)$$

when $y_j \in \left[-p_j(t), q_j(t)\right]$, $j \neq i$, the lower and the upper bounds in (13) being reachable, so that (12) is equivalent to

$$\begin{aligned} -\dot{p}_i(t) &\leq b_i p_i(t) + \sum_{j=1}^{n} w_{ij}^{d} g_j\left(-p_j(t)\right) + \\ &+ \sum_{j=1}^{n} \tilde{w}_{ij}^{+} g_j\left(-p_j(t)\right) - \sum_{j=1}^{n} \tilde{w}_{ij}^{-} g_j\left(q_j(t)\right),\\ \dot{q}_i(t) &\geq -b_i q_i(t) - \sum_{j=1}^{n} \tilde{w}_{ij}^{-} g_j\left(-p_j(t)\right) + \\ &+ \sum_{j=1}^{n} w_{ij}^{d} g_j\left(q_j(t)\right) + \sum_{j=1}^{n} \tilde{w}_{ij}^{+} g_j\left(q_j(t)\right), \end{aligned} \quad (14)$$

for $\forall t \geq 0$, $i=\overline{1,n}$. Evidently, relation (11) represents the matrix form of (14), which completes the proof. ■

Corollary 1. *Let* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *be two positive vectors from* $\mathbb{R}^n$. *The constant rectangular set (CRS) defined by* $I_{[-\boldsymbol{\alpha},\boldsymbol{\beta}]} = \left\{\boldsymbol{y}\in\mathbb{R}^n, -\boldsymbol{\alpha} \leq \boldsymbol{y} \leq \boldsymbol{\beta}\right\}$ *is flow invariant w.r.t.* (6) *if and only if the following algebraic inequality is fulfilled:*

$$-\begin{bmatrix}\boldsymbol{B} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{B}\end{bmatrix}\begin{bmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{bmatrix} + \begin{bmatrix}\boldsymbol{W}^d+\widetilde{\boldsymbol{W}}^+ & \widetilde{\boldsymbol{W}}^-\\ \widetilde{\boldsymbol{W}}^- & \boldsymbol{W}^d+\widetilde{\boldsymbol{W}}^+\end{bmatrix}\begin{bmatrix}-\boldsymbol{g}(-\boldsymbol{\alpha})\\ \boldsymbol{g}(\boldsymbol{\beta})\end{bmatrix} \leq \boldsymbol{0}.\ ■ \quad (15)$$

Remark 1. Reference [2, Remark 3] gives the following sufficient condition for a TDRS $I_{[-\boldsymbol{p},\boldsymbol{q}]}(t)$ to be flow-invariant w.r.t. (6) (property called *guaranteed trapping region* in [2]):

$$\begin{bmatrix}\dot{\boldsymbol{p}}(t)\\ \dot{\boldsymbol{q}}(t)\end{bmatrix} \geq -\begin{bmatrix}\boldsymbol{B} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{B}\end{bmatrix}\begin{bmatrix}\boldsymbol{p}(t)\\ \boldsymbol{q}(t)\end{bmatrix} + \begin{bmatrix}\boldsymbol{W}^+ & \boldsymbol{W}^-\\ \boldsymbol{W}^- & \boldsymbol{W}^+\end{bmatrix}\begin{bmatrix}-\boldsymbol{g}(-\boldsymbol{p}(t))\\ \boldsymbol{g}(\boldsymbol{q}(t))\end{bmatrix}, \quad (16)$$

for all $t \geq 0$. Obviously, the right hand side of (16) is greater than the one in (11), and this results in missing the necessity within the framework of [2]. In other words, there exist TDRSs $I_{[-\boldsymbol{p},\boldsymbol{q}]}(t)$ which are flow-invariant w.r.t. system (6) but the vector function $[\boldsymbol{p}^T(t), \boldsymbol{q}^T(t)]^T$ does not meet condition (16).■

Since it is rather difficult to exploit the system of nonlinear differential inequalities (11) in the practical study of a given model, a straightforward sufficient condition for a TDRS $I_{[-\boldsymbol{p},\boldsymbol{q}]}(t)$ defined by (8) to be FI w.r.t. (6) is presented below.

Theorem 2. *Let* $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *be two continuously differentiable vector functions,* $\boldsymbol{p},\boldsymbol{q}:\mathbb{R}_+ \to \mathbb{R}^n$, *with positive components. If the vector function* $\boldsymbol{\eta}:\mathbb{R}_+ \to \mathbb{R}^{2n}$, $\boldsymbol{\eta}(t) = [\boldsymbol{p}^T(t)\ \boldsymbol{q}^T(t)]^T$ *satisfies the linear differential inequality*

$$\dot{\boldsymbol{\eta}}(t) \geq \boldsymbol{\Pi}\boldsymbol{\eta}(t),\quad \forall t \geq 0, \quad (17)$$

where

$$\boldsymbol{\Pi} = -\begin{bmatrix}\boldsymbol{B} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{B}\end{bmatrix} + \begin{bmatrix}\boldsymbol{W}^+ & \widetilde{\boldsymbol{W}}^-\\ \widetilde{\boldsymbol{W}}^- & \boldsymbol{W}^+\end{bmatrix}\begin{bmatrix}\boldsymbol{\Lambda} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{\Lambda}\end{bmatrix}, \boldsymbol{\Lambda} = \mathrm{diag}\left[\lambda_1,\ldots,\lambda_n\right], \quad (18)$$

then the TDRS $I_{[-\boldsymbol{p},\boldsymbol{q}]}(t)$ (8) *is FI w.r.t. system* (6).

Proof. Because all the components of vector functions $\boldsymbol{p}$ and $\boldsymbol{q}$ are positive, condition (7) leads to

$$\begin{bmatrix}\boldsymbol{\Lambda} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{\Lambda}\end{bmatrix}\begin{bmatrix}\boldsymbol{p}(t)\\ \boldsymbol{q}(t)\end{bmatrix} \geq \begin{bmatrix}-\boldsymbol{g}(-\boldsymbol{p}(t))\\ \boldsymbol{g}(\boldsymbol{q}(t))\end{bmatrix},\quad \forall t \geq 0. \quad (19)$$

Since $w_{ii} g_i(s) \leq w_{ii}^{+}\lambda_i s$ and $w_{ii}[-g_i(-s)] \leq w_{ii}^{+}\lambda_i s$ for $s \geq 0$, $i=\overline{1,n}$, in addition to matrices $\widetilde{\boldsymbol{W}}^+$ and $\widetilde{\boldsymbol{W}}^-$ have nonnegative elements, relation (19) implies

$$\boldsymbol{\Pi}\begin{bmatrix}\boldsymbol{p}(t)\\ \boldsymbol{q}(t)\end{bmatrix} \geq -\begin{bmatrix}\boldsymbol{B} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{B}\end{bmatrix}\begin{bmatrix}\boldsymbol{p}(t)\\ \boldsymbol{q}(t)\end{bmatrix} + \\ +\begin{bmatrix}\boldsymbol{W}^d+\widetilde{\boldsymbol{W}}^+ & \widetilde{\boldsymbol{W}}^-\\ \widetilde{\boldsymbol{W}}^- & \boldsymbol{W}^d+\widetilde{\boldsymbol{W}}^+\end{bmatrix}\begin{bmatrix}-\boldsymbol{g}(-\boldsymbol{p}(t))\\ \boldsymbol{g}(\boldsymbol{q}(t))\end{bmatrix},\ \forall t \geq 0. \quad (20)$$

Taking (20) into account, every vector function that satisfies (17) also fulfills condition (11), and, consequently, $I_{[-\boldsymbol{p},\boldsymbol{q}]}(t)$ is flow-invariant w.r.t. system (6). ■

By applying Theorem 2 for the case of functions $\boldsymbol{p}$ and $\boldsymbol{q}$ being constant, the following corollary is obtained.

Corollary 2. *If* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *are two positive vectors from* $\mathbb{R}^n$ *so that the vector* $\boldsymbol{\mu} = [\boldsymbol{\alpha}^T\ \boldsymbol{\beta}^T]^T > \boldsymbol{0}$ *satisfies the inequality*

$$\boldsymbol{\Pi}\boldsymbol{\mu} \leq \boldsymbol{0}, \quad (21)$$

then the CRS $I_{[-\boldsymbol{\alpha},\boldsymbol{\beta}]} = \left\{\boldsymbol{y}\in\mathbb{R}^n, -\boldsymbol{\alpha} \leq \boldsymbol{y} \leq \boldsymbol{\beta}\right\}$ *is flow invariant w.r.t. system* (6). ■

Another particular situation which may be of interest is that of exponentially time-dependent rectangular sets. Theorem 2 applied for this case leads to:

Corollary 3. *Let* $\boldsymbol{p}(t) = \mathrm{e}^{\sigma t}\boldsymbol{\alpha}$, $\boldsymbol{q}(t) = \mathrm{e}^{\sigma t}\boldsymbol{\beta}$, $t \geq 0$, *with* $\sigma \in \mathbb{R}$ *and* $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{R}^n$, $\boldsymbol{\alpha},\boldsymbol{\beta} > \boldsymbol{0}$. *If the inequality*

$$\sigma\boldsymbol{\mu} \geq \boldsymbol{\Pi}\boldsymbol{\mu}, \quad (22)$$

is fulfilled by $\boldsymbol{\mu} = [\boldsymbol{\alpha}^T \boldsymbol{\beta}^T]^T$, *then* $I_{[-\boldsymbol{p},\boldsymbol{q}]}(t)$ *is FI w.r.t.* (6). ■

We may also take into consideration the case of symmetrical TDRSs, derived as a consequence of Theorem 2.

Corollary 4. *Let* $\boldsymbol{p}:\mathbb{R}_+ \to \mathbb{R}^n$ *be a continuously differentiable vector function, with positive components. If*

$$\dot{\boldsymbol{p}}(t) \geq \boldsymbol{\Pi}^{*}\boldsymbol{p}(t),\quad t \geq 0, \quad (23)$$

where

$$\boldsymbol{\Pi}^* = -\boldsymbol{B} + \left(\boldsymbol{W}^+ + \widetilde{\boldsymbol{W}}^-\right)\boldsymbol{\Lambda}, \tag{24}$$

then the symmetrical TDRS $I_{[-\boldsymbol{p},\boldsymbol{p}]}(t)$ *is FI w.r.t. system* (6).■

Remark 2. If the vector functions $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfy the premises of Theorem 2, then for every $c>0$ the TDRS defined by $\tilde{\boldsymbol{p}} = c\boldsymbol{p}$, and $\tilde{\boldsymbol{q}} = c\boldsymbol{q}$, namely $I_{[-c\boldsymbol{p},c\boldsymbol{q}]}(t)$, homothetic to $I_{[-\boldsymbol{p},\boldsymbol{q}]}(t)$, is FI w.r.t. system (6). Also, the vector function $\boldsymbol{p}+\boldsymbol{q}$ satisfies condition (23), therefore defining a symmetrical TDRS that is FI w.r.t. system (6). ■

Remark 3. Theorem 1 in [2] shows that $I_{[-\boldsymbol{p},\boldsymbol{q}]}(t)$ is flow-invariant w.r.t. system (6) if $\boldsymbol{\eta}(t) = [\boldsymbol{p}^T(t)\ \boldsymbol{q}^T(t)]^T$ satisfies

$$\dot{\boldsymbol{\eta}}(t) \geq \boldsymbol{\Psi}\boldsymbol{\eta}(t), \quad \forall t \geq 0, \tag{25}$$

where

$$\boldsymbol{\Psi} = -\begin{bmatrix} \boldsymbol{B} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{B} \end{bmatrix} + \begin{bmatrix} \boldsymbol{W}^+ & \boldsymbol{W}^- \\ \boldsymbol{W}^- & \boldsymbol{W}^+ \end{bmatrix}\begin{bmatrix} \boldsymbol{\Lambda} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Lambda} \end{bmatrix}. \tag{26}$$

Paper [2] also shows that, if vector function $\boldsymbol{p}$ fulfills

$$\dot{\boldsymbol{p}}(t) \geq \boldsymbol{\Psi}^*\boldsymbol{p}(t), \quad \forall t \geq 0, \tag{27}$$

with

$$\boldsymbol{\Psi}^* = -\boldsymbol{B} + |\boldsymbol{W}|\boldsymbol{\Lambda}, \tag{28}$$

then the symmetrical TDRS $I_{[-\boldsymbol{p},\boldsymbol{p}]}(t)$ is FI w.r.t. system (6). Since the following inequalities exist between matrices $\boldsymbol{\Pi}$ (18) and $\boldsymbol{\Psi}$ (26) on the one hand, and between $\boldsymbol{\Pi}^*$ (24) and $\boldsymbol{\Psi}^*$ (28) on the other hand,

$$\boldsymbol{\Pi} \leq \boldsymbol{\Psi}, \ \boldsymbol{\Pi}^* \leq \boldsymbol{\Psi}^*, \tag{29}$$

the fulfillment of differential inequality (25) implies the fulfillment of (17) (respectively, (27) implies (23)), but the converse statement is not true. Thus, the class of vector functions generated by differential inequality (17) is broader than the one generated by (25) (respectively, the class of vector functions generated by (23) is broader than the one generated by (27)). This happens because the results in [2] (the derivation of the sufficient conditions (25) and (27)) are based on the usage of a linear upper bound for the nonlinear system under study, and not on the upper bound provided by the FI theory. ■

3. Componentwise asymptotic stability

Definition 1 expresses a property of the state-space trajectories of system (6), which once initialized inside a rectangular set $I_{[-\boldsymbol{p},\boldsymbol{q}]}(t)$ which is FI w.r.t. (6), do not leave it any longer. In particular, in case that $I_{[-\boldsymbol{p},\boldsymbol{q}]}(t)$ approaches the state-space origin for $t \to \infty$, this property reflects a special type of asymptotic stability of system (6) as defined in the sequel.

Definition 2. *Let* $\boldsymbol{p},\boldsymbol{q}:\mathbb{R}_+ \to \mathbb{R}^n$ *be two continuously differentiable vector functions, with positive components and*

$$\lim_{t\to\infty}\boldsymbol{p}(t) = \lim_{t\to\infty}\boldsymbol{q}(t) = \boldsymbol{0}. \tag{30}$$

System (6) *is called componentwise asymptotically stable* (*CWAS*) *with respect to* $-\boldsymbol{p}$ *and* $\boldsymbol{q}$ *if* $I_{[-\boldsymbol{p},\boldsymbol{q}]}(t)$ *is FI w.r.t.* (6). ■

Noticeably, if system (6) is CWAS with respect to some functions $-\boldsymbol{p}$ and $\boldsymbol{q}$ satisfying the prerequisites in Definition 2, then (6) is also *asymptotically stable* in the sense of the standard definition based on any consistent norm in $\mathbb{R}^n$. The converse statement is, in general, not true. Some conditions for model (6) to be CWAS may be derived from the general results presented in the previous section regarding the characterization of the flow invariant sets.

Theorem 3. *System* (6) *is CWAS with respect to* $-\boldsymbol{p}$ *and* $\boldsymbol{q}$ *meeting the requirements in Definition 2 if and only if nonlinear differential inequality* (11) *is satisfied.*

Proof. This theorem is a direct consequence of Theorem 1 in case that functions $\boldsymbol{p}$ and $\boldsymbol{q}$ fulfill requirement (30). ■

Theorem 4. *If the continuously differentiable vector functions* $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *satisfy the prerequisites in Definition 2 and the linear differential inequality* (17)*, then system* (6) *is CWAS with respect to* $-\boldsymbol{p}$ *and* $\boldsymbol{q}$. ■

Remark 4. If the vector functions $\boldsymbol{p}$ and $\boldsymbol{q}$ comply with Definition 2 and satisfy the linear differential inequality (17), from Remark 2 we get that system (6) is CWAS with respect to $-c\boldsymbol{p}$ and $c\boldsymbol{q}$, for all $c>0$. Moreover, whenever inequality (17) is met, system (6) is globally CWAS because the positive vector functions $\boldsymbol{\eta}(t)$, which satisfy the linear differential equation $\dot{\boldsymbol{\eta}}(t) = \boldsymbol{\Pi}\boldsymbol{\eta}(t),\ \forall t \geq 0$, constrain all the state-trajectories of system (6) to the unique equilibrium point $\boldsymbol{0}$. ■

Theorem 5. *If matrix* $\boldsymbol{\Pi}$ (18) *is Hurwitz stable then there exist vector functions* $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *such that system* (6) *is CWAS with respect to* $-\boldsymbol{p}$ *and* $\boldsymbol{q}$. ■

Theorem 6. *If the continuously differentiable vector function* $\boldsymbol{p}$ *satisfies the linear differential inequality* (23) *and the condition* $\lim_{t\to\infty}\boldsymbol{p}(t) = \boldsymbol{0}$, *then system* (6) *is CWAS with respect to* $-\boldsymbol{p}$ *and* $\boldsymbol{p}$. ■

Theorem 7. *If matrix* $\boldsymbol{\Pi}^*$ (24) *is Hurwitz stable, then there exists a vector function* $\boldsymbol{p}$ *such that system* (6) *is CWAS with respect to* $-\boldsymbol{p}$ *and* $\boldsymbol{p}$. ■

4. Componentwise exponential asymptotic stability

In Corollary 2 we dealt with exponentially TDRSs which are FI w.r.t (6). If we consider only decaying exponentials, then $\boldsymbol{p}(t) = \mathrm{e}^{\sigma t}\boldsymbol{\alpha}$, $\boldsymbol{q}(t) = \mathrm{e}^{\sigma t}\boldsymbol{\beta}$, $t \geq 0$, with $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{R}^n$, $\boldsymbol{\alpha},\boldsymbol{\beta} > \boldsymbol{0}$, and $\sigma < 0$, comply with the requirements in Definition 2. This fact brings on a new concept.

Definition 3. *System* (6) *is called componentwise exponentially asymptotically stable* (CWEAS) *if there exist* $\sigma<0$ *and* $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{R}^n$, $\boldsymbol{\alpha},\boldsymbol{\beta} > \boldsymbol{0}$, *so that* (6) *is CWAS with respect to* $-\boldsymbol{p}(t) = -\mathrm{e}^{\sigma t}\boldsymbol{\alpha}$ *and* $\boldsymbol{q}(t) = \mathrm{e}^{\sigma t}\boldsymbol{\beta}$. ■

Obviously, if system (6) is CWEAS, then it is also *exponentially asymptotically stable* in the classical sense. Corollary 3 has the following theorem as a consequence.

Theorem 8. *If there exist* $\sigma<0$ *and* $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{R}^n$, $\boldsymbol{\alpha},\boldsymbol{\beta} > \boldsymbol{0}$, *so that vector* $\boldsymbol{\mu} = \left[\boldsymbol{\alpha}^T\boldsymbol{\beta}^T\right]^T$ *fulfills the linear inequality* (22), *then system* (6) *is CWEAS.* ■

According to Remark 4, the CWEAS property has a global character on which further comments are given in Section 5.

Remark 5. For $\boldsymbol{p}(t) = \mathrm{e}^{\sigma t}\boldsymbol{\alpha}$, $t \geq 0$, with $\sigma < 0$ and $\boldsymbol{\alpha} \in \mathbb{R}^n$, $\boldsymbol{\alpha} > \boldsymbol{0}$, Corollary 4 gives that (6) is CWEAS if

$$\sigma\boldsymbol{\alpha} \geq \boldsymbol{\Pi}^*\boldsymbol{\alpha}, \qquad (31)$$

where matrix $\boldsymbol{\Pi}^*$ is defined by (24). ■

Next, we discuss the compatibility of the systems of linear inequalities (22) and (31). Let us notice the special structure of matrices $\boldsymbol{\Pi}$ (18) and $\boldsymbol{\Pi}^*$ (24), whose all off-diagonal elements are nonnegative (that is $\boldsymbol{\Pi}$ and $\boldsymbol{\Pi}^*$ are *essentially nonnegative matrices*). The following result holds true.

Lemma 1. ([9]) *Let* $\boldsymbol{M}$ *be an essentially nonnegative square matrix of order l, and denote by* $\lambda_i(\boldsymbol{M})$, $i = \overline{1,l}$, *the eigenvalues of* $\boldsymbol{M}$.

a. *$\boldsymbol{M}$ has a real eigenvalue (simple or multiple), denoted by* $\lambda_{\max}(\boldsymbol{M})$, *which fulfills the dominance condition:* $\mathrm{Re}[\lambda_i(\boldsymbol{M})] \leq \lambda_{\max}(\boldsymbol{M})$, $i = \overline{1,l}$.

b. *For any square matrix* $\boldsymbol{N} = [n_{ij}]$ *which is componentwise dominated by* $\boldsymbol{M} = [m_{ij}]$, *namely* $|n_{ij}| \leq m_{ij}$, $i \neq j$, $i, j = \overline{1,l}$, *and* $n_{ii} \leq m_{ii}$, $i = \overline{1,l}$, *the eigenvalues* $\lambda_i(\boldsymbol{N})$ *fulfill the inequalities:* $\mathrm{Re}\left[\lambda_i(\boldsymbol{N})\right] \leq \lambda_{\max}(\boldsymbol{M})$, $i = \overline{1,l}$.

c. *The algebraic inequality* $\rho\boldsymbol{d} \geq \boldsymbol{M}\boldsymbol{d}$, $\rho \in \mathbb{R}$, $\boldsymbol{d} \in \mathbb{R}^l$, *has positive solutions* $\boldsymbol{d} > \boldsymbol{0}$ *if and only if* $\rho \geq \lambda_{\max}(\boldsymbol{M})$. ■

Remark 6. For an essentially nonnegative matrix $\boldsymbol{M}$, the algebraic inequality considered in Lemma 1.c may be equivalently written (by using a matrix measure built with the infinity norm $\|\bullet\|_\infty$, e.g. [3]) as

$$\mu_\infty(\boldsymbol{D}^{-1}\boldsymbol{M}\boldsymbol{D}) \overset{\Delta}{=} \lim_{\xi \to 0+} \frac{\|\boldsymbol{I} + \xi\boldsymbol{D}^{-1}\boldsymbol{M}\boldsymbol{D}\|_\infty - 1}{\xi} \leq \rho, \qquad (32)$$

where $\boldsymbol{D} = \mathrm{diag}\left[d_1, \ldots, d_l\right]$ denotes the diagonal matrix built with the elements of the positive vector $\boldsymbol{d}$. Moreover, the dominant eigenvalue $\lambda_{\max}(\boldsymbol{M})$ represents the spectral abscissa of $\boldsymbol{M}$, which can be also expressed as

$$\lambda_{\max}(\boldsymbol{M}) = \inf_{\boldsymbol{D} \in \mathcal{D}} \mu_\infty(\boldsymbol{D}^{-1}\boldsymbol{M}\boldsymbol{D}), \qquad (33)$$

where $\mathcal{D}$ stands for the set of all positive diagonal matrices. ■

Remark 7. If the activation functions are differentiable in a vicinity of the equilibrium point, one can construct the linear approximation which has no connection with linear inequality (17). Actually, the usage of the linear approximation is irrelevant for the CWEAS analysis of the nonlinear system. This results from a thorough examination of the flow-invariance condition and its proof as given in [6]. ■

5. CWEAS in the large

Lemma 1 draws attention to a way of ensuring the CWEAS property for (6), by imposing on matrix $\boldsymbol{\Pi}$ to be Hurwitz stable, equivalent to matrix $-\boldsymbol{\Pi}$ is a nonsingular M-matrix [1]. This leads to a stronger type of stability, as defined below.

Definition 4. *System* (6) *is called globally CWEAS* (*or CWEAS in the large*) *if for every initial state of system* (6), $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$, $t_0 \geq 0$, *there exist* $\sigma < 0$ *and* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$, $\boldsymbol{\alpha}, \boldsymbol{\beta} > \boldsymbol{0}$, *so that the corresponding state-space trajectory satisfies* $-\mathrm{e}^{\sigma t}\boldsymbol{\alpha} \leq \boldsymbol{y}(t; t_0, \boldsymbol{y}_0) \leq \mathrm{e}^{\sigma t}\boldsymbol{\beta}$ *for all* $t \geq t_0$. ■

Obviously, if system (6) is globally CWEAS, then it is also *globally exponentially asymptotically stable* in the classical sense, ensuring the uniqueness of the equilibrium point of (6).

The following hypothesis is considered further on, besides (H1) and (H2):

(H3) $-\boldsymbol{\Pi}^*$ is a M-matrix, with $\boldsymbol{\Pi}^*$ given by(24).

(H3) is also used in the notable paper [3] that investigates model (4), but the other hypotheses in [3] (the activation functions are continuously differentiable and for all $i = 1, \ldots, n$, there exists $\lambda_i > 0$ so that $0 \leq f_i'(x) \leq \lambda_i$, $\forall x \in \mathbb{R}$) are more restrictive than our (H1)-(H2). Nevertheless, our approach based on (H1)-(H3) guarantees for (6) stronger properties than resulting from [3], i.e. our qualitative analysis ensures global CWEAS, whereas [3] ensures only global exponential stability. This statement can be easily proved by examples, as for instance Example 1 in [3]. To give a theoretical demonstration, we first establish the following results.

Lemma 2. *Let* $\boldsymbol{C} \in \mathbb{R}^{2n \times 2n}$ *be a matrix written in block-form as* $\boldsymbol{C} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{A} \end{bmatrix}$ *with* $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n \times n}$ *and let* $S(\boldsymbol{M})$ *denote the spectrum of a square matrix* $\boldsymbol{M}$ (*the set of all the eigenvalues of matrix* $\boldsymbol{M}$). *The spectrum of matrix* $\boldsymbol{C}$ *is given by*

$$S(\boldsymbol{C}) = S(\boldsymbol{A} + \boldsymbol{B}) \bigcup S(\boldsymbol{A} - \boldsymbol{B}). \qquad (34)$$

Proof. Let $\boldsymbol{J}$ be the square matrix of order $2n$ defined by $\boldsymbol{J} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{I} \\ \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix}$ with $\boldsymbol{I}$ the identity matrix of order n. Matrices $\boldsymbol{C}$ and $\tilde{\boldsymbol{C}} = \boldsymbol{J}\boldsymbol{C}\boldsymbol{J}^{-1}$ have the same eigenvalues, $S(\boldsymbol{C}) = S(\tilde{\boldsymbol{C}})$. Since $\boldsymbol{J}^{-1} = \frac{1}{2}\boldsymbol{J}$, matrix $\tilde{\boldsymbol{C}}$ becomes $\tilde{\boldsymbol{C}} = \begin{bmatrix} \boldsymbol{A} + \boldsymbol{B} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A} - \boldsymbol{B} \end{bmatrix}$, so that $S(\tilde{\boldsymbol{C}}) = S(\boldsymbol{A} + \boldsymbol{B}) \bigcup S(\boldsymbol{A} - \boldsymbol{B})$. ■

Theorem 9. *Let matrices* $\boldsymbol{\Pi}$ *and* $\boldsymbol{\Pi}^*$ *be defined by* (18) *and* (24), *respectively. Matrix* $-\boldsymbol{\Pi}$ *is a nonsingular M-matrix if and only if* $-\boldsymbol{\Pi}^*$ *is a nonsingular M-matrix.*

Proof. Taking into account that both $-\boldsymbol{\Pi}$ and $-\boldsymbol{\Pi}^*$ are essentially nonpositive matrices, the statement in Theorem 9 is equivalent to *matrix* $\boldsymbol{\Pi}$ *is Hurwitz stable if and only if* $\boldsymbol{\Pi}^*$ *is Hurwitz stable* [1]. By writing matrix $\boldsymbol{\Pi}$ (18) as

$$\boldsymbol{\Pi} = \begin{bmatrix} -\boldsymbol{B} + \boldsymbol{W}^+\boldsymbol{\Lambda} & \widetilde{\boldsymbol{W}}^-\boldsymbol{\Lambda} \\ \widetilde{\boldsymbol{W}}^-\boldsymbol{\Lambda} & -\boldsymbol{B} + \boldsymbol{W}^+\boldsymbol{\Lambda} \end{bmatrix}, \qquad (35)$$

and taking Lemma 2 into account, the spectrum of matrix $\boldsymbol{\Pi}$ is:

$$S(\boldsymbol{\Pi}) = S(\boldsymbol{\Pi}^*) \bigcup S(\boldsymbol{\Omega}), \qquad (36)$$

where $\boldsymbol{\Omega} = -\boldsymbol{B} + (\boldsymbol{W}^+ - \widetilde{\boldsymbol{W}}^-)\boldsymbol{\Lambda}$. By applying the result in Lemma 1.a to matrices $\boldsymbol{\Pi}$ and $\boldsymbol{\Pi}^*$, we have to prove that the dominant eigenvalues of these matrices satisfy: $\lambda_{\max}(\boldsymbol{\Pi}) < 0$ *if and only if* $\lambda_{\max}(\boldsymbol{\Pi}^*) < 0$. From (36) we get that:

$$\lambda_{\max}(\boldsymbol{\Pi}) = \max\left\{\mathrm{Re}(\lambda_i(\boldsymbol{\Pi}^*)), \mathrm{Re}(\lambda_i(\boldsymbol{\Omega}))\right\} = \lambda_{\max}(\boldsymbol{\Pi}^*) \quad (37)$$

since $\boldsymbol{\Pi}^*$ dominates Ω in the sense of Lemma 1.b, that is

$$\mathrm{Re}(\lambda_i(\boldsymbol{\Pi}^*)) \le \lambda_{\max}(\boldsymbol{\Pi}^*),\ \mathrm{Re}(\lambda_i(\boldsymbol{\Omega})) \le \lambda_{\max}(\boldsymbol{\Pi}^*), i = \overline{1,n}\,. \blacksquare (38)$$

Based on these preliminaries, we are now able to show that relying on hypotheses (H1)-(H3) our approach provides a more refined result than the one presented in [3].

Theorem 10. *If hypotheses* (H1)-(H3) *are fulfilled, then system* (6) *is globally CWEAS.*

Proof. According to Theorem 9, hypothesis (H3) is equivalent to matrix $\boldsymbol{\Pi}$ being Hurwitz stable, therefore its dominant eigenvalue is negative, $\lambda_{\max}(\boldsymbol{\Pi}) < 0$. Choosing σ as

$$\lambda_{\max}(\boldsymbol{\Pi}) \le \sigma < 0\,, \quad (39)$$

Lemma 1.c shows that there exist $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$, $\boldsymbol{\alpha}, \boldsymbol{\beta} > \boldsymbol{0}$, such that (6) is CWAS w.r.t. $-\boldsymbol{p}(t) = -\mathrm{e}^{\sigma t}\boldsymbol{\alpha}$ and $\boldsymbol{q}(t) = \mathrm{e}^{\sigma t}\boldsymbol{\beta}$. Considering an arbitrary initial condition of (6), $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$, for some $t_0 \ge 0$, we can always find a constant $c \ge 1$ so that $-c\,\mathrm{e}^{\sigma t_0}\boldsymbol{\alpha} \le \boldsymbol{y}_0 \le c\,\mathrm{e}^{\sigma t_0}\boldsymbol{\beta}$. Taking Remark 2 into account, (6) is CWAS w.r.t. $-c\boldsymbol{p}(t) = -c\,\mathrm{e}^{\sigma t}\boldsymbol{\alpha}$ and $c\boldsymbol{q}(t) = c\,\mathrm{e}^{\sigma t}\boldsymbol{\beta}$; therefore the trajectory $\boldsymbol{y}(t) = \boldsymbol{y}(t; t_0, \boldsymbol{y}_0)$ of (6) satisfies:

$$-c\,\mathrm{e}^{\sigma t}\boldsymbol{\alpha} \le \boldsymbol{y}(t) \le c\,\mathrm{e}^{\sigma t}\boldsymbol{\beta},\ \forall t \ge t_0\,. \quad \blacksquare (40)$$

Remark 8. Inequality (39) gives the possibility to relate the CWEAS decaying rate to the system parameters concisely reflected by the spectral abscissa $\lambda_{\max}(\boldsymbol{\Pi}) = \lambda_{\max}(\boldsymbol{\Pi}^*)$. ■

Another improvement brought by our paper to the qualitative analysis of nonlinear system (6) refers to the following comparison with the results in the recent paper [2]. In [2], the compatibility of the following algebraic inequalities is given as a sufficient condition for (6) to be globally CWEAS:

$$\sigma\boldsymbol{\mu} \ge \boldsymbol{\Psi}\boldsymbol{\mu}\,,\ \boldsymbol{\mu} \in \mathbb{R}^{2n},\ \ \boldsymbol{\mu} > \boldsymbol{0}\,,\ \sigma < 0\,, \quad (41)$$

where $\boldsymbol{\Psi}$ is the essentially nonnegative matrix defined by (26). According to Lemma 1.c, (41) is compatible if and only if $\lambda_{\max}(\boldsymbol{\Psi}) \le \sigma$. The componentwise inequality (29) shows that matrix $\boldsymbol{\Psi}$ dominates our matrix $\boldsymbol{\Pi}$ in the sense of Lemma 1.b, leading to the conclusion that $\lambda_{\max}(\boldsymbol{\Pi}) \le \lambda_{\max}(\boldsymbol{\Psi})$. Hence, it is obvious that our sufficient condition for CWEAS is better than the one formulated in [2]. This conclusion can be obtained by using the matrix measure (32) instead of the linear inequalities (41) and (22). Moreover, in [2], the case of symmetrical constraints, characterized by the linear inequality

$$\sigma\boldsymbol{\alpha} \ge \boldsymbol{\Psi}^*\boldsymbol{\alpha}\,,\ \boldsymbol{\alpha} \in \mathbb{R}^n,\ \ \boldsymbol{\alpha} > \boldsymbol{0}\,,\ \sigma < 0\,, \quad (42)$$

with matrix $\boldsymbol{\Psi}^*$ given by (28), is treated as a separate situation, without observing that $\lambda_{\max}(\boldsymbol{\Psi}^*) = \lambda_{\max}(\boldsymbol{\Psi})$.

6. Conclusions

For the class of Hopfield-type neural networks we have developed new results which offer a refinement in the analysis of their behavior. The key element of this refinement is the componentwise monitoring of the state-trajectories by considering time-dependent rectangular sets that are invariant with respect to the dynamics of the investigated systems.

At a first stage, the rectangular sets are assumed to exhibit an arbitrary time-dependence (Definition 1) and they are characterized via a nonlinear differential inequality (Theorem 1). It is shown that a simplified condition can be also formulated (Theorem 2) by using a linear differential inequality which ensures only the sufficiency. The second stage focuses on those time-dependent rectangular sets that approach the equilibrium point and introduces the concept of CWAS (Definition 2). Two theorems (Theorem 3 and Theorem 4) provide a characterization of CWAS and an easy to exploit sufficient criterion, respectively. Next, CWAS is explored for the particular case when the invariant sets approach the equilibrium point in an exponential manner yielding the stronger concept of CWEAS (Definition 3) whose existence is guaranteed by the fulfillment of an algebraic condition (Theorem 8). Finally, by introducing a supplementary hypothesis (similar to other papers) we demonstrate that the CWEAS property has a global character (Theorem 10). Actually, for weaker conditions than in other works establishing global exponential stability, we are able to prove the stronger property of global CWEAS.

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