ROBUST STABILITY ANALYSIS OF SIMPLE CONTROL ALGORITHMS IN COMMUNICATION NETWORKS *

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Abstract

There is increasing interest in the control of communication networks. This paper reveals that a simple method, the dual-locus diagram method, is very effective in analyzing the robust stability of simple time-delay systems, which are often met in communication systems. As to a single connection studied in [15], the same results as those obtained by Niculescu have been obtained, but in a much simpler and clearer graphical way. As to the web-based control system studied in [2], the results have been extended to a much broader class of systems, i.e. from a line in the parameter plane to the whole plane. The stability region is divided into a delay-dependent stability region and a delay-independent stability region, which offers a nice graphical view on the conservativeness of the delay-independent stability criteria.

1 Introduction

In recent years, there is increasing interest in the control of communication networks [23] because communication networks are among the fastest-growing areas in engineering. Thanks to high-speed networks, control-via-Internet is now available [20, 11]. These controlled communication networks and/or the systems controlled via network are frequently modeled from the control point of view as time-delay systems because of the inherent propagation delays, see, for example, [9, 12]. These delays are crucial to the stability of the congestion control and the quality-of-service (QoS). As is well known, the presence of delays makes the control design and system analysis much more complicated. For details on the control of communication networks, see [9, 12, 19, 4] and the references therein. In this paper, we focus on the stability analysis of the communication networks, specifically, the systems studied in two very recent papers [15, 2].

The robust stability analysis of time-delay systems is not well established and has become a very active research field in recent years. Current efforts can be divided into two categories: delay-dependent stability criteria [5, 6, 13, 18] and delay-independent stability criteria [10]. Although delay-dependent stability criteria are in general less conservative than delay-independent criteria, they may still be quite conservative. One reason is that delay-dependent stability criteria were frequently obtained by using a model transformation, which introduces additional dynamics [7, 8]. An interesting case where the delay-independent stability criteria are not conservative at all will be shown.

The system studied in [15] is a high-speed network controlled by a simple proportional control algorithm. More specifically, it is a single connection between a source node (controlled by an access regulator) and a distant node. The system model derived in [9] and treated in [14] can be described by the second-order delay differential equation:

$$\text{System I}: \quad \ddot{y}(t) + ay(t-\tau) + by(t-\tau-r) = 0,$$

where $y(t)$ is the congestion status of the remote node (if $y(t) > 0$, then the remote node is said to be congested), τ is the round-trip delay (equal to the sum of the forward propagation delay and the backward propagation delay), r is the control interval artificially introduced to stabilize the system, and a and b are proportional control gains. The stability criteria proposed there are very elegant. However, the reasoning is not very transparent and is difficult to follow.

The system studied in [2] is a special case of a mass-spring-

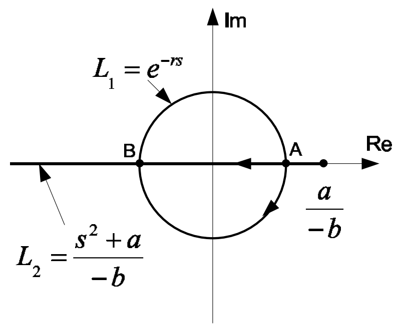
*An abridged and expanded version of this paper can be found in [25].

damper system controlled over the network using a simple proportional controller. The system can be described by the second-order delay differential equation:

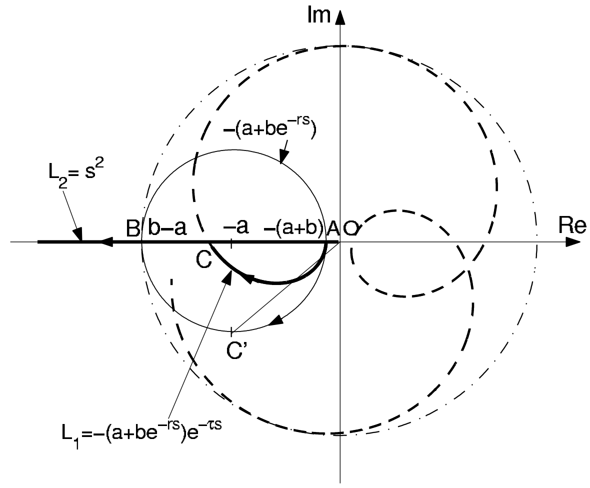
$$\text{System II}: \quad \ddot{y}(t) + 2\zeta\alpha\dot{y}(t) + \alpha^2 y(t) - K_p y(t-\tau) = 0,$$

where $y(t)$ is the position of the mass, $\zeta > 0$ is the damping ratio, $\alpha > 0$ is the natural frequency, $K_p > 0$ is the proportional control gain and $\tau > 0$ is the network communication delay. The stability bound is obtained by using the *Lambert W* function [3, 26, 1]. The approach is effective for high-order systems with a delay as well. However, the assumption $\zeta = 1$ (equivalent to the condition $b_m = \frac{a_m^2}{4}$ in the paper) or the assumption having repeating poles for high-order systems considerably limits the applicability of the results.

In this paper, the stability analysis of these two systems is reconsidered using a very simple method — the dual-locus diagram (also called Satche diagram) method [21, 22, 17, 24], which is an extension or variant of the well-known Nyquist diagram [16, 25]. It is very effective in analyzing the stability of simple time-delay systems. The advantages consist in the simplicity of the approach and the easy understanding of the reasoning, as demonstrated in Section 2, where the stability of System I is analyzed. The stability of System II is analyzed in Section 3, where the assumption $\zeta = 1$ has been removed and thus the result can be applied to systems having various ζ and/or α. The stability region of the system is divided into a delay-dependent stability region and a delay-independent stability region on the parameter plane $\frac{K_p}{\alpha^2} - \zeta$. This offers a nice graphical view on the conservativeness of the delay-independent stability criteria: the delay-independent stability criteria are *not* conservative *at all* when $\zeta \geq \frac{1}{\sqrt{2}}$.



(a) $\tau = 0$



(b) $\tau \geq 0$

Figure 1: The dual-locus diagram of System I

2 Stability analysis of System I

The assumptions in [15], *i.e.* $b < 0$ and $a > |b|$, are retained to reduce the length of this paper.

2.1 Case 1: Round-trip delay $\tau = 0$

In this case, System I can be reformulated as

$$e^{-rs} = \frac{s^2 + a}{-b}.$$

The corresponding dual-locus diagram is shown in Figure 1(a). In this paper, in order to simplify the exposition, the bounded locus is denoted by L_1 and the unbounded one by L_2. When ω increases from 0 to $+\infty$, locus $L_1 = e^{-rs}$ is the clockwise unity circle starting at $(1, 0)$ and locus $L_2 = \frac{s^2+a}{-b}$ is a straight line originating at $(\frac{a}{-b}, 0)$, which is at the right side of the unity circle, and extending to $(-\infty, 0)$. L_2 intersects with L_1 at points A and B, at which the corresponding frequencies on L_2 are denoted by ω_A and ω_B, respectively. Using magnitude conditions, ω_A and ω_B can be obtained as

$$\omega_A = \sqrt{a+b} \text{ and } \omega_B = \sqrt{a-b}. \quad (1)$$

In order to guarantee the stability of the system, L_2 (the locus which approaches ∞ faster) should arrive at B earlier than L_1 according to the dual-locus diagram method [22, 24]. This means that the phase shift of e^{-rs} due to ω_B should be larger than $-\pi$ (*i.e.* $-r\omega_B > -\pi$), in other words,

$$0 < r < \frac{\pi}{\sqrt{a-b}}.$$

This is the major branch of the delay stability bound. As a matter of fact, if the control interval r is large, L_1 may have already traveled several cycles along the unity circle before L_2 arrives at A. In this case, in order to guarantee the stability of the system, L_2 should have traveled from A to B while L_1 is still traveling from A to B during the same cycle, *i.e.* to guarantee that L_2 lies at the right side of L_1[22, 24]. This means that

$$\begin{cases} -r\omega_A < -2i\pi \\ -r\omega_B > -2i\pi - \pi \end{cases} \quad or \quad \frac{2i\pi}{\sqrt{a+b}} < r < \frac{(2i+1)\pi}{\sqrt{a-b}},$$

where i is the traveled cycles of L_1 before L_2 arrives at A. This condition is exactly the same as Proposition 1 in [15], but quite easily obtained here. It is worth noting that the admissible i satisfies the following condition:

$$0 \leq i < \frac{0.5}{\sqrt{\frac{a-b}{a+b}} - 1}.$$

Hence, there only exists the major delay stability bound to guarantee the system stability if $-b < a < -2.6b$.

2.2 Case 2: Round-trip delay $\tau \geq 0$

In this case, System I can be described as

$$-(a + be^{-rs})e^{-\tau s} = s^2.$$

The corresponding dual-locus diagram is shown in Figure 1(b). When ω increases from 0 to $+\infty$, locus $L_2 = s^2$ is a straight line originating at $(0, 0)$ and extending to $(-\infty, 0)$. Locus $L_1 = -(a+be^{-rs})e^{-\tau s}$, starting at $A(-(a+b), 0)$, can be obtained by rotating the points on the circle $(-a, 0, |b|)$ clockwise with respect to the origin $O(0, 0)$ by an angle of $\omega\tau$. Hence, L_2 never exceeds the circle $(0, 0, a - b)$. The corresponding frequencies at A and B on L_2 are the same as those in the case when $\tau = 0$ given in (1) and independent of the round-trip delay τ.

The following sufficient condition for the system stability can be obtained with ease [22, 24]: If L_2 arrives at B before L_1 intersects with L_2, then the system is stable. The latter part of this condition can be divided into two sub-conditions:

(i) $-(a + be^{-rs})$ is still traveling from A to B, say, at C';

(ii) L_1 has not arrived at the would-be intersection C on the negative real axis.

At the would-be intersection C (of which the corresponding frequency is denoted by ω_C), the following magnitude condition is satisfied:

$$\omega_C^2 = \sqrt{(a + b\cos(r\omega_C))^2 + (b\sin(r\omega_C))^2}.$$

This is equivalent to

$$\omega_C^4 = a^2 + b^2 + 2ab\cos(r\omega_C).$$

As can be seen in Figure 1(b), the second sub-condition means that the phase shift (absolute value) due to $e^{-\tau s}$ plus the phase angle of point C' should be less than π, *i.e.*

$$\tau\omega_C + (\pi - \arctan\frac{-b\sin(r\omega_C)}{a + b\cos(r\omega_C)}) < \pi,$$

while the first sub-condition means that

$$-r\omega_B > -\pi.$$

These two conditions can then be represented as

$$\begin{cases} 0 < r < \frac{\pi}{\sqrt{a-b}} \\ 0 \leq \tau < \frac{1}{\omega_C}\arctan\frac{-b\sin(r\omega_C)}{a+b\cos(r\omega_C)} \end{cases}.$$

This is the same as the major delay bound given by Proposition 2 in [15], but easily obtained here.

As can be seen in Figure 1(b), there always exists an intersection C between A and B, which means there always exists a solution $\omega_C > 0$. Moreover, C moves towards A when the control interval r decreases and moves towards B when the round-trip delay τ decreases.

The other possible branches can be obtained in a similar way and are thus omitted in this paper.

3 Stability analysis of System II

System II studied in [2] can be represented in s-domain as

$$e^{-\tau s} = \frac{s^2 + 2\zeta\alpha s + \alpha^2}{K_p},$$

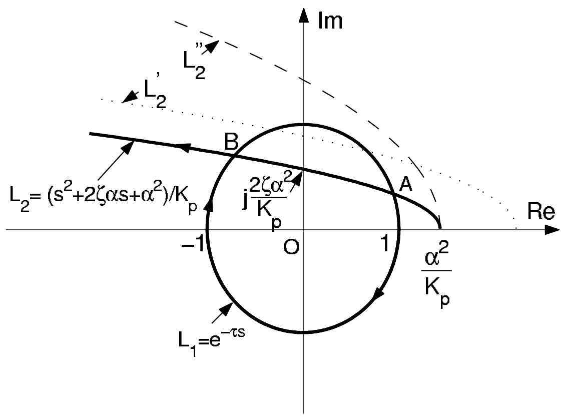
of which the dual-locus diagram is shown in Figure 2(a). When ω increases from 0 to $+\infty$, locus $L_1 = e^{-\tau s}$ is the clockwise unity circle starting at $(1, 0)$ and locus $L_2 = \frac{s^2+2\zeta\alpha s+\alpha^2}{K_p}$ is a parabola originating at $(\frac{\alpha^2}{K_p}, 0)$, which is at the right side of the unity circle, and extending towards the left. It is assumed that $\zeta > 0$, $\alpha > 0$ and $K_p > 0$ as in [2] but ζ is not limited to 1.

When decreasing K_p and/or increasing α, L_2 (denoted as L_2' in Figure 2(a)) moves towards the outside of the unity circle in parallel and, when $\frac{\alpha^2}{K_p}$ is large enough, no longer intersects with L_1. Hence, for some large $\frac{\alpha^2}{K_p}$, L_2 always stays at the right side of L_1 and the system is delay-independently stable. When increasing the damping ratio ζ, the intersection of L_2 (denoted as L_2'' in Figure 2(a)) with the imaginary axis moves up, but the starting point of L_2 remains still. Hence, the system is delay-independently stable for large ζ.

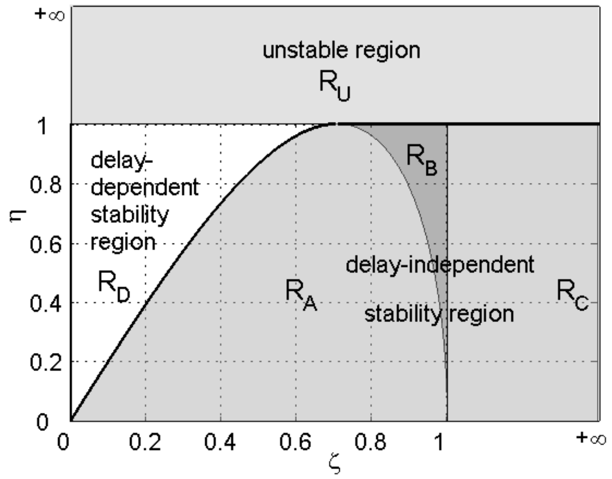
If $K_p > \alpha^2$, then the starting point $(\frac{\alpha^2}{K_p}, 0)$ of L_2 lies inside L_1 and the system is not stable because the total rotation of the spider-web vector is $-360°$ [22]. Hence, the proportional gain guaranteeing the system stability is limited by α^2, *i.e.* $K_p < \alpha^2$. In order to simplify later expositions, the proportional control gain K_p is normalized as

$$\eta = \frac{K_p}{\alpha^2},$$

and then it is assumed that $0 < \eta < 1$ in the sequel.



(a) the dual-locus diagram



(b) the stability region

Figure 2: Analysis of System II

The parabola L_2 may have two intersections or no intersection with L_1, corresponding to the solution condition of the following equation with respect to ω:

$$\frac{(\alpha^2-\omega^2)^2+(2\zeta\alpha\omega)^2}{K_p^2}=1.$$

Assuming that there exist two positive solutions ω_A and ω_B, which are actually the corresponding frequencies on L_2 at the two intersections A and B respectively, then ω_A and ω_B can be solved from the last equation as

$$\omega_A=\alpha\sqrt{1-2\zeta^2-\sqrt{\eta^2-4\zeta^2+4\zeta^4}}, \tag{2}$$

$$\omega_B=\alpha\sqrt{1-2\zeta^2+\sqrt{\eta^2-4\zeta^2+4\zeta^4}}. \tag{3}$$

The conditions on the existence of ω_A and ω_B will now be analyzed and the delay-dependent and/or delay-independent stability criteria will be developed as follows:

(i) If $\zeta \geq 1$, then $\eta^2-4\zeta^2+4\zeta^4$ is always positive. However, $1-2\zeta^2-\sqrt{\eta^2-4\zeta^2+4\zeta^4} < 0$ and $1-2\zeta^2+\sqrt{\eta^2-4\zeta^2+4\zeta^4} < 0$ for any $0<\eta<1$. Hence, either ω_A or ω_B does not exist (hereafter, "to exist" means the existence of a *positive* solution) and the system is delay-independently stable. The corresponding stability region is denoted as R_C in Figure 2(b).

(ii) If $0<\zeta<1$ and $\eta<2\zeta\sqrt{1-\zeta^2}$, then $\eta^2-4\zeta^2+4\zeta^4<0$. Either ω_A or ω_B does not exist and the system is delay-independently stable. This stability region is denoted as R_A in Figure 2(b). When $\zeta=\frac{1}{\sqrt{2}}$ the delay-independent stability region reaches the maximum because L_2 never intersects with L_1 and the system is stable for any $0<\eta<1$.

(iii) If $0<\zeta<1$ and $\eta \geq 2\zeta\sqrt{1-\zeta^2}$ (and $\eta<1$ by assumption), $\eta^2-4\zeta^2+4\zeta^4 \geq 0$. Either ω_A or ω_B does not exist when $\frac{1}{\sqrt{2}}<\zeta<1$ but both exist when $0<\zeta<\frac{1}{\sqrt{2}}$. Hence, when $0<\zeta<\frac{1}{\sqrt{2}}$ and $\eta \geq 2\zeta\sqrt{1-\zeta^2}$, there are two intersections. As can be seen later, this provides the unique delay-dependent stability region, denoted as R_D in Figure 2(b); when $\frac{1}{\sqrt{2}}<\zeta<1$ and $\eta \geq 2\zeta\sqrt{1-\zeta^2}$, there is no intersection and the system is delay-independently stable. This stability region is denoted as R_B in Figure 2(b).

It is trivial that the system is stable when $K_p=0$ because the open-loop system is stable. The $\zeta-\eta$ plane shown in Figure 2(b) is then divided into an unstable region (R_U), a delay-dependent stability region (R_D) and a delay-independent stability region (including R_A, R_B and R_C). This offers a nice view on the conservativeness of the delay-independent stability criteria. For $\frac{1}{\sqrt{2}}<\zeta<1$, the delay-independent region consists of two complementary parts: R_B and the right portion of R_A. When $\zeta \geq \frac{1}{\sqrt{2}}$, the delay-independent stability criteria are *not* conservative *at all*: the system is stable for all possible gains in $0<K_p<\alpha^2$.

In region R_D, $0<\zeta<\frac{1}{\sqrt{2}}$ and $2\zeta\sqrt{1-\zeta^2} \leq \eta < 1$, L_2 intersects with L_1 at points A and B. The system is stable if L_2 arrives at B before L_1 [22, 24]. In other words, the phase shift of L_1 should be less than the phase angle of point B on L_2. This provides the major delay bound as

$$0 \leq \tau < \frac{1}{\omega_B}\left(\frac{3\pi}{2}+\arctan\frac{\alpha^2-\omega_B^2}{2\zeta\alpha\omega_B}\right).$$

Similarly as in the previous section, L_1 may have already traveled several cycles along the unity circle before L_2 arrives at A, then the following condition is required:

$$\begin{cases} \tau\omega_A > -\frac{\pi}{2}+\arctan\frac{\alpha^2-\omega_A^2}{2\zeta\alpha\omega_A}+2i\pi \\ \tau\omega_B < \frac{3\pi}{2}+\arctan\frac{\alpha^2-\omega_B^2}{2\zeta\alpha\omega_B}+2i\pi \end{cases}.$$

This provides the following theorem:

Theorem 1. *If $0<\zeta<\frac{1}{\sqrt{2}}$ and $2\zeta\sqrt{1-\zeta^2} \leq \eta < 1$, System II is delay-*dependently *stable. The stability delay bounds ($\tau \geq 0$) are given by*

$$\frac{1}{\omega_A}\left(-\frac{\pi}{2}+\arctan\frac{\alpha^2-\omega_A^2}{2\zeta\alpha\omega_A}+2i\pi\right)<\tau<\frac{1}{\omega_B}\left(\frac{3\pi}{2}+\arctan\frac{\alpha^2-\omega_B^2}{2\zeta\alpha\omega_B}+2i\pi\right)$$

where $i = 0, 1, 2, ...$, *until the right side is no longer larger than the left, and* ω_A *and* ω_B *are given in (2) and (3) respectively.*

The delay-independently criteria can be summarized as:

Theorem 2. *System II is delay*-independently *stable: (i) for* $0 < K_p < \alpha^2$ *if* $\zeta \geq \frac{1}{\sqrt{2}}$, *(ii) for* $0 < K_p < 2\zeta\alpha^2\sqrt{1-\zeta^2}$ *if* $0 < \zeta < \frac{1}{\sqrt{2}}$.

Remarks:

(i) The damping ratio ζ is a crucial parameter for the system stability. The larger the damping ratio ζ, the better the stability (but, of course, the slower the system). If ζ is too small, then the delay-independent stability region is very narrow. For example, the damping ratio in System I is 0. Hence, there is no delay-independent stability region and the control interval r is (very likely, has to be) artificially introduced to stabilize the system.

(ii) The network communication delays impose very strict limitations on the system performance. The control gain is considerably limited (although the system may still be delay-independently stable). This is one of the reasons why control-via-Internet requires a reliable high-speed communication network.

(iii) When $0 < \zeta < \frac{1}{\sqrt{2}}$, the delay-dependent stability criteria offer a larger control gain and, hence, a better dynamic performance but the allowable communication delays are limited. On the other hand, the delay-independent stability criteria offer a smaller control gain but allow a broad range of communication delays (theoretically, $0 \sim +\infty$). Hence, the compromise between the network requirements and the control performance requirements is necessary when designing a system to be controlled via network. The delay-dependent stability criteria are suitable for the control over a high-performance communication network, e.g. a local area network (LAN), and the delay-independent stability criteria are suitable for the control over a less reliable and/or low speed communication network, e.g. the Internet.

4 Conclusions

In this paper, the stability analysis of two time-delay systems related to communication networks are re-considered using a simple method — the dual-locus diagram method. Both delay-independent and delay-dependent stability criteria for a mass-spring-damper system which is controlled via Internet are derived. A nice graphical view on the conservativeness of the delay-independent stability criteria is obtained. In some cases ($\zeta \geq \frac{1}{\sqrt{2}}$), the delay-independent stability criteria are not conservative at all. It is revealed that the dual-locus diagram method is very effective in the robust stability analysis of simple time-delay systems.

Acknowledgments

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