FURTHER RESULTS ON DYNAMIC FEEDBACK LINEARIZATION

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Abstract

The paper deals with dynamic feedback linearization of two input continuous time systems. A constructive procedure based on prolongations is proposed. The algorithm, which relies on necessary geometric conditions, computes a set of prolongation indices.

1 Introduction

The feedback linearization problem has been widely studied both in continuous and in discrete time (see [10] and the reference therein). The first works on the topic regard the exact linearization problem [2], [11], [8], [7], [12], [14]. Successive works concern the computation of the largest linearizable subsystem of a given dynamics [13], [3].

Dynamic feedback solutions were first considered in [9]. In [6], sufficient geometric conditions were given for the solvability of the problem via prolongations and diffeomorphism. The sufficiency regards two aspects: the apriori knowledge of a candidate set of prolongation indices; the requirement that the associated distributions, are the projection of the corresponding distributions defined on the extended space. In the same work necessary and sufficient conditions were given for systems where the number of states minus the number of inputs is equal to one.

The equivalence of differentially flat nonlinear systems to dynamic feedback linearizable systems was first addressed in [4], [5] where the concept of endogenous feedback was introduced. Roughly speaking the solution is in this case based on the knowledge of a set of flat outputs, i.e. linearizing output functions depending on the state, the control and its derivatives.

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Algebraic necessary and sufficient conditions for the solvability of the problem were given in [1] where it was shown how to associate an infinitesimal Brunovskii form with a nonlinear system; the existence of a set of linearizing outputs is equivalent to the existence of a differential operator transforming the infinitesimal Brunovskii form into a system of exact one-form. However these conditions are existence conditions and are not constructive thus not allowing a direct computation of the dynamic compensator. Finally, a bound on the number of integrators necessary to achieve linearization was given in [15].

In this paper we propose an algorithm for the computation of a dynamic compensator consisting of prolongations. The algorithm is based on a set of necessary geometric conditions which are also sufficient when the prolongation indices are at most equal to 2. The procedure is illustrated on an example used in [6] to enlighten the sufficiency of the proposed result.

The Problem: Consider the continuous time system

$$\dot{x} = f(x) + \sum_{i=1}^{2} g_i(x) u_i \tag{1}$$

where $x \in \mathbb{R}^n$, and $f(x), g_1(x), g_2(x)$ are smooth maps defined on a open set of $\mathbb{R}^n$. Find, if there exists, a dynamic feedback

$$\dot{\zeta}_i \;=\; A_i\zeta_i + B_i v_i, \qquad i = 1, 2 \tag{2}$$

with $\zeta_i = (\zeta_{i,1}, \cdots, \zeta_{i,\mu_i})^T$, $\zeta = (\zeta_1^T, \zeta_2^T)^T$, $u_i = \zeta_{i,1}$, for $i = 1, 2$, such that the extended system

$$\begin{pmatrix} \dot{x} \\ \dot{\zeta} \end{pmatrix} = F(x, \zeta) + \sum_{i=1}^{2} G_i(x, \zeta) v_i \tag{3}$$

is static feedback equivalent to a linear system, i.e. there exists a regular static feedback $v = \alpha(x, \zeta) + \beta(x, \zeta) w$, such that the closed–loop system is diffeomorphic to a linear system. In (2) A_i and B_i of dimension respectively $\mu_i \times \mu_i$ and $\mu_i \times 1$, are given by $A_i = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$, $B_i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. If $u_i = v_i$ we set $\mu_i = 0$.

Remark.The indices μ_i will be called prolongation indices and system (1) will be said dynamic feedback linearizable with prolongation indices (μ_1, μ_2). If the system is dynamic feedback linearizable with prolongations only, then at least one prolongation index can be set to zero, i.e. $0 = \mu_1 \leq \mu_2$ ([15]). The number of necessary integrators is related to the dimension n of system (1) and the number of inputs. In the two input case $\sum_{i=1}^{2} \mu_i \leq 2n - 3$ ([15]). $\triangleleft$

2 Preliminaries

We will first recall the well known result concerning the state space exact linearization problem and then we will study the properties of the extended system (1-2). Some preliminary results on the properties of the distributions associated with the original system (1) are enlightened. These properties are at the basis of the proposed algorithm for the computation of a dynamic compensator solving the problem.

2.1 Recalls and notations

The following notation is used. Given a number p, $\lfloor p \rfloor$ is its inferior integer; $p! := p(p-1)\cdots 1$ is the factorial number; given two smooth vector fields f and g_i, $ad_f g_i := [f, g_i] = \frac{\partial g_i}{\partial x} f - \frac{\partial f}{\partial x} g_i$ is the standard Lie brackets of vector fields and $ad_f^k g_i = ad_f(ad_f^{k-1} g_i)$; we denote by $g = (g_1, g_2)$, by $G = (G_1, G_2)$, by $\mathcal{G}_i$ the distribution $\mathcal{G}_i := \text{span}\{g, \cdots, ad_f^i g\}$, by $\bar{\mathcal{G}}_i$, the involutive closure of $\mathcal{G}_i$. The set of distributions $\mathcal{G}_i$ plays a crucial role in the solution of the regular static feedback linearization problem. We recall the following result, stated in the general m–input case.

Theorem 1 [7] Suppose that the matrix $g(x_0)$ has *rank* m. Then the state space exact linearization problem is solvable if and only if

i.) $\mathcal{G}_i$ has constant dimension near x_0, for $0 \leq i \leq n-1$,

ii.) $\mathcal{G}_{n-1}$ has dimension n

iii.) $\mathcal{G}_i$ is involutive, for $0 \leq i \leq n-2$.

The loss of involutivity of any of the distributions $\mathcal{G}_i$ does not allow to achieve exact linearization via regular static state feedback. As for dynamic feedback linearization, in [6] the problem is solved by showing that if it is possible to define a partition of the input vector defined by the set of integers $0 = \mu_1 \leq \cdots \leq \mu_m$, s.t. the distributions

$$\begin{aligned} \Delta_0 &= \text{span}\{g_j : \mu_j = 0\} \\ \Delta_{i+1} &= \Delta_i + ad_f \Delta_i + \text{span}\{g_j : \mu_j = i+1\},\ i \geq 0 \end{aligned}$$

satisfy appropriate properties, then the problem is solvable via prolongations. More precisely

Theorem 2 [6] If locally in a neighborhood U_0 of x_0

i.) Δ_i is involutive and of constant dimension for $0 \leq i \leq n + \mu_m - 1$

ii.) $\dim(\Delta_{n+\mu_m-1}) = n$

iii.) $[g_s, \Delta_i] \subset \Delta_{i+1}$, $\forall s$, such that $\mu_s \geq 1$ and $\forall i$, $0 \leq i \leq n + \mu_m - 1$

then system (1) is locally dynamic feedback linearizable with prolongation indices $\mu_1 \cdots \mu_m$.

A part from the apriori knowledge of the prolongation indices, the sufficiency of Theorem 2 is essentially linked to condition iii), while conditions i) and ii) are necessary. In fact, consider the two–input case. Then, for $l = 0, \cdots, n + \mu_2 - 1$,

$$\Delta_l = \text{span}\left\{g_1, \cdots ad_f^l g_1, g_2, \cdots, ad_f^{l-\mu_2} g_2\right\}. \quad (4)$$

Consider the distributions $\mathcal{G}_i^e := \text{span}\left\{G, \cdots, ad_F^i G\right\}$ associated with the extended system (3).

Condition iii) corresponds to verify that the vector fields $ad_F^s G_j$ have the form

$$Ad_F^s G_j = \begin{pmatrix} \sum_{j=1}^{2} \sum_{i=0}^{s-\mu_j} \eta_{ij}(\zeta) ad_f^i g_j \\ * \end{pmatrix}$$

so that the projection of $\mathcal{G}_i^e$ is given by Δ_i, i.e. $\pi(\mathcal{G}_i^e) = \Delta_i \circ \pi$, which is a particular case.

With respect to the present result we will propose an algorithm which allows the following improvements:

– a constructive computation of a set of prolongation indices with respect to which the distributions Δ_i satisfy condition i) and ii) of Theorem 2.

– we will relax condition iii), so that the projection of $\mathcal{G}_i^e$ may not coincide with Δ_i.

2.2 The properties of the extended system

Let us thus consider the extended dynamics (1)-(2), and assume it feedback linearizable. After a possible reordering of the inputs, $0 = \mu_1 < \mu_2$. The closed–loop system is given by

$$\begin{array}{rcl} \dot{x} & = & f(x) + g_1(x)v_1 + g_2(x)\zeta_{21} \\ \dot{\zeta}_2 & = & A_2\zeta_2 + B_2 v_2 \end{array} \quad (5)$$

The extended system (3) is thus characterized by the following vector fields

$$F = \begin{pmatrix} f(x) + g_2(x)\zeta_{21} \\ A_2\zeta_2 \end{pmatrix}, G_1 = \begin{pmatrix} g_1(x) \\ 0 \end{pmatrix}, G_2 = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}.$$

The generic term $ad_F^{l+\mu_i} G_i$, $l + \mu_i \geq 0$, has the form

$$ad_F^{l+\mu_i} G_i = (-1)^{l+\mu_i} \begin{pmatrix} \tau_i^l(x, \zeta) \\ A_i^{l+\mu_i} B_i \end{pmatrix}. \quad (6)$$

In (6), for $-\mu_i \leq l < 0$, $\tau_i^l = 0$, and $A_i^{l+\mu_i} B_i = e_l$ the canonical vector whose elements are all zero but the l–th which is equal to one. Instead, when $l \geq 0$, $A_i^{l+\mu_i} B_i = 0$ and

$$\tau_i^l(x,\zeta) := ad_f^l g_i + \sum_{k=1}^{min(\mu_2,l)} \sum_{r=0}^{l-k} c_{ir}^k ad_f^r [g_2, ad_f^{l-k-r} g_i]\zeta_{2k} + \mathcal{O}(\|\zeta\|^2)$$

where $\mathcal{O}(\|\zeta\|^2)$ represents terms of order greater than one in ζ, and $c_{ir}^j = \frac{(r+j-1)!}{r!(j-1)!}$.

For the first terms one gets $\tau_i^0(x, \zeta) = g_i(x)$, $\tau_i^1(x, \zeta) = ad_{f(x)} g_i(x) + c_{i0}^1 ad_{g_2(x)} g_i(x)\zeta_{21}, \cdots$

The distribution $\mathcal{G}_0^e$ associated with the extended system (1-2), is then given by

$$\mathcal{G}_0^e = \text{span}\left\{ \begin{pmatrix} g_1(x) \\ 0 \end{pmatrix} \right\} + \text{span}\left\{ \frac{\partial}{\partial \zeta_{2,\mu_2}} \right\}$$

and for $i > 0$,

$$\begin{array}{rcl} \mathcal{G}_i^e & = & \mathcal{G}_{i-1}^e + \text{span}\left\{ \begin{pmatrix} \tau_l^{(i-\mu_l)}(\cdot) \\ 0 \end{pmatrix}, l = 1, 2 \right\} \\ & & + \text{span}\left\{ \frac{\partial}{\partial \zeta_{2,\mu_2 - i}}, \text{ if } \mu_2 - i > 0 \right\} \end{array} \quad (7)$$

Since the closed–loop system is linearizable via regular static feedback, the distributions $\mathcal{G}_i^e$ must be involutive and regular (Theorem 1). This fact induces a certain number of properties on the Lie brackets of the vector fields defined on the original system. More precisely the following result holds true.

Proposition 1 Let system (1) be dynamic feedback linearizable with prolongation indices $0 = \mu_1 \leq \mu_2$ around $(x_0, \zeta_0) = (x_0, 0)$. Consider for $i = 0 \cdots, n + \mu_2 - 1$ the distribution Δ_i defined in (4). Locally around x_0, the following properties hold true

a) Δ_i has constant dimension and is involutive, for $0 \leq i \leq n + \mu_2 - 1$

b) $dim\ \Delta_{n+\mu_2-1} = n$

c) $\forall i \geq 0$, $[ad_f^{r_2} g_2, ad_f^{r_1} g_1] \in \Delta_i$, $\forall (r_1, r_2) : r_1 + r_2 = i - \mu_2 + 1, \cdots, i - \mu_2 + \left\lfloor \frac{\mu_2}{2} \right\rfloor$.

The proof of the result with its technical details is omitted. Let us instead note that if $0 = \mu_1 \leq \mu_2 \leq 2$ the conditions of Proposition 1 become also sufficient as underlined in the following Theorem.

Theorem 3 System (1) is locally dynamic feedback linearizable around $(x_0, 0)$ with prolongation indices $0 = \mu_1 \leq \mu_2 \leq 2$ if and only if the conditions of Proposition 1 are satisfied.

3 Main Result

The algorithm proposed hereafter aims to compute a set of prolongation indices $\mu_1 \leq \mu_2$ with respect to which the associated distributions Δ_i satisfy the necessary conditions of Proposition 1. At the generic step s we will denote by μ_i^s, the prolongation index μ_i and by Δ_i^s the distribution Δ_i.

The algorithm starts with $\mu_1^0 = \mu_2^0 = 0$ and considers the first index k such that dim $\Delta_k^0 = n$, while dim $\Delta_{k-1}^0 < n$, which certainly exists due to the controllability assumption. The generic Step s starts with a given set of prolongation indices $\mu_1^{s-1} \leq \mu_2^{s-1}$. The corresponding distributions Δ_j^{s-1} satisfy the conditions of Proposition 1 for $j \geq k_{s-1}$. The algorithm checks if they are satisfied for $j = k_{s-1} - 1$ also. If not, the prolongation indices are modified accordingly. This is done in two phases:

Phase 1 concerns condition a) of Proposition 1: if $\Delta_{k_{s-1}-1}^{s-1}$ is not involutive, the algorithm computes its involutive closure which is contained in $\Delta_{k_{s-1}}^{s-1}$. By construction, after a possible reordering of the inputs

$$\Delta_{k_{s-1}}^{s-1} = \Delta_{k_{s-1}-1}^{s-1} + \text{span}\{ad_f^{k_{s-1}} g_1, ad_f^{k_{s-1}-\mu_2^{s-1}} g_2\},$$

and $\bar{\Delta}_{k_{s-1}-1}^{s-1} = \Delta_{k_{s-1}-1}^{s-1} \oplus \tau_1^{s-1}$. Assume that $\tau_1^{s-1} = \alpha_i ad_f^{k_{s-1}-\mu_i^{s-1}} g_i$, with $\alpha_i \neq 0$; then the algorithm sets $\Delta_{k_s}^s := \Delta_{k_{s-1}-1}^{s-1} \oplus ad_f^{k_{s-1}-\mu_i^{s-1}} g_i$, which is by construction involutive, and accordingly $\mu_i^s := \mu_i^{s-1} + 1$.

Phase 2 concerns condition c) of Proposition 1. The algorithm checks the condition and adds the necessary elements $ad_f^{\bar{k}_{s-1}-\mu_i^{s-1}} g_i$ in order to satisfy it. This operation changes the prolongation indices which are updated accordingly. As a consequence the operation must be iterated on the new obtained distribution until condition c) is satisfied. The previous operation may in general not preserve the involutivity for the new distributions, so that the algorithm computes the first index r such that $\tilde{\Delta}_{\tilde{k}_s+r} \equiv \Delta_{k_s+r}$.

The algorithm

Suppose that the system is locally controllable around $u = 0$, i.e. dim $\mathcal{G}_{n-1} = n$, and let k be the first integer such that dim $\mathcal{G}_k =$ dim span$\{g, \cdots, ad_f^k g\} = n$ and dim $\mathcal{G}_{k-1} =$ dim span$\{g, \cdots, ad_f^{k-1} g\} < n$. Note that $\mathcal{G}_k = \mathcal{G}_{k-1} +$ span $\{ad_f^k g\}$.

Step **0** Set $\mu_1^0 = \mu_2^0 = 0$, $k_0 = k$ and consider $\Delta_{k_0}^0 = \{g,\ ad_f g, \cdots, ad_f^{k_0} g\}$. By construction $\Delta_{k_0}^0$ satisfies the conditions Proposition 1 since it has constant dimension equal to n.

Step **1** Consider $\Delta_{k_0-1}^0 = \{g,\ ad_f g, \cdots, ad_f^{k_0-1} g\}$ and let $\bar{\Delta}_{k_0-1}^0$ be its involutive closure. Note that by construction $\Delta_{k_0}^0 = \Delta_{k_0-1}^0 + \text{span}\{ad_f^{k_0} g\}$.

Phase 1: Set $p_1 =$ dim $\bar{\Delta}_{k_0-1}^0 -$ dim $\Delta_{k_0-1}^0 \leq 2$. If $p_1 = 0$ ($\Delta_{k_0-1}^0$ is involutive) set $k_1 = k_0 - 1$, $\mu_i^1 = \mu_i^0$ and go to next step; else there exists p_1 vector fields, σ_i, $1 \leq i \leq p_1$, s.t.

$$\bar{\Delta}_{k_0-1}^0 = \Delta_{k_0-1}^0 \oplus \text{span}\{\sigma_i,\ 1 \leq i \leq p_1\} \subseteq \Delta_{k_0}^0.$$

Assume that after a possible reordering of the inputs, $\Delta_{k_1}^1 := \bar{\Delta}_{k_0-1}^0 \equiv \Delta_{k_0-1}^0 + \text{span}\{ad_f^{k_0} g_1\}$ with $\Delta_{k_1}^1$ of constant dimension, so that one gets the following table associated with $\Delta_{k_1}^1$:

	g_1	$ad_f g_1$	$\cdots$	$ad_f^{k_1-1} g_1$	$ad_f^{k_1} g_1$
$\frac{\partial}{\partial \xi_{2,1}}$	g_2	$ad_f g_2$	$\cdots$	$ad_f^{k_1-1} g_2$	

$\Delta_{k_1}^1$

On the left-hand side a column has been added which takes into account the integrators to be set on the second input channel. This is done by imposing that the table second row has the same number of elements of the first row. Accordingly set $\mu_1^1 = 0$ and $\mu_2^1 = 1$.

Step **s** Let $0 = \mu_1^{s-1} \leq \mu_2^{s-1}$ be the prolongation indices computed at Step $s-1$ and consider the associated distribution $\Delta_{k_{s-1}}^{s-1}$,

$$\Delta_{k_{s-1}}^{s-1} = \Delta_{k_{s-1}-\mu_2^{s-1}}^0 + \text{span}\{ad_f^{k_{s-1}-\mu_2^{s-1}+i} g_1, i = 1, \cdots, \mu_2^{s-1}\}$$

to which is associated the following table:

			g_1	$\cdots$		$\cdots$	$ad_f^{k_{s-1}} g_1$
$\frac{\partial}{\partial \xi_{2,\mu_2^{s-1}}}$	$\cdots$	$\frac{\partial}{\partial \xi_{2,1}}$	g_2	$\cdots$	$ad_f^{k_{s-1}-\mu_2^{s-1}} g_2$		

$\Delta_{k_{s-1}}^{s-1}$

Again the left-hand side of the table takes into account the integrators to be added on the second input channel. At each step an integrator is added or eliminated according to the rule that the number of elements of any row must be the same.

Phase 1. Consider now $\Delta_{k_{s-1}-1}^{s-1}$ which is given by

$$\Delta_{k_{s-1}}^{s-1} = \Delta_{k_{s-1}-1}^{s-1} + \text{span}\{ad_f^{k_{s-1}} g_1, ad_f^{k_{s-1}-\mu_2^{s-1}} g_2, k_{s-1} \geq \mu_2^{s-1}\}$$

Note that the table associated with $\Delta_{k_{s-1}-1}^{s-1}$ is simply obtained from the table associated with $\Delta_{k_{s-1}}^{s-1}$ by eliminating the last element of each row.

Let $\bar{\Delta}_{k_{s-1}-1}^{s-1}$ be the involutive closure of $\Delta_{k_{s-1}-1}^{s-1}$ and set $p_s =$ dim $\bar{\Delta}_{k_{s-1}-1}^{s-1} -$ dim $\Delta_{k_{s-1}-1}^{s-1} \leq 2$. If $p_s = 0$ ($\Delta_{k_{s-1}-1}^{s-1}$ is involutive) set $k_s = k_{s-1} - 1$, $\mu_i^s = \mu_i^{s-1}$ and go to Phase 2; else there exist p_s vector fields, σ_i, $1 \leq i \leq p_s$, linear combination of elements of $\Delta_{k_{s-1}}^{s-1}$, such that

$$\bar{\Delta}_{k_{s-1}-1}^{s-1} = \Delta_{k_{s-1}-1}^{s-1} \oplus \text{span}\{\sigma_i,\ 1 \leq i \leq p_s\} \subseteq \Delta_{k_{s-1}}^{s-1}$$

Assume that $\bar{\Delta}_{k_{s-1}-1}^{s-1} = \Delta_{k_{s-1}-1}^{s-1} \oplus \text{span}\{ad_f^{k_{s-1}-\mu_i^{s-1}} g_i\}$ $:= \Delta_{k_s}^s$, with $\Delta_{k_s}^s$ of constant dimension. The associated table is

			g_1	$\cdots$		$\cdots$	$ad_f^{k_s} g_1$
$\frac{\partial}{\partial \xi_{2,\mu_2^s}}$	$\cdots$	$\frac{\partial}{\partial \xi_{2,1}}$	g_2	$\cdots$	$ad_f^{k_s-\mu_2^s} g_2$		

$\Delta_{k_s}^s$

obtained from the table associated with $\Delta_{k_{s-1}-1}^{s-1}$ by adding $ad_f^{k_{s-1}-\mu_i^{s-1}} g_i$. The prolongation indices $0 = \mu_1^s \leq \mu_2^s$, defined by the left-hand-side of the above table, are updated accordingly.

Phase 2. Check condition c) of Proposition 1, i.e. if $\forall (r_1, r_2):\ r_1 + r_2 = k_s - \mu_2^s + 1, \cdots k_s - \mu_2^s + \left\lfloor \frac{\mu_2^s}{2} \right\rfloor$

$$[ad_f^{r_1} g_1, ad_f^{r_2} g_2] \in \Delta_{k_s}^s. \qquad (8)$$

If condition (8) is not satisfied for some (r_1, r_2), i.e. $[ad_f^{r_1} g_1, ad_f^{r_2} g_2] \in \Delta_{k_{s-1}}^{s-1}$, while $[ad_f^{r_1} g_1, ad_f^{r_2} g_2] \notin \Delta_{k_s}^{s}$, then there exists a vector field $\tau_2 = \alpha_1 ad_f^{k_s} g_1 + \alpha_2 ad_f^{k_s-\mu_2^s} g_2$ such that $[ad_f^{r_1} g_1, ad_f^{r_2} g_2] \in \Delta_{k_s}^s \oplus \tau_2$.

Assume that $\tau_2 = \alpha_1 ad_f^{k_s-\mu_i^s} g_i$, $\alpha_1 \neq 0$. Update the table of $\Delta_{k_s}^s$ by adding $ad_f^{k_s-\mu_i^s} g_i$, and the prolongation indices accordingly. Let $0 = \tilde{\mu}_1^s \leq \tilde{\mu}_2^s$ be the new prolongation indices, and let

$$\tilde{\Delta}_{\tilde{k}_s}^s = \Delta_{k_s}^s \oplus \operatorname{span}\{ad_f^{k_{s-1}-\mu_i^s} g_i\}$$

be the associated distribution. Let r be the first index such that $ad_f^r ad_f^{k_{s-1}} g_1 \in \Delta_{\tilde{k}_s+r}^s$. Set $\hat{k}_s = \tilde{k}_s + r$, rename $\hat{k}_s$ as k_s and go back to Phase 2 of Step s.

Step $\mathbf{k}^*$ Suppose that the algorithm ends with prolongation indices $0 = \mu_1 \leq \mu_2$. Consider the extended system obtained by adding the dynamic compensator (2) with the above prolongation indices. If $\mu_2 \leq 2$ stop else apply the algorithm on the new system.

Note that at Step s, Phase 1, it is assumed that if $\Delta_{k_{s-1}}^{s-1}$ is not involutive then $\bar{\Delta}_{k_{s-1}}^{s-1} = \Delta_{k_{s-1}}^{s-1} \oplus \tau_1^{s-1}$ with $\tau_1^{s-1} = \alpha_1 ad_f^{k_{s-1}} g_1$. The more general case when $\tau_1^{s-1} = \alpha_1 ad_f^{k_{s-1}} g_1 + \alpha_2 ad_f^{k_{s-1}-\mu_2^{s-1}} g_2$, $\alpha_1, \alpha_2 \neq 0$, can be handled as follows: if $\mu_2^{s-1} = 0$ set $\tilde{g}_1 = g_1 + \frac{\alpha_2}{\alpha_1}\tilde{g}_2$, and replace accordingly $ad_f^i g_1$ by $ad_f^i \tilde{g}_1$. Add to the table the element $ad_f^{k_{s-1}} \tilde{g}_1$. Instead if $\mu_2^{s-1} > 0$ set, μ_2^{s-1} integrators on the second input channel and iterate the algorithm on the new system. Note also that if the system is static feedback equivalent to a linear system the algorithm ends with prolongation indices $\mu_1 = \mu_2 = 0$.

The next result states that at the generic step s the algorithm computes a partition of the inputs and prolongation indices $0 = \mu_1^s \leq \mu_2^s$ with respect to which the associated distributions Δ_i^s for $i \geq k_s$ satisfy the necessary conditions of Proposition 1. If the algorithm ends and $\mu_2 \leq 2$ then these conditions are also sufficient (Theorem 3) and the dynamic compensator defined by the given prolongation indices solves the problem. Else one must iterate the procedure on the extended system. In this case the final compensator may not necessarily have $\mu_1 = 0$.

Theorem 4 Suppose that Step s ends with prolongation indices $0 = \mu_1^s \leq \mu_2^s$. Then $\Delta_{k_s+i}^s$ is involutive and satisfies the conditions of Proposition 1 $\forall i \geq 0$.

The following example is issued from [6] where it was used to show that it didn't satisfy the sufficient conditions of Theorem 2 though dynamic feedback linearizable with prolongation indices $\mu_1 = 0$, $\mu_2 = 3$. We will show how the proposed algorithm computes the above prolongation indices, thus solving the problem.

Example. Consider the continuous time system

$$\begin{aligned} \dot{x}_1 &= x_2 + x_3 u_2 \\ \dot{x}_2 &= x_3 + x_1 u_2 \\ \dot{x}_3 &= u_1 + x_2 u_2 \\ \dot{x}_4 &= u_2 \end{aligned}$$

Step **0**. Set $\mu_1^0 = \mu_2^0 = 0$ and compute the distributions $\Delta_0^0, \cdots \Delta_3^0$ which are respectively

$$\Delta_0^0 = \operatorname{span}\left\{\frac{\partial}{\partial x_3}, x_3\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\right\}$$

$$\Delta_1^0 = \Delta_0^0 + \operatorname{span}\left\{-\frac{\partial}{\partial x_2}, -x_1\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_3}\right\},$$

$$\Delta_2^0 = \Delta_1^0 + \operatorname{span}\left\{\frac{\partial}{\partial x_1}\right\} = \mathbb{R}^4$$

Locally around the origin $\dim(\Delta_2^0) = 4$, whereas the origin is a singular point for Δ_1^0.

Step 1. [Phase 1] $\Delta_2^0 = \Delta_1^0 + ad_f^2 g_1$ and $[g_1, g_2] = ad_f^2 g_1$, so that Δ_1^0 is not involutive and we must compute $\bar{\Delta}_1^0$. We have $\bar{\Delta}_1^0 = \Delta_1^0 + \operatorname{span}\left\{\frac{\partial}{\partial x_1}\right\} := \Delta_2^1$,

	g_1	$ad_f g_1$	$ad_f^2 g_1$
$\frac{\partial}{\partial \zeta_{21}}$	g_2	$ad_f g_2$	

(columns g_1 to $ad_f^2 g_1$: Δ_2^1) with $\begin{cases} \mu_1^1 = 0 \\ \mu_2^1 = 1 \end{cases}$

Step 2. [Phase 1] Consider $\Delta_1^1 = \{g_1, g_2, ad_f g_1\}$. We have that $\bar{\Delta}_1^1 = \Delta_0 + \operatorname{span}\left\{-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\right\} := \Delta_2^2$,

		g_1	$ad_f g_1$	$ad_f^2 g_1$
$\frac{\partial}{\partial \zeta_{22}}$	$\frac{\partial}{\partial \zeta_{21}}$	g_2		

(columns g_1 to $ad_f^2 g_1$: Δ_2^2) with $\begin{cases} \mu_1^2 = 0 \\ \mu_2^2 = 2 \end{cases}$

[Phase 2] Condition c) of Proposition 1 is satisfied, since $[ad_f g_2, g_1] = g_1 \in \Delta_2^2$. We go to the next step.

Step 3. [Phase 1] Compute $\Delta_1^2 = \{g_1, ad_f g_1\}$ which is involutive so that $\bar{\Delta}_1^2 \equiv \Delta_1^2 := \Delta_1^3$,

		g_1	$ad_f g_1$
$\frac{\partial}{\partial \zeta_{22}}$	$\frac{\partial}{\partial \zeta_{21}}$		

(columns g_1 to $ad_f g_1$: Δ_1^3) with $\begin{cases} \mu_1^3 = 0 \\ \mu_2^3 = 2 \end{cases}$

[Phase 2] Condition c) of Proposition 1 is not satisfied, since $[g_2, g_1] \notin \Delta_1^3$. As $[g_2, g_1] \in \Delta_1^3 \oplus \text{span}\{ad_f^2 g_1\}$, we set $\tilde{\Delta}_2^3 = \Delta_1^3 \oplus \text{span}\left\{\frac{\partial}{\partial x_1}\right\}$,

			g_1	$\cdots$	$ad_f^2 g_1$
$\frac{\partial}{\partial \zeta_{23}}$	$\cdots$	$\frac{\partial}{\partial \zeta_{21}}$			

(the last three columns form $\tilde{\Delta}_2^3$) with $\begin{cases} \tilde{\mu}_1^3 = 0 \\ \tilde{\mu}_2^3 = 3 \end{cases}$

Condition c) of Proposition 1 is satisfied for the new distribution $\tilde{\Delta}_2^3$, with the new prolongation indices. In fact $r_1 + r_2 = 0$ and $[g_2, g_1] \in \tilde{\Delta}_2^3$.

According to the algorithm we must now compute the first index r such that $ad_f^r[g_2, g_1] = ad_f^{r+2} g_1 \in \Delta_{1+r}^3$. In the present case, $r = 1$ since $ad_f^3 g_1 = 0$. We thus set $k_3 = 2$ and set $\Delta_2^3 = \tilde{\Delta}_2^3$. The new distribution Δ_2^3 is characterized by $\rho = 0$ and satisfies condition c) so that we can go to next step.

Step 4. [Phase 1] Compute $\Delta_1^3 = \{g_1, ad_f g_1\}$ which is involutive, so that $\bar{\Delta}_1^3 = \Delta_1^3 := \Delta_1^4$,

			g_1	$ad_f g_1$
$\frac{\partial}{\partial \zeta_{23}}$	$\frac{\partial}{\partial \zeta_{22}}$			

(the last two columns form Δ_1^4) with $\begin{cases} \mu_1^4 = 0 \\ \mu_2^4 = 3 \end{cases}$

[Phase 2] Condition c) of Proposition 1 is trivially satisfied, s.t. we go to next step

Step 5. [Phase 1] compute Δ_0^4 which is involutive so that $\bar{\Delta}_0^4 = \Delta_0^4 := \Delta_0^5$,

			g_1
$\frac{\partial}{\partial \zeta_{23}}$			

(the last column forms Δ_0^5) with $\begin{cases} \mu_1^5 = 0 \\ \mu_2^5 = 3 \end{cases}$

[Phase 2] Condition c) of Proposition 1 is trivially satisfied. The algorithm ends with $\mu_1 = \mu_1^5 = 0$ and $\mu_2 = \mu_2^5 = 3$, the searched prolongation indices. $\triangleleft$

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