Locally positive nonlinear systems

Tadeusz Kaczorek

Warsaw University of Technology, Faculty of Electrical Engineering
Institute of Control and Industrial Electronics
00-662 Warszawa, Koszykowa 75, Poland
e-mail: kaczorek@isep.pw.edu.pl

Abstract. The notion of locally positive nonlinear time-varying linear systems is introduced. Necessary and sufficient conditions for the local positivity of nonlinear time-varying systems are established. The concept of local reachability in the direction of a cone is introduced and sufficient conditions for the local reachability in the direction of a cone of this class of nonlinear systems are presented.

1. Introduction

Roughly speaking positive systems are systems whose trajectories are entirely in the nonnegative orthant R_+^n whenever the initial state and input are nonnegative. Positive systems arise in modelling of systems in engineering, economics, social sciences, biology, medicine and other areas [7,1-3,11,20,19]. The single-input single-output externally positive and internally positive linear time-invariant systems have been investigated in [7,2,3]. The notions of externally positive and internally positive systems have been extended for singular continuous-time and discrete-time and two-dimensional linear systems in [11]. The reachability and controllability of standard and singular internally positive linear systems have been analysed in [6,14,16,22]. The notions of weakly positive discrete-time and continuous-time linear systems have been introduced in [11,12]. Recently the positive two-dimensional (2D) linear systems have been extensively investigated by Fornasini and Valcher [23,22] and in [11].
Necessary and sufficient conditions for the external and internal positivities and sufficient conditions for reachability of time-varying linear systems have been established in [10,15]. The notion of the controllability of a dynamic system in the direction of a cone was introduced by Walczak in [21] and a sufficient condition for local controllability of nonlinear systems was established.
In this paper the notion of local positive in the neighborhood of zero of nonlinear time-varying systems will be introduced and the necessary and sufficient conditions for the local positivity will be established. The reachability of nonlinear time-varying systems will be also investigated. To the best knowledge of the author this class of locally positive nonlinear systems has not been considered yet.

2. Preliminaries

Let $R^{n\times m}$ be the set of real matrices with non-negative entries and $R^n := R^{n\times 1}$.
Consider a nonlinear system described by the equations

(1a) $$\dot{x} = f(x,u,t),\ \ x(t_0) = x_0$$

(1b) $$y = h(x,u,t)$$

where $\dot{x} = \frac{dx}{dt}$, $x \in R^n$, $u \in R^m$ and $y \in R^p$ are the state, input and output vectors, respectively, and

(2) $$f(x,u,t) = \begin{bmatrix} f_1(x,u,t) \\ f_2(x,u,t) \\ \cdots\cdots \\ f_n(x,u,t) \end{bmatrix}, \; h(x,u,t) = \begin{bmatrix} h_1(x,u,t) \\ h_2(x,u,t) \\ \cdots\cdots \\ h_p(x,u,t) \end{bmatrix}$$

are R^n- and R^p - valued mappings defined on open sets. It is assumed that the functions $f_1(x,u,t),..., f_n(x,u,t)$ and $h_1(x,u,t),...,h_p(x,u,t)$ are smooth in their arguments, i.e. they are real-valued functions of $x_1,...,x_n,u_1,...,u_m,t$ with continuous partial derivatives of any order, where $x = [x_1, x_2 ..., x_n]^T, u = [u_1, u_2 ..., u_m]^T$ and T denotes the transpose. It is also assumed that the system (1a) possesses a solution for any admissible input u.

Let

(3) $$f(0,0,t) = 0 \text{ and } h(0,0,t) = 0 \text{ for all } t$$

and

(4a) $$\dot{x} = A(t)x + B(t)u + N_f(x,u,t)$$

(4a) $$y = C(t)x + D(t)u + N_h(x,u,t)$$

where

(5)

$$A(t) = \frac{\partial f}{\partial x}\Big|_{\substack{x=0\\u=0}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\substack{|x=0\\u=0}}, \quad B(t) = \frac{\partial f}{\partial x}\Big|_{\substack{x=0\\u=0}} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{\substack{|x=0\\u=0}}$$

$$C(t) = \frac{\partial f}{\partial x}\Big|_{\substack{x=0\\u=0}} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_p}{\partial x_1} & \cdots & \frac{\partial h_p}{\partial x_n} \end{bmatrix}_{\substack{|x=0\\u=0}}, \quad D(t) = \frac{\partial f}{\partial x}\Big|_{\substack{x=0\\u=0}} = \begin{bmatrix} \frac{\partial h_1}{\partial u_1} & \cdots & \frac{\partial h_1}{\partial u_m} \\ \frac{\partial h_p}{\partial u_1} & \cdots & \frac{\partial h_p}{\partial u_m} \end{bmatrix}_{\substack{|x=0\\u=0}}$$

$N_f(x,u,t)$, $N_h(x,u,t)$ are the nonlinear parts of $f(x,u,t)$ and $h(x,u,t)$, respectively and

(6) $$\lim_{\|x\|,\|u\| \to 0} \frac{N_f(x,u,t)}{\|x\|\|u\|} = 0, \; \lim_{\|x\|,\|u\| \to 0} \frac{N_h(x,u,t)}{\|x\|\|u\|} = 0$$

The linear system

(7a) $$\dot{x} = A(t)x + B(t)u$$

(7b) $$y = C(t)x + D(t)u$$

is called a linear approximation of the nonlinear system (1) in the neighborhood of the zero $(x = 0, u = 0)$

Example 1. Consider the nonlinear system

(8a) $$\begin{aligned} \dot{x}_1 &= x_1 t + x_2 + \sin x_2^2 + u + u^2 \\ \dot{x}_2 &= x_2 e^{x_1} + x_2 + 2u \end{aligned}$$

(8b) $$y = x_1 + ut + u^3$$

Using (5) we obtain

(9)

$$A(t)=\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{bmatrix}_{\substack{|x=0\\u=0}}=\begin{bmatrix}t\ ,1\\0\ ,1\end{bmatrix},\ B(t)=\begin{bmatrix}\frac{\partial f_1}{\partial u}\\ \frac{\partial f_2}{\partial u}\end{bmatrix}_{\substack{|x=0\\u=0}}=\begin{bmatrix}1\\2\end{bmatrix}$$

$$C(t)=\begin{bmatrix}\frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2}\end{bmatrix}_{\substack{|x=0\\u=0}}=[1\ \ 0],\ D(t)=\begin{bmatrix}\frac{\partial h}{\partial u}\end{bmatrix}_{\substack{|x=0\\u=0}}=[t]$$

and

(10)

$$N_f(x,u,t)=\begin{bmatrix}f_1(x,u,t)\\f_2(x,u,t)\end{bmatrix}-A(t)x-B(t)u=\begin{bmatrix}\sin x_2^2+u^2\\x_2e^{x_1}\end{bmatrix}$$

$$N_h(x,u,t)=h(x,u,t)-C(t)x-D(t)u=u^3$$

It is easy to check that (10) satisfy the conditions (6).

Main result

Lemma

Let

(11) $$\dot{x}=A(t)x$$

be the linear approximation of the nonlinear autonomous system

(12) $$\dot{x}=f(x,t)=A(t)x+N_f(x,t)$$

where

(13) $$A(t)=[a_{ij}(t)]_{\substack{i=1,\dots,n\\j=1,\dots n}}=\frac{\partial f}{\partial x}_{|x=0}=\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}\end{bmatrix}_{x=0}$$

and

(14) $$\lim_{\|x\|\to 0}\frac{N_f(x,t)}{\|x\|}=0$$

If the components $f_1(x,t),\dots,f_n(x,t)$ of $f(x,t)$ satisfy the condition

(15) $f_i(x,t)\ge 0$ for $x_j\ge 0, i\ne j, x_i=0$ and all $t\ge 0$

then

(16) $a_{ij}(t)\ge 0$ for $i\ne j$ and all $t\ge 0$

where $i,j=1,\dots,n$.

Proof. From (13) we have

(17)

$$a_{ij}(t)=\begin{bmatrix}\frac{\partial f_i(x,t)}{\partial x_j}\end{bmatrix}_{\|x=0}=\lim_{x_j\to 0^+}\frac{f_i(0,\dots,0,x_j,0,\dots,0,t)-f_i(0,0,\dots,0,t)}{x_j}=$$

$$=\lim_{x_j\to 0^+}\frac{f_i(0,\dots,0,x_j,0,\dots,0,t)}{x_j}\ge 0$$

since $f_i(0,\dots,0,x_j,0,\dots,0,t)\ge 0$. □

Remark 1. In particular case when $f(x,t)$ is explicitly independent of time t, $f(x,t) = f(x)$ then $A(t) = A$ and the time-invariant matrix (13) is a Metzler matrix satisfying the condition

$$e^{At} \in R_+^{n\times n} \qquad \text{for all} \qquad t \geq 0$$

Definition 1. The nonlinear system (1) is called locally positive in the neighborhood of zero $(x=0,\, u_0)$ if there exists a neighbourhood of the zero U_0 such that for any $x_0 \in U_0 \cap R_+^n$ we have $x(t) \in U_0 \cap R_+^n$ for (or at least $t \in [0, \varepsilon)$ for some $\varepsilon > 0$).

Theorem 1. The nonlinear system (1) is locally positive in the neighborhood of zero $(x=0,\, u=0)$ if and only if

(18a) $$\int_0^t \frac{\partial f_i}{\partial x_j}(\tau)d\tau \geq 0 \ \text{ for } i \neq j\,; i,j = 1,...,n \ \text{ and } t \geq 0$$

(18b) $$\frac{\partial f}{\partial u}(t)_{\substack{|x=0\\u=0}} \in R_+^{n\times m},\ \frac{\partial h}{\partial x}(t)_{\substack{|x=0\\u=0}} \in R_+^{p\times n},\ \frac{\partial h}{\partial u}(t)_{\substack{|x=0\\u=0}} \in R_+^{p\times m} \ \text{ for } t \geq 0$$

Proof. Note that the condition (18a) is equivalent to

(19) $$\int_0^t a_{ij}(\tau)d\tau \geq 0 \ \text{ for } i \neq j\,; i,j = 1,...,n \ \text{ and } t \geq 0$$

In [10] it has been shown that the linear approximation (7) is positive if and only if the conditions (19) and (18b) are satisfied. Under the assumptions and (6) it is easy to show that the nonlinear system (1) is locally positive in the neighborhoad of zero if and only if the linear approximation (7) is positive. □

Example 2. (continuation of Example 1)

We shall show that the nonlinear system (8) is locally positive in the neighborhood of zero ($x = 0, u = 0$).

The nonlinear system (8) satisfies the conditions (18) since

$$\int_0^t \frac{\partial f_1}{\partial x_1}(\tau)d\tau = \int_0^t \tau\, d\tau \geq 0,\ \int_0^t \frac{\partial f_1}{\partial x_2}(\tau)d\tau = \int_0^t 1 d\tau \geq 0$$

$$\int_0^t \frac{\partial f_2}{\partial x_2}(\tau)d\tau = \int_0^t 0\, d\tau = 0,\ \int_0^t \frac{\partial f_2}{\partial x_2}(\tau)d\tau = \int_0^t 1 d\tau \geq 0 \quad \textit{for} \ \ t \geq 0$$

and

$$C(t) = [1\ \ 0] \in R_+^{1\times 2}\ ,\ D(t) = t \geq 0$$

Therefore, by Theorem 1 the nonlinear system (8) is locally positive in the neighborhood of zero.

Let $C_+ \in R_+^n$ be a cone in the neighborhoad of zero ($x = 0, u = 0$). Following Walczak [21] the notion of the local reachability in the direction of a cone will be introduced.

Definition 2. The nonlinear system (1) is called locally reachable in the cone direction of a C_+ if for every state $x_f \in C_+$ there exists a time $t_f - t_0 > 0$ and input $u(t) \in R_+^m, t \in [t_0, t_f]$ such that $x(t_f) = x_f$ for $x(t_0) = x_0 = 0$.

A matrix is called the monomial matrix if its every row and its every column contains only one positive entry and the remaining entries are zero.

The inverse matrix A^{-1} of a positive matrix $A \in R_+^{n\times n}$ is the positive matrix if and only if A is the monomial matrix [11].

Theorem 2. The nonlinear system (1) is locally reachable in the direction of a cone $C_+ \subset R_+^n$ if the matrix

(20) $$R_f = \int_{t_0}^{t_f} \Phi(t_f,\tau)B(\tau)B^T(\tau)\Phi^T(t_f,\tau)d\tau,\ t_f > t_0$$

is a monomial matrix.

The input that steers the state of the system (1) in time $t_f - t_0$ from $x(t_0) = 0$ to the final state x_f is given by

(21) $$u(t) = B^T(t)\Phi(t_f,t)R_f^{-1}x_f \text{ for } t \in [t_0, t_f]$$

Proof. If R_f is a monomial matrix then there exists the inverse matrix $R_f^{-1} \in R_+^{n\times n}$ which is also monomial. Hence the output (21) is well defined and $u(t) \in R_+^m$ for $t \in [t_0, t_f]$.

Substituting of (21) into the solution

(22) $$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)d\tau$$

of (7a) for $t = t_f$ and $x_0 = 0$ we obtain

$$x(t_f) = \int_{t_0}^{t_f} \Phi(t_f,\tau)B(\tau)B^T(\tau)\Phi^T(t_f,\tau)R_f^{-1}x_f d\tau = \left[\int_{t_0}^{t_f} \Phi(t_f,\tau)B(\tau)B^T(\tau)\Phi^T(t_f,\tau)d\tau\right]R_f^{-1}x_f = x_f$$

If the matrix (20) is a monomial matrix then the linear approximation (7a) of the nonlinear system (1) is reachable. Under the assumptions and (6) it is easy to show that the nonlinear system (1) is locally reachable in the cone C_+. □

In particular case if the system (1) is linear then from Theorem 1 and Theorem 2 we obtain the known results given in [10,11,14,15].

3. Concluding remarks.

The notion of local positive in the neighborhood of zero nonlinear time-varying system has been introduced. Necessary and sufficient conditions for the local positivity have been established by the use of the linear approximation (7) of the nonlinear system (1). The concept of local reachability in the direction of a cone of the positive nonlinear systems (1) has been introduced and sufficient conditions for the local reachability in the direction of a cone by the use of the linear approximation (7) have been also derived.

With minor modifications the considerations can be extended to nonlinear discrete time-varying systems. Open problem are the extensions of the considerations to singular nonlinear systems and to 2D nonlinear systems.

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