SYNTHESIS OF ANTI-WINDUP LOOPS FOR ENLARGING THE STABILITY REGION OF TIME-DELAY SYSTEMS WITH SATURATING INPUTS

J.M. Gomes da Silva Jr[a], S. Tarbouriech[b], G.Garcia[b,*]

a. UFRGS - Department of Electrical Engineering,
Av. Osvaldo Aranha 103, 90035-190 Porto Alegre-RS, Brazil.
E-mail: jmgomes@eletro.ufrgs.br

b. LAAS-CNRS,
7 Avenue du Colonel Roche, 31077 Toulouse cedex 4, France.
E-mail: tarbour@laas.fr, garcia@laas.fr

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Abstract

This paper focus on the study and the characterization of regions of stability for linear systems with delayed states and subject to input saturation through anti-windup strategies. In particular, the synthesis of anti-windup gains in order to guarantee the stability of the closed-loop system for a region of admissible initial states as large as possible is addressed. Based on the modeling of the closed-loop system, resulting from the controller plus the anti-windup loop, as a linear time-delay system with a deadzone nonlinearity, stability conditions are stated for both the delay independent and delay dependent contexts by using quadratic functionals.

1 Introduction

In the last few years, the study of systems presenting time-delays has received a special attention in the control systems literature. This interest comes from the fact that time-delays appear in many kinds of control systems (e.g. chemical, mechanical and communication systems) and their presence can be source of performance degradation and instability. In this sense, we can found in the literature many works giving conditions for ensuring stability as well as performance and robustness requirements, considering or not the delay dependence. Concerning the delay independent results, the stability is ensured no matter the size of the delay [4], [12], [18]. On the other hand, in the delay dependent results, the size of the delay is directly taken into account and this fact can lead to less conservative results [2], [5].

Since physical actuators cannot deliver unlimited signals to the controlled plants, the problem of control saturation and its impact on the stability and the performance of the closed-loop system has also received a lot of attention in the last years. The studies on the analysis and controller design problems for linear systems with input saturation have followed two main approaches. In the first one, the effects of the saturation are directly taken into account in the design of the control law. We can identify methods dealing with the stabilization of the closed-loop system in global, semi-global and local contexts (see among others [19], [13], [6]). The second approach assumes that a controller was previously designed, in order to guarantee some performances. The effects of the saturation on the stability and the performance of the closed-loop system are then considered *a posteriori*. The anti-windup technique fits in this last approach as it consists in introducing control modifications in order to recover, as much as possible, the performance induced by a previous design carried out on the basis of the unsaturated system (see, for example, [1, 9, 11, 21]). It should be pointed out that several results on the anti-windup problem are concerned with achieving global stability properties [9, 14]. Since global results cannot be achieved for open-loop unstable linear systems in the presence of actuator saturation, local results have to be developed. In this context, a key issue is the determination of domains of stability for the closed-loop system, i.e., sets of admissible initial states for which the asymptotic convergence of the corresponding trajectories to the origin is ensured. On the other hand, it should be highlighted that these global stability results do not consider the case of systems with time-delays.

Considering that many practical systems present both time-delays and saturating inputs, from the considerations above, it becomes important to study the stability issues regarding this kind of systems. In this sense, we can identify in the literature some results proposed mainly in the context of the stabilization via state feedback for systems with delays in the state. In [16], a globally stabilizing state observer based controller is proposed. In [3], [15] and [22], conditions of stability or stabilization are proposed with state feedback and sampled state feedback. However, in these papers, the set of admissible initial conditions for which the asymptotic stability is ensured in the presence of control saturation is not mentioned or explicitly defined. In [20], considering the synthesis of state feedback control laws, it was underlined the importance of describing a set of admissible initial conditions associated to the stabilizing

*This author is also with INSA, 135 Avenue de Rangueil, 31077 Toulouse cedex 4, France.

control law. On the other hand, considering an anti-windup approach we can cite [17]. In that paper, it is proposed a dynamic anti-windup method for linear systems with control input delays and output measurement delays that ensures bounded input - bounded state stability. It should be highlighted that this method cannot be applied to open-loop unstable systems.

The objective of this paper is the study and the characterization of regions of stability for linear systems with delayed states and subject to input saturation through anti-windup strategies. Differently of the most anti-windup techniques cited above, where the synthesis of the anti-windup loop is introduced with the objective of minimizing the performance degradation, we are particularly interested in the synthesis of anti-windup gains in order to guarantee the stability of the closed-loop system for a region of admissible initial states as large as possible. With this aim we propose results both in the delay independent and dependent contexts. These results allow to formulate optimization problems in order to search anti-windup gains that lead to the maximization of the size of the region of stability associated to the closed-loop system.

Notations. $x_{(i)}$, $A_{(i)}$ and $A_{(i,j)}$ denotes respectively: the ith entry of vector x, the ith row of matrix A and the element (i,j) of A. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. A' denotes the transpose of A. I_m denotes the m-order identity matrix. $\lambda_{max}(P)$ and $\lambda_{min}(P)$ denote respectively the maximal and minimal eigenvalues of matrix P. $\mathcal{C}_\tau = \mathcal{C}([-\tau,0],\Re^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau,0]$ into $\Re^n$ with the topology of uniform convergence. $\|\cdot\|$ refers to either the Euclidean vector norm or the induced matrix 2-norm. $\|\phi\|_c = \sup_{-\tau\le t\le 0} \|\phi(t)\|$ stands for the norm of a function $\phi \in \mathcal{C}_\tau$. When the delay is finite then "sup" can be replaced by "max". $\mathcal{C}_\tau^v$ is the set defined by $\mathcal{C}_\tau^v = \{\phi \in \mathcal{C}_\tau \,;\, \|\phi\|_c < v,\ v > 0\}$.

2 Problem Statement

Consider the linear continuous-time delay system:

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_d x(t-\tau) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \right. \quad (1)$$

with the initial conditions

$$x(t_0+\theta) = \phi_x(\theta), \forall\theta \in [-\tau,0],\ t_0, \phi_x \in \Re_+ \times \mathcal{C}_\tau^v \quad (2)$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, the input and the measured output vectors, respectively. Matrices A, A_d, B and C are real constant matrices of appropriate dimensions. We suppose also that the input vector u is subject to amplitude limitations defined as follows:

$$\mathcal{U}_0 = \{u \in \Re^m; |u_{(i)}| \le u_{0(i)},\ i = 1,...,m\} \quad (3)$$

Considering system (1), assume therefore that an n_c-order dynamic output stabilizing compensator

$$\begin{array}{l} \dot{\eta}(t) = A_c\eta(t) + B_c y(t) \\ y_c(t) = C_c\eta + D_c y(t) \end{array} \quad (4)$$

where $\eta(t) \in \Re^{n_c}$ is the controller state, $u_c(t) = y(t) \in \Re^p$ is the controller input and $y_c(t) \in \Re^m$ is the controller output, has been designed in order to guarantee some performance requirement and the stability of the closed-loop system in the absence of the control saturation. Matrices A_c, B_c, C_c et D_c are of appropriate dimensions. In consequence of the control bounds, the control signal to be injected in the system is a saturated one:

$$u(t) = sat(y_c(t)) = sat(C_c\eta(t) + D_c Cx(t)) \quad (5)$$

where each component of $sat(y_c)$ is defined, $\forall i = 1,...,m$, as

$$sat(y_c)_{(i)} = sat(y_{c(i)}) = sign(y_{c(i)}) \min(|y_{c(i)}|, u_{0(i)}) \quad (6)$$

In order to mitigate the undesirable effects of windup, caused by input saturation, an anti-windup term $E_c(sat(y_c(t)) - y_c(t))$, $E_c \in \Re^{n_c \times m}$, can be added to the controller [9]. Thus, considering the dynamic controller and this anti-windup strategy, the closed-loop system reads:

$$\begin{array}{l} \dot{x}(t) = Ax(t) + A_d x(t-\tau) + Bsat(y_c(t)) \\ y(t) = Cx(t) \\ \dot{\eta}(t) = A_c\eta(t) + B_c y(t) + E_c(sat(y_c(t)) - y_c(t)) \\ y_c(t) = C_c\eta(t) + D_c y(t) \end{array} \quad (7)$$

Define now an extended state vector $\xi(t) = [x(t)' \quad \eta(t)']' \in \Re^{n+n_c}$ and the following matrices: $\mathbb{A} = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}$; $\mathbb{A}_d = \begin{bmatrix} A_d & 0 \\ 0 & 0 \end{bmatrix}$; $\mathbb{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$; $\mathbb{R} = \begin{bmatrix} 0 \\ I_{n_c} \end{bmatrix}$; $\mathbb{K} = \begin{bmatrix} D_cC & C_c \end{bmatrix}$ Hence, the closed-loop system reads:

$$\dot{\xi}(t) = \mathbb{A}\xi(t) + \mathbb{A}_d\xi(t-\tau) - (\mathbb{B} + \mathbb{R}E_c)\psi(\mathbb{K}\xi(t)) \quad (8)$$

with $\psi(\mathbb{K}\xi(t)) = y_c(t) - sat(y_c(t)) = \mathbb{K}\xi(t) - sat(\mathbb{K}\xi(t))$. By definition, $\psi(\mathbb{K}\xi)$ is a decentralized deadzone nonlinearity and satisfies, for any positive definite diagonal matrix T, the following sector condition [7]:

$$2\psi(\mathbb{K}\xi)'T[\psi(\mathbb{K}\xi) - \Lambda\mathbb{K}\xi] \le 0,\ \forall\xi \in S(\mathbb{K},\Lambda,u_0) \quad (9)$$

where $\Lambda \in \Re^{m*m}$ is a positive definite diagonal matrix, with $\Lambda_{(i,i)} < 1$, and, considering $u_{0(i)}^\Lambda \triangleq u_{0(i)}/(1 - \Lambda_{(i,i)})$, the set $S(\mathbb{K},\Lambda,u_0)$ is defined by

$$S(\mathbb{K},\Lambda,u_0) = \{\xi \in \Re^{n+n_c}; |\mathbb{K}_{(i)}\xi| \le u_{0(i)}^\Lambda,\ i = 1,...,m\} \quad (10)$$

Considering an augmented initial condition $\xi(t_0+\theta) = \phi_\xi(\theta) = \begin{bmatrix} x(t_0+\theta) \\ \eta(t_0+\theta) \end{bmatrix} = \begin{bmatrix} \phi_x(\theta) \\ \phi_\eta(\theta) \end{bmatrix}, \forall\theta \in [-\tau,0]$ where $\phi_\xi(\theta)$ is supposed to satisfy $\|\phi_\xi\|_c < v$, $v > 0$, system (8) will be said globally asymptotically stable if for any initial condition satisfying $\|\phi_\xi\|_c \le v$ with any finite v, the trajectories of system (8) converge asymptotically to the origin [16], [10]. Similar to the case of delay-free ($\tau = 0$), the determination of a global stabilizing controller is possible only when some stability hypothesis are verified by the open-loop system ($u(t)=0$) [13]. When this hypothesis is not verified, it is only possible to achieve local stabilization. In fact, in the generic case, given a stabilizing matrix $\mathbb{K}$, we associate a *basin of attraction* to the equilibrium point $\xi_e=0$ of system (8). The basin of attraction corresponds to all initial conditions $\phi_\xi(\theta)\in\mathcal{C}_\tau^v$ such that the corresponding trajectories of system (8) converge asymptotically to the origin. Since the determination of the exact basin of attraction is practically impossible, a problem of interest is to ensure the asymptotic stability for a set $B(\delta)=\{\phi_\xi\in\mathcal{C}_\tau;\|\phi_\xi\|_c^2\leq\delta\}$ of admissible initial conditions $\phi_\xi(\theta)$ [20]. Of course, the set $B(\delta)$ is included in the basin of attraction. Throughout the paper we will refer a set $B(\delta)$ as *a region of stability* for system (8). The problem we aim to solve can then be summarized as follows.

Problem 1 *Determine the anti-windup gain matrix E_c and a scalar δ, as large as possible, such that the asymptotic stability of system (8) is ensured for all initial conditions $\phi_\xi(\theta)\in B(\delta)=\{\phi_\xi\in\mathcal{C}_\tau;\|\phi_\xi\|_c^2\leq\delta\}$, $\forall\theta\in[-\tau,0]$.*

Since $B(\delta)$ can be viewed as an estimate of the basin of attraction of the system (8), the implicitly idea behind Problem 1 is to enlarge this basin over the choice of the anti-windup gain matrix E_c. Throughout the paper, we address Problem 1 both in the delay dependent and independent contexts.

3 Delay-independent results

Consider the following Lyapunov candidate function:

$$V(\xi_t)=\xi(t)'P\xi(t)+\int_{t-\tau}^{\tau}\xi(\theta)S\xi(\theta)d\theta \qquad (11)$$

with $P=P'>0$, $S=S'>0$ and where ξ_t, $\forall t\geq t_0$, denotes the restriction of ξ to the interval $[t-\tau,\,t]$ translated to $[-\tau,\,0]$, that is, $\xi_t(\theta)=\xi(t+\theta)$, $\forall\theta\in[-\tau,\,0]$.

Proposition 1 *If there exist symmetric positive definite matrices $W\in\Re^{(n+n_c)\times(n+n_c)}$ and $R\in\Re^{(n+n_c)\times(n+n_c)}$, a diagonal matrix $\Lambda\in\Re^{m\times m}$, a diagonal positive definite matrix $G\in\Re^{m\times m}$, a matrix $Z\in\Re^{n_c\times m}$, and a positive scalar γ satisfying:*

$$\begin{bmatrix} W\mathbb{A}'+\mathbb{A}W+R & \mathbb{A}_dW & \mathbb{B}G+\mathbb{R}Z-W\mathbb{K}'\Lambda \\ W\mathbb{A}_d' & -R & 0 \\ G\mathbb{B}'+Z'\mathbb{R}'-\Lambda\mathbb{K}W & 0 & -2G \end{bmatrix}<0 \qquad (12)$$

$$\begin{bmatrix} W & (1-\Lambda_{(i,i)})W\mathbb{K}_{(i)}' \\ (1-\Lambda_{(i,i)})\mathbb{K}_{(i)}W & \gamma u_{0(i)}^2 \end{bmatrix}\geq 0,\ i=1,\ldots,m \qquad (13)$$

$$0\leq\Lambda_{(i,i)}<1,\ i=1,\ldots,m \qquad (14)$$

then, for $E_c=ZG^{-1}$, it follows that for all initial conditions $\phi_\xi(\theta)\in B(\delta)=\{\phi_\xi\in\mathcal{C}_\tau;\|\phi_\xi\|_c^2\leq\delta\}$, $\forall\theta\in[-\tau,0]$ with

$$\delta=\frac{\gamma^{-1}}{\lambda_{max}(W^{-1})+\tau\lambda_{max}(W^{-1}RW^{-1})} \qquad (15)$$

the corresponding trajectories of system (8) converge asymptotically to the origin.

Proof. The satisfaction of relations (13) with $0<\Lambda_{(i,i)}\leq 1$ implies that the set $\mathcal{E}(W^{-1},\gamma^{-1})\stackrel{\triangle}{=}\{\xi\in\Re^{n+n_c};\ \xi'W^{-1}\xi\leq\gamma^{-1}\}$ is contained in $S(\mathbb{K},\Lambda,u_0)$. Hence, $\forall\xi(t)\in\mathcal{E}(W^{-1},\gamma^{-1})$ it follows that $\psi(\mathbb{K}\xi(t))=\mathbb{K}\xi(t)-sat(\mathbb{K}\xi(t))$ satisfies the sector condition (9).

By considering the Lyapunov candidate function as defined in (11), and by computing its time-derivative along the trajectories of system (8) one gets: $\dot V(\xi_t)=\xi(t)'(\mathbb{A}'P+P\mathbb{A})\xi(t)-2\xi(t)'P(\mathbb{B}+\mathbb{R}E_c)\psi(\mathbb{K}\xi(t))+2\xi(t)'P\mathbb{A}_d\xi(t-\tau)+\xi(t)'S\xi(t)-\xi(t-\tau)'S\xi(t-\tau)$. Thus, by using the sector condition (9), it follows that $\forall\xi(t)\in S(\mathbb{K},\Lambda,u_0)$

$$\dot V(\xi_t)\leq\dot V(\xi_t)-2\psi(\mathbb{K}\xi(t))'T[\psi(\mathbb{K}\xi(t))-\Lambda\mathbb{K}\xi(t)] \qquad (16)$$

Since $S>0$ and $T>0$ it follows that $-\xi(t-\tau)'S\xi(t-\tau)+2\xi(t)'P\mathbb{A}_d\xi(t-\tau)\leq\xi(t)'P\mathbb{A}_dS^{-1}\mathbb{A}_d'P\xi(t)$ and $-2\psi(\mathbb{K}\xi(t))'T\psi(\mathbb{K}\xi(t))+2\xi'(t)(\mathbb{K}'\Lambda T-P(\mathbb{B}+\mathbb{R}E_c))'\psi(\mathbb{K}\xi(t))\leq 0.5\xi(t)'(\mathbb{K}'\Lambda T-P(\mathbb{B}+\mathbb{R}E_c))'T^{-1}(\mathbb{K}'\Lambda T-P(\mathbb{B}+\mathbb{R}E_c))\xi(t)$. Hence, from (16) one has

$$\dot V(\xi_t)\leq\xi(t)'\mathcal{L}\xi(t)\ ,\quad\forall\xi(t)\in S(\mathbb{K},\Lambda,u_0) \qquad (17)$$

with $\mathcal{L}\stackrel{\triangle}{=}\xi(t)'[\mathbb{A}'P+P\mathbb{A}+S+P\mathbb{A}_dS^{-1}\mathbb{A}_d'P+0.5(\mathbb{K}'\Lambda T-P(\mathbb{B}+\mathbb{R}E_c))'T^{-1}(\mathbb{K}'\Lambda T-P(\mathbb{B}+\mathbb{R}E_c))]\xi(t)$

Consider now inequality (12). Pre and post multiplying this inequality by $\begin{bmatrix} P & 0 & 0\\ 0 & P & 0\\ 0 & 0 & T\end{bmatrix}$, and considering $P^{-1}=W$, $T^{-1}=G$, $E_c=ZG^{-1}$ and $S=PRP$, it follows that (12) is equivalent to

$$\begin{bmatrix} \mathbb{A}'P+P\mathbb{A}+S & P\mathbb{A}_d & P\mathbb{B}+P\mathbb{R}E_c-\mathbb{K}'\Lambda T \\ \mathbb{A}_d'P & -S & 0 \\ \mathbb{B}'P+E_c'\mathbb{R}'P-T\Lambda\mathbb{K} & 0 & -2T \end{bmatrix}<0$$

and, from Schur's complement, we can conclude that (12) is equivalent to $\mathcal{L}<0$. Hence, provided that $\xi(t)\in S(\mathbb{K},\Lambda,u_0)$, if (12) is verified one gets: $\dot V(\xi_t)<\pi_1||\xi(t)||^2<0$ and $\pi_2||\xi(t)||^2\leq V(\xi_t)\leq\pi_3||\xi_t||_c^2$ with $\pi_2=\lambda_{min}(P)$ and $\pi_3=\lambda_{max}(P)+\tau\lambda_{max}(S)$. From (15) and (ii), it follows that for $\phi_\xi(\theta)\in B(\delta)$, $\theta\in[-\tau,0]$, one gets $\xi(t)'P\xi(t)\leq V(\xi_t)\leq V(\xi_{t_0})\leq\gamma^{-1}$, $\forall t\geq t_0$. Hence, for any initial condition in the ball $B(\delta)$, one has $\xi(t)\in\mathcal{E}(W^{-1},\gamma^{-1})$, $\forall t\geq t_0$. Since relations (13) and (14) are satisfied, it follows that $\xi(t)\in\mathcal{S}(\mathbb{K},\Lambda,u_0)$, $\forall t\geq t_0$. Thus, for any initial condition belonging to $B(\delta)$ conditions (i) and (ii) of the Krasovskii Theorem [8] are verified, ensuring the asymptotic stability of the closed-loop system (8). □

Remark 1 *For $\Lambda=0$, saturation does not occur $\forall\phi_\xi(\theta)\in B(\delta)$ and the anti-windup loop does not act. On the other hand, if it is possible to verify (12) with $\Lambda=I_{n+n_c}$, the sector condition is verified globally and the global stability follows.*

4 Delay-dependent results

Since $\xi(t)$ is continuously differentiable for $t\geq 0$, from the Leibniz-Newton formula, it follows that $\xi(t-\tau)=\xi(t)-$

$\int_{-\tau}^{0}\dot{\xi}(t+\beta)d\beta$. Hence, from [8], to ensure the stability of the closed-loop system (8) it suffices to ensure the stability for the following system:

$$\begin{array}{c}\dot{\xi}(t)=(\mathbb{A}+\mathbb{A}_d)\xi(t)-(\mathbb{B}+\mathbb{R}E_c)\psi(\mathbb{K}\xi(t))\\ -\int_{-\tau}^{0}[\mathbb{A}_d\mathbb{A}\xi(t+\beta)-\mathbb{A}_d(\mathbb{B}+\mathbb{R}E_c)\psi(\mathbb{K}\xi(t+\beta))]d\beta\\ -\int_{-\tau}^{0}\mathbb{A}_d\mathbb{A}_d\xi(t+\beta-\tau)d\beta\end{array} \quad (18)$$

with the initial data $\xi(t_0+\theta)=\phi_\xi(\theta), \forall\theta\in[-2\tau,0]$.

Consider the Lyapunov-Krasovskii functional

$$V(\xi_t)=\xi(t)'P\xi(t)+Q(\xi_t) \quad (19)$$

where $P=P'>0$ and $Q(\xi_t)$ is a positive definite quadratic form that will be defined in the sequel.

Proposition 2 *If there exist symmetric positive definite matrices $W\in\Re^{(n+n_c)\times(n+n_c)}$, $X\in\Re^{(n+n_c)\times(n+n_c)}$, $R\in\Re^{(n+n_c)\times(n+n_c)}$ and $H\in\Re^{(n+n_c)\times(n+n_c)}$, a diagonal matrix $\Lambda\in\Re^{m\times m}$, a diagonal positive definite matrix $G\in\Re^{m\times m}$, a matrix $Z\in\Re^{n_c\times m}$ and a positive scalar γ satisfying*

$$\begin{bmatrix}\Gamma & \tau W\mathbb{A}' & \tau W\mathbb{A}_d' & \Xi & 0\\ \star & -\tau X & 0 & 0 & 0\\ \star & \star & -\tau R & 0 & 0\\ \star & \star & \star & -2G & \tau(G\mathbb{B}'+Z'\mathbb{R}')\\ \star & \star & \star & \star & -\tau H\end{bmatrix}<0 \quad (20)$$

$$\begin{bmatrix} W & (1-\Lambda_{(i,i)})W\mathbb{K}_{(i)}'\\ (1-\Lambda_{(i,i)})\mathbb{K}_{(i)}W & \gamma u_{0(i)}^2\end{bmatrix}\geq 0,\ i=1,\ldots,m \quad (21)$$

$$0\leq\Lambda_{(i,i)}<1,\ i=1,\ldots,m \quad (22)$$

where $\Gamma\triangleq W(\mathbb{A}+\mathbb{A}_d)'+(\mathbb{A}+\mathbb{A}_d)W+\tau\mathbb{A}_d(X+R+H)\mathbb{A}_d'$ and $\Xi=-W\mathbb{K}'\Lambda+\mathbb{B}G+\mathbb{R}Z$, then, for $E_c=ZG^{-1}$, it follows that for all initial conditions $\phi_\xi(\theta)\in B(\delta)=\{\phi_\xi\in\mathcal{C}_{2\tau};\|\phi_\xi\|_c^2\leq\delta\}$, $\forall\theta\in[-\tau,0]$ with

$$\delta=\frac{\gamma^{-1}}{\alpha} \quad (23)$$

$\alpha\triangleq\lambda_{max}(W^{-1})+\frac{3\tau^2}{2}\lambda_{max}(\mathbb{A}_d'R^{-1}\mathbb{A}_d)+\frac{\tau^2}{2}\lambda_{max}(\mathbb{A}'X^{-1}\mathbb{A})+\frac{\tau^2}{2}\lambda_{max}((\mathbb{B}+\mathbb{R}E_c)'H^{-1}(\mathbb{B}+\mathbb{R}E_c))\|\Lambda\mathbb{K}\|^2$, *the corresponding trajectories of system (8) converge asymptotically to the origin.*

Proof. The satisfaction of relations (21) with $0\leq\lambda_{(i)}<1$ implies that $\mathcal{E}(W^{-1},\gamma^{-1})\subset S(\mathbb{K},\Lambda,u_0)$. Hence, for all $\xi(t)\in\mathcal{E}(W^{-1},\gamma^{-1})$ it follows that $\psi(\mathbb{K}\xi(t))=\mathbb{K}\xi(t)-sat(\mathbb{K}\xi(t))$ satisfies the sector condition (9).

By considering the Lyapunov candidate function as defined in (19), and by computing its time-derivative along the trajectories of system (18) one gets[1]: $\dot{V}(\xi_t)=\xi(t)'[P(\mathbb{A}+\mathbb{A}_d)+(\mathbb{A}+\mathbb{A}_d)'P]\xi(t)-2\xi(t)'P(\mathbb{B}+\mathbb{R}E_c)\psi(t)+\mu(\xi_t)+\eta(\xi_t)+\zeta(\xi_t)+\dot{Q}(\xi_t)$, where $\mu(\xi_t)=-2\int_{-\tau}^{0}\xi(t)'P\mathbb{A}_d\mathbb{A}\xi(t+\beta)d\beta$, $\eta(\xi_t)=-2\int_{-\tau}^{0}\xi(t)'P\mathbb{A}_d\mathbb{A}_d\xi(t-\tau+\beta)d\beta$ and $\zeta(\xi_t)=2\int_{-\tau}^{0}\xi(t)'P\mathbb{A}_d(\mathbb{B}+\mathbb{R}E_c)\psi(t+\beta)d\beta$

[1]For notational simplicity, we denote $\psi(\mathbb{K}\xi(t))$ as $\psi(t)$ throughout the proof.

Using now the fact that $2u'v\leq u'Xu+v'X^{-1}v$ where X is any symmetric positive definite matrix and u and v are vectors of appropriate dimensions, it follows that $\mu(\xi_t)\leq\tau\xi(t)'P\mathbb{A}_dX\mathbb{A}_d'P\xi(t)+\int_{-\tau}^{0}\xi(t+\beta)'\mathbb{A}'X^{-1}\mathbb{A}\xi(t+\beta)d\beta$, $\eta(\xi_t)\leq\tau\xi(t)'P\mathbb{A}_dR\mathbb{A}_d'P\xi(t)+\int_{-\tau}^{0}\xi(t-\tau+\beta)'\mathbb{A}_d'R^{-1}\mathbb{A}_d\xi(t-\tau+\beta)d\beta$ and $\zeta(\xi_t)\leq\tau\xi(t)'P\mathbb{A}_dH\mathbb{A}_d'P\xi(t)+\int_{-\tau}^{0}\psi(t+\beta)'(\mathbb{B}+\mathbb{R}E_c)'H^{-1}(\mathbb{B}+\mathbb{R}E_c)\psi(t+\beta)d\beta$

Defining now $Q(\xi_t)=\int_{-\tau}^{0}\int_{t+\beta}^{t}\xi(\theta)'\mathbb{A}'X^{-1}\mathbb{A}\xi(\theta)d\theta d\beta+\int_{-\tau}^{0}\int_{t-\tau+\beta}^{t}\xi(\theta)'\mathbb{A}_d'R^{-1}\mathbb{A}_d\xi(\theta)d\theta d\beta+\int_{-\tau}^{0}\int_{t+\beta}^{t}\psi(\theta)'(\mathbb{B}+\mathbb{R}E_c)'H^{-1}(\mathbb{B}+\mathbb{R}E_c)\psi(\theta)d\theta d\beta$ one obtains $\dot{Q}(\xi_t)=-\int_{-\tau}^{0}\xi(t+\beta)'\mathbb{A}'X^{-1}\mathbb{A}\xi(t+\beta)d\beta+\tau\xi(t)'\mathbb{A}'X^{-1}\mathbb{A}\xi(t)-\int_{-\tau}^{0}\xi(t-\tau+\beta)'\mathbb{A}_d'R^{-1}\mathbb{A}_d\xi(t-\tau+\beta)d\beta+\tau\xi(t)'\mathbb{A}_d'R^{-1}\mathbb{A}_d\xi(t)-\int_{-\tau}^{0}\psi(t+\beta)'(\mathbb{B}+\mathbb{R}E_c)'H^{-1}(\mathbb{B}+\mathbb{R}E_c)\psi(t+\beta)d\beta+\tau\psi(t)'(\mathbb{B}+\mathbb{R}E_c)'H^{-1}(\mathbb{B}+\mathbb{R}E_c)\psi(t)$.

Hence,considering the sector condition,it follows that $\dot{V}(\xi_t)\leq\xi(t)'[P(\mathbb{A}+\mathbb{A}_d)+(\mathbb{A}+\mathbb{A}_d)'P]\xi(t)-2\xi(t)'P(\mathbb{B}+\mathbb{R}E_c)\psi(t)+\tau\xi(t)'P\mathbb{A}_dX\mathbb{A}_d'P\xi(t)+\tau\xi(t)'\mathbb{A}'X^{-1}\mathbb{A}\xi(t)+\tau\xi(t)'P\mathbb{A}_dR\mathbb{A}_d'P\xi(t)+\tau\xi(t)'\mathbb{A}_d'R^{-1}\mathbb{A}_d\xi(t)+\tau\xi(t)'P\mathbb{A}_dH\mathbb{A}_d'P\xi(t)+\tau\psi(t)'(\mathbb{B}+\mathbb{R}E_c)'H^{-1}(\mathbb{B}+\mathbb{R}E_c)\psi(t)-2\psi(t)'T[\psi(t)-\Lambda\mathbb{K}\xi(t)]$, $\forall\xi(t)\in S(\mathbb{K},\Lambda,u_0)$.

Following a similar reasoning to the one done in the proof of the Proposition 1, it is easy to show that

$$\dot{V}(\xi_t)\leq\xi(t)'\mathcal{L}\xi(t)\ ,\quad\forall\xi(t)\in S(\mathbb{K},\Lambda,u_0), \quad (24)$$

with $\mathcal{L}\triangleq\xi(t)'[P(\mathbb{A}+\mathbb{A}_d)+(\mathbb{A}+\mathbb{A}_d)'P+\tau P\mathbb{A}_d(X+R+H)\mathbb{A}_d'P+\tau\mathbb{A}'X^{-1}\mathbb{A}+\tau'\mathbb{A}_d'R^{-1}\mathbb{A}_d+(T\Lambda\mathbb{K}-(\mathbb{B}+\mathbb{R}E_c)'P)'(2T-\tau(\mathbb{B}+\mathbb{R}E_c)'H^{-1}(\mathbb{B}+\mathbb{R}E_c))^{-1}(T\Lambda\mathbb{K}-(\mathbb{B}+\mathbb{R}E_c)'P)]\xi(t)$, and that inequality (20) is equivalent to $\mathcal{L}<0$.

Hence, provided that $\xi(t)\in S(\mathbb{K},\Lambda,u_0)$, if (20) is verified one gets: $\dot{V}(\xi_t)<\pi_1||\xi(t)||^2<0$. and $\pi_2||\xi(t)||^2\leq V(\xi_t)\leq\pi_3||\xi_t||_c^2$ with $\pi_2=\lambda_{min}(P)$. The computation of π_3 needs to study the overbounding of $V(\xi_t)$ and therefore that one of $Q(\xi_t)$. Thus, we have to express the upper bound on the norm of $\psi(t)$. For $\xi(t)\in S(\mathbb{K},\Lambda,u_0)$, one can verify that $\psi(\mathbb{K}\xi(t))$ satisfies $||\psi(\mathbb{K}\xi(t))||\leq||\Lambda\mathbb{K}\xi(t)||\leq||\Lambda\mathbb{K}||||\xi(t)||$.

Hence, from (19) one gets $\pi_3=\alpha$, and, for $\phi_\xi(\theta)\in B(\delta)$, $\forall\theta\in[-2\tau,0]$, one gets $\xi(t)'P\xi(t)\leq V(\xi_t)\leq V(\xi_{t_0})\leq\gamma^{-1}$, $\forall t\geq t_0$. Thus, following the same reasonig used in the proof of Proposition 1, we conclude that $\forall\phi_\xi(\theta)\in B(\delta)$, $\theta\in[-2\tau,0]$, the asymptotic stability of the system (18) is ensured and, as consequence, the asymptotic stability of system (8) is ensured $\forall\phi_\xi(\theta)\in B(\delta)$, $\theta\in[-\tau,0]$. □

5 Computational issues

5.1 Delay-independent case

In Proposition 1, one gets one bilinearity (appearing in both conditions) due to the product between decision variables W and Λ. The basic idea to overcome this difficulty is to iterate between two steps where we fix Λ or W. Since the implicit objective is to obtain a set $B(\delta)$ with a significant size, we can consider an optimization problem with the following criterion:

$$\min\{\beta_0\gamma + \beta_1\lambda_{max}(W^{-1}) + \beta_2\lambda_{max}(R)\}$$

where β_i, $i = 0,1,2$ are tuning parameters. Note that by minimizing the function above we are, implicitly, maximizing δ. Thus, we propose the following algorithm for providing a solution to Problem 1.

Algorithm 1 *(Delay-independent case)*

1. *Initialization of* Λ.
2. *Fix* Λ *and solve for* W, λ_W, R, λ_R, G, Z *and* γ:

$$\begin{array}{l} \min\{\beta_0\gamma + \beta_1\lambda_W + \beta_2\lambda_R\} \\ \textit{subject to} \\ \textit{relations } (12),(13) \\ \left[\begin{array}{cc} \lambda_W I_{n+n_c} & I_{n+n_c} \\ I_{n+n_c} & W \end{array}\right] \geq 0, \;\; \lambda_R I_{n+n_c} \geq R \end{array} \quad (25)$$

3. *Fix* W *obtained in step 1 and solve for* Λ, R, λ_R, G, Z *and* γ:

$$\begin{array}{l} \min\{\beta_0\gamma + \beta_1\lambda_W + \beta_2\lambda_R\} \\ \textit{subject to} \\ \textit{relations } (12),(13),(14) \\ \lambda_R I_{n+n_c} \geq R \end{array} \quad (26)$$

4. *Go to step 2 until no significant change on the value of* $J_{di} = \beta_0\gamma + \beta_1\lambda_W + \beta_2\lambda_R$ *is obtained.*

5.2 Delay-dependent case

In Proposition 2, for a given τ, there exists a bilinearity due to the product between Λ and W. As in the previous case, we choose to proceed through some relaxation schemes associated to some optimization problems. In particular, we consider the following optimization criterion, related, implicitly, to the maximization of the δ: $\min\{\beta_0\gamma + \beta_1\lambda_{max}(W^{-1}) + \beta_2\lambda_{max}(H^{-1}) + \beta_3\lambda_{max}(\mathbb{A}_d' R^{-1}\mathbb{A}_d) + \beta_4\lambda_{max}(\mathbb{A}' X^{-1}\mathbb{A})\}$ where β_i, $i = 0,...,4$ are weighting parameters.

Algorithm 2 *(Delay-dependent case)*

1. *Initialization of* Λ.
2. *Fix* Λ *and solve for* W, λ_W, λ_X, λ_R, λ_H, X, R, H, G, Z,γ:

$$\begin{array}{l} \min\{\beta_0\gamma + \beta_1\lambda_W + \beta_2\lambda_H + \beta_3\lambda_R + \beta_4\lambda_X)\} \\ \textit{subject to} \\ \textit{relations } (20),(21) \\ \left[\begin{array}{cc} \lambda_W I_{n+n_c} & I_{n+n_c} \\ I_{n+n_c} & W \end{array}\right] \geq 0, \left[\begin{array}{cc} \lambda_X I_{n+n_c} & \mathbb{A}' \\ \mathbb{A} & X \end{array}\right] \geq 0 \\ \left[\begin{array}{cc} \lambda_R I_{n+n_c} & \mathbb{A}_d' \\ \mathbb{A}_d & R \end{array}\right] \geq 0, \left[\begin{array}{cc} \lambda_H I_{n+n_c} & I_{n+n_c} \\ I_{n+n_c} & H \end{array}\right] \geq 0 \end{array} \quad (27)$$

3. *Fix* W *obtained in step 1 and solve for* Λ, λ_X, λ_R, λ_H, X, R, H, G, Z *and* γ:

$$\begin{array}{l} \min\{\beta_0\gamma + \beta_1\lambda_W + \beta_2\lambda_H + \beta_3\lambda_R + \beta_4\lambda_X\} \\ \textit{subject to} \\ \textit{relations } (20),(21),(22) \\ \left[\begin{array}{cc} \lambda_X I_{n+n_c} & \mathbb{A}' \\ \mathbb{A} & X \end{array}\right] \geq 0, \left[\begin{array}{cc} \lambda_R I_{n+n_c} & \mathbb{A}_d' \\ \mathbb{A}_d & R \end{array}\right] \geq 0, \\ \left[\begin{array}{cc} \lambda_H I_{n+n_c} & I_{n+n_c} \\ I_{n+n_c} & H \end{array}\right] \geq 0 \end{array} \quad (28)$$

4. *Go to step 2 until no significant change on the value of* $J_{dd} = \beta_0\gamma + \beta_1\lambda_W + \beta_2\lambda_H + \beta_3\lambda_R + \beta_4\lambda_X$ *is obtained.*

6 Illustrative examples

Example 1 *Consider system (1) borrowed in [20] described by the following data:* $u_0 = 15$, $\tau = 1$, $C = \left[\begin{array}{cc} 5 & 1 \end{array}\right]$, $A = \left[\begin{array}{cc} 1 & 1.5 \\ 0.3 & -2 \end{array}\right]$, $A_d = \left[\begin{array}{cc} 0 & -1 \\ 0 & 0 \end{array}\right]$ *and* $B = \left[\begin{array}{c} 10 \\ 1 \end{array}\right]$. *Note that the open-loop matrix* A *is strictly unstable since its eigenvalues are* $1.1432; -2.1432$.

Consider the following dynamic controller for the linear system (without saturation): $C_c = \left[\begin{array}{cc} -0.9165 & 0.1091 \end{array}\right]$, $A_c = \left[\begin{array}{cc} -20.2042 & 2.5216 \\ 2.1415 & -4.4821 \end{array}\right]$ *and* $B_c = \left[\begin{array}{c} 1.9516 \\ -0.0649 \end{array}\right]$. *Note that matrix* $\mathbb{A}$ *is asymptotically stable. Hence, we try to solve our problem in the delay-independent context. Applying Algorithm 1, with all tuning parameters* $\beta_i = 1$, *one gets* $\delta = 637.8241$; $\Lambda = 0.9348$; $E_c = \left[\begin{array}{c} 18.4898 \\ -4.8331 \end{array}\right]$.

Table 1 shows the values of δ *resulting from the application of Algorithms 1 and 2 for different values of* τ.

τ	0.1	0.5	1
δ (Alg. 1)	923.9005	775.3730	637.8241
δ (Alg. 2)	1.8826×10^3	662.3369	285.1510

Table 1: δ both in the context delay-independent and delay-dependent contexts for different values of τ.

From table 1, we can notice that Algorithm 2 provides greater δ *for small* τ ($\tau < 0.4$), *whereas Algorithm 1 provides better results for* $\tau \geq 0.4$. *Hence, even when the closed-loop stability is delay-independent, it may be interesting to evaluate the set of admissible initial conditions* $B(\delta)$ *via a delay-dependent procedure.*

Example 2 *Consider system (1) described by the following data:* $u_0 = 15$, $C = \left[\begin{array}{cc} 5 & 1 \end{array}\right]$, $A = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right]$, $A_d = \left[\begin{array}{cc} 0 & 1.5 \\ 0.3 & -2 \end{array}\right]$ *and* $B = \left[\begin{array}{c} 10 \\ 1 \end{array}\right]$. *Consider now the same dynamic controller as that one in Example 1. Note that matrix* $\mathbb{A}$ *is now unstable. Hence, we have to solve our problem in the delay-dependent context. We apply Algorithm 2 with all the tuning parameters* $\beta_i = 1$. *Table 2 presents the corresponding values of* E_c, δ *(computed from (23)) and* Λ *for different values of* τ.

τ	0.1	0.2	0.4
E_c	$\left[\begin{array}{c} 15.6191 \\ -16.5437 \end{array}\right]$	$\left[\begin{array}{c} 13.5927 \\ -15.2705 \end{array}\right]$	$\left[\begin{array}{c} 8.0816 \\ -1.4875 \end{array}\right]$
Λ	0.8385	0.7675	0.5750
δ	1.5115×10^3	977.8398	288.8288

Table 2: E_c and δ for different values of τ.

We can notice that the value of δ *is strongly dependent on the* *value of the delay τ. In particular, we can identify the trade-off between the size of the stability region and the size of the delay: larger is τ, smaller is the obtained δ. Furthermore, by searching the maximal value of τ for which Proposition 2 provides a solution to Problem 1, we obtain* $\tau_{max} = 0.4430$.

7 Concluding remarks

In this paper, we have addressed the problem of designing anti-windup gains in order to obtain a region of stability, as large as possible, for linear state delayed systems with saturating inputs. On the other hand, given a gain E_c (computed, for example for satisfying performance requirements), the proposed methodology can be straightforwardly adapted for computing regions of stability in order to estimate the region of attraction of the closed-loop system. In this case it is always possible to take the E_c obtained with the proposed algorithms and try to enlarge $B(\delta)$ in an analysis algorithm, based on the same theoretical conditions.

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