ASYMPTOTICALLY EXACT INPUT-OUTPUT LINEARIZATION USING CARLEMAN LINEARIZATION

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Abstract

On the basis of the Carleman linearization this contribution presents a new approach to the design of observers with linear error dynamics for nonlinear systems. The presented observer design is extended to a dynamic output feedback controller achieving asymptotically exact input-output linearization. A simple example demonstrates the proposed observer and controller design procedure.

1 Introduction and problem formulation

Consider a single-input single-output nth order nonlinear system

$$\dot{x} = f(x) + g(x)u \quad (1)$$
$$y = h(x) \quad (2)$$

Using the *Carleman linearization* [7, 9] system (1)-(2) can be approximated by a bilinear system of order $n_{bil} > n$

$$\dot{z} = Az + (b + Nz)u \quad (3)$$
$$y = c^T z \quad (4)$$

Obviously the bilinear system (3)-(4) has a simpler structure than system (1)-(2), such that the design of nonlinear controllers for the nonlinear system (1)-(2) is simplified by using the bilinear approximation model (3)-(4). The Carleman linearization of the nonlinear system (1)-(2) to obtain the bilinear approximation model (3)-(4) is based on a Taylor expansion of the system nonlinearities $f(x)$, $g(x)$ and $h(x)$ in (1)-(2). The multivariable monomials of this Taylor series are defined as the states z of the bilinear approximation model, i.e.

$$z_k = x^{(k)}, \quad k = 1(1)M \quad (5)$$

where the multivariable monomials are given by

$$x^{(i)} = \underbrace{x \otimes \ldots \otimes x}_{i \text{ times}} \quad (6)$$

and "$\otimes$" denoting the *Kronecker product* (see e.g. [2]). Since the state equation (3) of the bilinear approximation model of order $n_{bil} = \binom{n+M}{n}$ is determined by omitting higher order terms the resulting bilinear system is a local approximation of the nonlinear system under consideration. In this contribution the simple structure of the bilinear approximation model is exploited in order to design nonlinear observers and controllers for the nonlinear system (1)-(2). As far as observer design is concerned the possibly high order of the bilinear approximation model to obtain an accurate system approximation seems to lead to observers with an undesirable high order. This can be circumvented by using the properties of the states z of the bilinear approximation model (3)-(4). On the one hand only the first n states of the bilinear approximation model need to be estimated to reconstruct the whole system state x, since $z_1 = x$ holds (see (5)). On the other hand, all states z of the bilinear approximation model, that can be expressed by the output y of system (1)-(2) need not be estimated by the observer. Both properties can be used to reduce the observer order significantly. The application of the resulting nonlinear observer in the closed loop system is problematic, since for nonlinear systems it is not easy to predict the effect of the estimation error on the closed loop dynamics. However if the system has relative degree one and is minimum phase (i.e. has stable zero dynamics) the design of the observer and the closed loop dynamics can be carried out independently using *asymptotically exact input-output linearization* (see [4]). This approach exactly linearizes the input-ouput behaviour of the observer using feedback of the observer states. The resulting control input is also applied to plant, such that a linear input-ouput behaviour is achieved for the closed loop system if the estimation error decays to zero. Using the results in [4] the presented observer is extended to a dynamic output feedback controller achieving asymptotically exact input-output linearization. By way of an simple example it is shown that the proposed nonlinear control design yields an exact observer or controller in the case of systems with polynomial and measurable nonlinearities. If nonpolynomial but measurable nonlinearities appear in (1)-(2), the presented results can also be used by introducing the nonpolynomial nonlinearities as new states yielding a higher order system with polynomial nonlinearities (see [6]).

The next section presents the new approach for the observer design along with a design procedure for solving the design equations. Section 3 contains the design of dynamic output feedback controllers achieving asymptotically exact input-output linearization. In Section 4 the presented observer and controller design procedure is demonstrated by means of simple example.

2 Observer design for nonlinear systems

2.1 Observer for a linear functional of the state

In the following a state observer for the nonlinear system (1)-(2) is developed. The proposed observer uses the bilinear approximation model of order n_{bil}

$$\begin{aligned} \dot{z} &= Az + (b + Nz)\, u && (7) \\ y &= c^T z && (8) \end{aligned}$$

as system description, which results from the Carleman linearization of system (1)-(2) (see Section 1). The states $z = z(x)$ of the bilinear approximation model (7)-(8) defined as functions of the states x of the nonlinear system (1)-(2) (see Section 1) can be subdivided into m *measurable states* $\tilde{y}$ (i.e. the elements of $\tilde{y}$ are states z_μ, that satisfy $z_\mu = z_\mu(y)$) and $n_{bil} - m$ *unmeasurable states* $\tilde{z}$. The measurable states $\tilde{y}$ can be employed to reduce the order of the observer to be designed. Using the (n_{bil}, n_{bil}) permutation matrix U the state vector z is subdivided into the measurable and the unmeasurable states

$$\begin{bmatrix} \tilde{y} \\ \tilde{z} \end{bmatrix} = Uz = \begin{bmatrix} C \\ \overline{C} \end{bmatrix} z \tag{9}$$

with the (m, n_{bil}) matrix C, such that the states z can be expressed as

$$z = U^T \begin{bmatrix} \tilde{y} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \tilde{y} \\ \tilde{z} \end{bmatrix} = V_1 \tilde{y} + V_2 \tilde{z} \tag{10}$$

in which the property $U^{-1} = U^T$ of the permutation matrix U is used and V_1 is an (n_{bil}, m) matrix and V_2 is an $(n_{bil}, n_{bil} - m)$ matrix. Since one is interested in obtaining an estimate for the states x of the nonlinear system (1)-(2) it suffices to estimate the first n states of the bilinear approximation model (see Section 1). To this end consider a linear functional of the states z of the bilinear system (7)-(8)

$$v = Ez = G\zeta + H\tilde{y} \tag{11}$$

in which the $(\bar{n}, 1)$ vector $\zeta = Tz$ is to be reconstructed by the observer and the matrix E in (11) is chosen as

$$E = \begin{bmatrix} I_n & 0 \end{bmatrix} \tag{12}$$

such that $v = x$ in (11) holds. The following theorem shows under which conditions an $\bar{n}$th order observer for the linear functional (11) exists.

Theorem 1 *The $\bar{n}$th order observer for the linear functional (11)*

$$\begin{aligned} \dot{\hat{\zeta}} &= F\hat{\zeta} + T\,(b + NV_1\tilde{y})\, u + L\tilde{y} && (13) \\ \hat{v} &= G\hat{\zeta} + H\tilde{y} && (14) \end{aligned}$$

has a linear and asymptotically stable error dynamics

$$\dot{e} = Fe \tag{15}$$

with the estimation error $e = Tz - \hat{\zeta}$, such that

$$\lim_{t \to \infty} (v - \hat{v}) = 0 \tag{16}$$

if the following relations hold

$$\begin{aligned} \mathrm{Re}\,\lambda(F) &< 0 && (17) \\ TA - FT &= LC && (18) \\ TNV_2 &= 0 && (19) \\ GT + HC &= E && (20) \end{aligned}$$

Proof. The differential equation for the estimation error

$$e = Tz - \hat{\zeta} \tag{21}$$

(see Theorem 1) reads

$$\begin{aligned} \dot{e} &= T\dot{z} - \dot{\hat{\zeta}} \\ &= TAz + T(b + Nz)u - F\hat{\zeta} - T(b + NV_1\tilde{y})u - L\tilde{y} && (22) \end{aligned}$$

in light of (7)-(8) and (13)-(14). By substituting (10) in TNz and by a simple rearrangement the relation

$$TNz - TNV_1\tilde{y} = TNV_2\tilde{z} \tag{23}$$

is obtained. Adding $FTz - FTz$ to (22) and using $\tilde{y} = Cz$ (see (9)) as well as (23) the error dynamics (22) can be rewritten as

$$\dot{e} = F(Tz - \hat{\zeta}) + (TA - FT - LC)z + TNV_2\tilde{z}u \tag{24}$$

giving (15) if (18) and (19) is satisfied. Next consider the error

$$v - \hat{v} = Ez - G\hat{\zeta} - H\tilde{y} \tag{25}$$

following from (11) and from (14) and solve the estimation error e (see (21)) for $\hat{\zeta}$ yielding

$$\hat{\zeta} = Tz - e \tag{26}$$

Substituting (26) and $\tilde{y} = Cz$ (see (9)) in (25) leads to

$$v - \hat{v} = (E - GT - HC)z + Ge \tag{27}$$

If (20) is satisfied (27) takes the form

$$v = Ge \tag{28}$$

With (17) follows

$$\lim_{t \to \infty} e = 0 \tag{29}$$

from (15), such that

$$\lim_{t \to \infty} (v - \hat{v}) = 0 \tag{30}$$

in view of (28) and (29). ◁

Remark 1 *The observer presented in Theorem 1 is reminiscient of an unknown input observer (see e.g. [1]). In fact the input nonlinearity "Nzu" of the bilinear approximation model (7)-(8) can be regarded as an unknown input, which has to be decoupled from the error dynamics to obtain a linear error differential equation (15).*

2.2 Design procedure

This section presents a procedure for solving the design equations (17)-(20) of the observer in Theorem 1. At first a parametric characterization of the solution (T,L) for given (C,A) of the Sylvester equation (18) is provided in Theorem 2 .

Theorem 2 *Assume that the $(\bar{n},\bar{n})$ matrix F has distinct eigenvalues $\bar{\lambda}_\mu$, $\mu = 1(1)\bar{n}$, which are different from the eigenvalues of A, and let the matrix F be represented by*

$$F = W^{-1}\operatorname{diag}\left(\bar{\lambda}_\mu\right)W \tag{31}$$

in which W is the matrix of left eigenvectors of F. Further suppose, that the pair (C,A) is observable. Then the solution T with $\operatorname{rank}(T) = \bar{n}$ of the Sylvester equation

$$TA - FT = LC \tag{32}$$

for

$$L = W^{-1}\begin{bmatrix}\bar{p}_1^T \\ \vdots \\ \bar{p}_{\bar{n}}^T\end{bmatrix} \tag{33}$$

is given by

$$T = W^{-1}\begin{bmatrix}\bar{p}_1^T C(A-\bar{\lambda}_1 I)^{-1} \\ \vdots \\ \bar{p}_{\bar{n}}^T C(A-\bar{\lambda}_{\bar{n}} I)^{-1}\end{bmatrix} \tag{34}$$

with $\bar{n}$ freely assignable (1,m) parameter vectors $\bar{p}_\mu^T$, $\mu = 1(1)\bar{n}$.

Proof. First it is shown, that any solution of (32) can be represented by (34) under the assumptions of Theorem 2. To this end substitute (31) in (32) and premultiply the result with W giving

$$WTA - \operatorname{diag}\left(\bar{\lambda}_\mu\right)WT = WLC \tag{35}$$

By introducing the parameter vectors

$$\begin{bmatrix}\bar{p}_1^T \\ \vdots \\ \bar{p}_{\bar{n}}^T\end{bmatrix} = WL \tag{36}$$

equation (35) can rewritten rowwise as

$$w_\mu^T T(A-\bar{\lambda}_\mu I) = \bar{p}_\mu^T C, \quad \mu = 1(1)\bar{n} \tag{37}$$

Since the eigenvalues $\bar{\lambda}_\mu$ are assumed to be different from the eigenvalues of A relation (37) is solvable for $w_\mu^T T$ yielding

$$w_\mu^T T = \bar{p}_\mu^T C(A-\bar{\lambda}_\mu I)^{-1} \tag{38}$$

such that (34) follows from writing (38) in matrix form and solving for T. The matrix L then results from solving (36) for L. Next one has to prove, that T given by (34) is a solution of the Sylvester equation (32) for L in (33). This is readily verified by substituting (34) and (33) in (32). The proof that T in (34) has full rank is implied by the fact, that W is nonsingular and that with the pair (C,A) observable the vectors $\bar{p}_\mu^T C(A-\bar{\lambda}_\mu I)^{-1}$ are the left eigenvector of the corresponding full order observer, which are linearly independent for distinct observer eigenvalues (see [8]). $\triangleleft$

Remark 2 *For conjugate complex observer eigenvalues $\bar{\lambda}_\mu$ the corresponding parameter vectors $\bar{p}_\mu^T$ have to be chosen also conjugate complex, such that T is a real valued matrix.*

Remark 3 *If all observer eigenvalues $\bar{\lambda}_\mu$ are real, the matrix W can be chosen as $W = I$, which simplifies the parametrization of T in Theorem 2.*

Theorem 2 provides a parametric solution to the design equations (17) and (18). The degrees of freedom contained in the solution T of the Sylvester equation have to determined, such that condition (19) is satisfied. Inserting (34) in (19) yields rowwise

$$\bar{p}_\mu^T C(A-\bar{\lambda}_\mu I)^{-1}NV_2 = 0^T, \quad \mu = 1(1)\bar{n} \tag{39}$$

after premultiplying with W, which does not change the result. This is a condition to be met by the parameter vectors $\bar{p}_\mu^T$ and the corresponding observer eigenvalues $\bar{\lambda}_\mu$, such that (19) is fullfilled assuring a linear error dynamics (15) (see (24)). Since $n_{bil} - m > m$ holds in most cases, the observer eigenvalues $\bar{\lambda}_\mu$ can be assigned arbitrarily (but different from the eigenvalues of A; see Theorem 2) if the $(m, n_{bil} - m)$ transfer matrix

$$F(s) = C(A-sI)^{-1}NV_2 \tag{40}$$

satisfies

$$\operatorname{rank} F(s) < m \quad \forall s \tag{41}$$

If the transfer matrix (40) is not rank deficient for all s a solution of (39) can be found by determining the zeros η_j of the transfer matrix (40). As a consequence the matrix $F(\eta_j)$ is rank deficient, such that a nontrivial solution $\bar{p}_\mu^T$ of (39) exists. But in this case the observer eigenvalues $\bar{\lambda}_\mu$ are fixed by condition (39) and a stable observer only exists if the transfer matrix (40) has all its zeros in the open left half plane.

With the matrix T resulting from (34) the matrices G and H in (20) have to be computed by solving the linear equation

$$\begin{bmatrix}G & H\end{bmatrix}\begin{bmatrix}T \\ C\end{bmatrix} = E \tag{42}$$

Remark 4 *Since a solution of equation (42) only exists if*

$$\operatorname{rank}\begin{bmatrix}T^T & C^T\end{bmatrix}^T = \operatorname{rank}\begin{bmatrix}T^T & C^T & E^T\end{bmatrix}^T \tag{43}$$

a lower bound for the observer order is attainable, since from (43) follows $m + \bar{n} \geq n$ in view of $\operatorname{rank} E = n$ (see (12)) implying $\bar{n} \geq n - m$. This result shows, that even though the bilinear approximation model is of high order n_{bil} the corresponding observer order may be considerably smaller.

Remark 5 *If no solution to the design equations (17)-(20) exists, at least an observer with approximately linear error dynamics*

$$\dot{e} = Fe + O^{[q+]}(x,u) \tag{44}$$

may be attainable with $O^{[q+]}(x,u)$ denoting nonlinear terms in x and u of order strictly larger than q. Condition (39) then takes the form

$$\bar{p}_{\mu}^{T} C(A - \bar{\lambda}_{\mu} I)^{-1} N V_{21} = 0^{T}, \quad \mu = 1(1)\bar{n} \tag{45}$$

where the (n_{bil},$n_{bil} - m - q$) matrix V_{21} is given by

$$V_{21} = V_2 \begin{bmatrix} I_{n_{bil}-m-q} \\ 0 \end{bmatrix} \tag{46}$$

(see (10)). The differential equation (44) is implied by the definition of the states z in (5).

3 Asymptotically exact input-output linearization

In this section a dynamic output feedback controller for relative degree one and minimum phase systems is derived, which asymptotically input-output linearizes the resulting first order transfer behaviour of the closed loop system. The proposed input-output linearization approach is based on the results in [4], where an extended Luenberger observer is employed to obtain the dynamic output feedback controller.

3.1 Exact input-output linearization of the observer

For the derivation of the output feedback controller, which asymptotically exact input-output linearizes the transfer behaviour of the closed loop system, consider the observer of Theorem 1 in the form

$$\dot{\hat{\zeta}} = F\hat{\zeta} + T\left(b + NV_1\tilde{y}\right)u + L\tilde{y} \tag{47}$$

$$\hat{y} = g^{T}\hat{\zeta} + h^{T}\tilde{y} \tag{48}$$

with the row vectors g^T and $h^T$and $\hat{y}$ being an estimate for $y = c^T z$ in (8). The basic idea of the input-output linearization approach in [4] is to linearize the input-output behaviour of the observer (47)-(48) yielding

$$\dot{\hat{y}} + \tilde{a}_0\hat{y} = \tilde{a}_0 w \tag{49}$$

where w is the reference input of the closed loop system. If the estimate $\hat{y}$ of the observer (47)-(48) converges with the linear observer dynamics (15) asymptotically to the value of the real output y, i.e.

$$\lim_{t\to\infty}(y - \hat{y}) = 0 \tag{50}$$

then also the input-output behaviour of the closed loop system with respect to the real ouput y satisfies

$$\dot{y} + \tilde{a}_0 y = \tilde{a}_0 w \tag{51}$$

asymptotically. An important advantage of this approach is, that the linear dynamics of the input-output behaviour and the linear dynamics of the related estimation error can be assigned independently for the closed loop system.

The input-output linearizing feedback for the observer (47)-(48) is obtained by differentiating the output $\hat{y}$ in (48) with respect to time yielding

$$\dot{\hat{y}} = g^{T}\dot{\hat{\zeta}} + h^{T}\dot{\tilde{y}} \tag{52}$$

Substituting the right hand side of (47) for $\dot{\hat{\zeta}}$ and the right hand side of (7) for $\dot{z}$ in $\dot{\tilde{y}} = C\dot{z}$ (see (9)) leads to

$$\begin{aligned} \dot{\hat{y}} &= g^{T}F\hat{\zeta} + \left((g^{T}T + h^{T}C)b + (g^{T}T + h^{T}C)NV_1\tilde{y}\right)u \\ &\quad + g^{T}L\tilde{y} + h^{T}CAV_1\tilde{y} + h^{T}C(AV_2 + NV_2u)\tilde{z} \end{aligned} \tag{53}$$

after a simple rearrangement and using (10). Since the observer (47)-(48) estimates the linear functional $y = c^T z$ of the states z the relation

$$g^{T}T + h^{T}C = c^{T} \tag{54}$$

results from (20) in Theorem 1 with $E = c^T$ (see (11) and (48)). Using (54) expression (53) simplifies to

$$\begin{aligned} \dot{\hat{y}} &= g^{T}F\hat{\zeta} + \left(c^{T}b + c^{T}NV_1\tilde{y}\right)u \\ &\quad + (g^{T}L + h^{T}CAV_1)\tilde{y} + h^{T}\left(CAV_2 + CNV_2u\right)\tilde{z} \end{aligned} \tag{55}$$

Thus the observer has relative degree one at $\tilde{z}_0$ (see [5]) if

$$c^{T}b + c^{T}NV_1\tilde{y}_0 \neq 0 \tag{56}$$

holds. Note that condition (56) implies, that the system (7)-(8) has relative degree one at z_0, i.e. with (10)

$$c^{T}b + c^{T}Nz_0 = c^{T}b + c^{T}NV_1\tilde{y}_0 + c^{T}NV_2\tilde{z}_0 \neq 0 \tag{57}$$

if $c^T b + c^T NV_1\tilde{y}_0 \neq -c^T NV_2\tilde{z}_0$.

Remark 6 *If the order n_{bil} of the bilinear approximation model (7)-(8) (i.e. the approximation degree) is chosen appropriately, the relative degree of the nonlinear system (1)-(2) and the bilinear system (7)-(8) coincide.*

If condition (56) is fulfilled, the exact input-output linearizing feedback

$$u = \frac{1}{c^{T}b + c^{T}NV_1\tilde{y}}\left(-g^{T}F\hat{\zeta} - (g^{T}L + h^{T}CAV_1)\tilde{y} + \bar{u}\right) \tag{58}$$

follows from (55). The feedback law (58) compensates the nonlinearities in the direct input-ouput channel of the observer achieving the linear input-output behaviour

$$\dot{\hat{y}} = \bar{u} \tag{59}$$

if a vector h^T can be found satisfying

$$h^{T}\left[CAV_2 \;\; CNV_2\right] = 0^{T} \tag{60}$$

Condition (60) assures that the unmeasurable states $\tilde{z}$ do not effect the input-output behaviour (see (55)). The expression $(g^T L + h^T CAV_1)\tilde{y}$ in (58) can be regarded as a feedforward of the measurable "disturbance" $\tilde{y}$, such that $\tilde{y}$ does not effect the input-ouput behaviour of the observer. The closed loop eigenvalue $\tilde{\lambda} = -\tilde{a}_0$, $\tilde{a}_0 > 0$, is assigned to the dynamics (49) of the input-output behaviour by the feedback portion

$$\bar{u} = -\tilde{a}_0\hat{y} + \tilde{a}_0 w \tag{61}$$

in (58). Note that the feedback (58) is directly implementable since only the measurable states $\hat{\zeta}$ of the observer and the measurable states $\tilde{y}$ of the bilinear approximation model are fed back.

3.2 Design procedure

For the design of the dynamic output feedback controller derived in this section the design equations (17)-(19) for the observer of Section 2 have to be solved using the results of Section 2.2. The only difference is equation (20), which is replaced by (54). Equation (54) and (60) can be written in one equation as

$$\begin{bmatrix} g^T & h^T \end{bmatrix} \begin{bmatrix} T & 0 & 0 \\ C & CAV_2 & CNV_2 \end{bmatrix} = \begin{bmatrix} c^T & 0^T \end{bmatrix} \tag{62}$$

In view of the discussion in Remark 4 the order $\bar{n}$ of the dynamic output feedback controller obviously may be smaller than the observer order, since the matrix at the right hand side of (62) has rank 1 yielding the lower bound $\bar{n} \geq 1$.

4 Example

Consider the following nonlinear second order system

$$\begin{aligned} \dot{x}_1 &= x_2 + x_2^2 - 2u && (63)\\ \dot{x}_2 &= -2x_1 - 3x_2 + u && (64)\\ y &= x_2 && (65) \end{aligned}$$

with relative degree one. First a state observer for the system (63)-(65) is designed using the Carleman linearization [7, 9]. To this end introduce the new states

$$z = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \end{bmatrix}^T = \begin{bmatrix} x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{bmatrix}^T \tag{66}$$

yielding a bilinear approximation model (see (7)-(8)) of order 5 with the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ -2 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & -3 & 1 \\ 0 & 0 & 0 & -4 & -6 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} b &= \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \end{bmatrix}^T \\ c^T &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Since the state x_2 is measurable (see (65)) one obtains the following subdivision of the states z (see (66)) into measurable states $\tilde{y}$ and unmeasurable states $\tilde{z}$

$$\begin{aligned} z &= V_1\tilde{y} + V_2\tilde{z} \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_2 \\ z_5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_3 \\ z_4 \end{bmatrix} \end{aligned} \tag{67}$$

Since two states of the bilinear approximation model are measurable ($m = 2$) and since the transfer matrix (40) satisfies $\operatorname{rank} F(s) = 1 < m$, for all s (NV_2 contains two zero columns) the observer eigenvalues can be assigned arbitrarily. By choosing the observer eigenvalue $\bar{\lambda} = -6$ the corresponding parameter vector $\bar{p}^T = \begin{bmatrix} 0.9950 & 0.0995 \end{bmatrix}$ results from solving (39), such that the matrices L and T are given by (33) and (34). The output equation (14) of the observer is obtained by computing the solution of (42) yielding the reduced order observer of order one

$$\dot{\hat{\zeta}} = -6\hat{\zeta} + 0.0995u + \begin{bmatrix} 0.9950 & 0.0995 \end{bmatrix} \begin{bmatrix} x_2 \\ x_2^2 \end{bmatrix} \tag{68}$$

$$\hat{x} = \begin{bmatrix} 10.0499 \\ 0 \end{bmatrix} \hat{\zeta} + \begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_2^2 \end{bmatrix} \tag{69}$$

Figure 1 shows the simulation of the observer with initial condition $\hat{\zeta}(0) = 0.2985$ and step input $u = 1$.

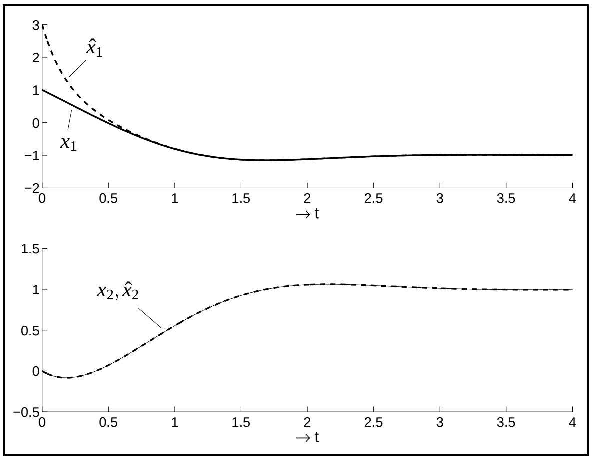

Figure 1: States of the system and the corresponding estimates.

Obviously the observer (68)-(69) yields an exact estimate for x_1 asymptotically, whereas the system ouput y is directly used to reconstruct the state x_2 without error. These results are due to the fact, that the nonlinearity "x_2^2" of the system under consideration is measurable and a polynomial. As a consequence an exact observer with linear error dynamics and the nonlinearity "x_2^2" as input exists, which is reproduced exactly by the proposed approach though a bilinear approximation model is employed in the design.

Next a dynamic output feedback controller, which achieves an asymptotically exact input-output linearization of the system (63)-(65), is designed. Since the design equation (62) is not solvable for the previous first order observer, one has to increase the observer order. Thus a second order observer with eigenvalues $\tilde{\lambda}_1 = -12$ and $\tilde{\lambda}_2 = -10$ is investigated yielding the parameter vectors $\bar{p}_1^T = \begin{bmatrix} 0.9998 & 0.0182 \end{bmatrix}$ and $\bar{p}_2^T = \begin{bmatrix} 0.9996 & 0.0278 \end{bmatrix}$ by evaluating (39). The matrices T and L are determined by (33) and (34) with $W = I$. For the resulting second order observer the design equation (62) is solvable, such that by assigning the closed loop eigenvalue $\tilde{\lambda} = -3$ to the closed loop input-output behaviour the dynamic output feedback controller

$$\begin{aligned} \dot{\hat{\zeta}} &= \begin{bmatrix} -12 & 0 \\ 0 & -10 \end{bmatrix} \hat{\zeta} + \begin{bmatrix} 0.0727 \\ 0.0833 \end{bmatrix} u + \begin{bmatrix} 0.9998 & 0.0182 \\ 0.9996 & 0.0278 \end{bmatrix} \begin{bmatrix} x_2 \\ x_2^2 \end{bmatrix} \\ \hat{y} &= \begin{bmatrix} 55.0091 & -36.0139 \end{bmatrix} \hat{\zeta} \end{aligned}$$

with the feedback law

$$u = -\begin{bmatrix} -660.1091 & 360.1389 \end{bmatrix} \hat{\zeta} - \begin{bmatrix} 19.0000 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_2^2 \end{bmatrix} - 3\hat{y} + 3w \tag{70}$$

is obtained. Note that the feedback (70) is also applied to the plant (63)-(65).

Figure 2 depicts the step response of the resulting closed loop system.

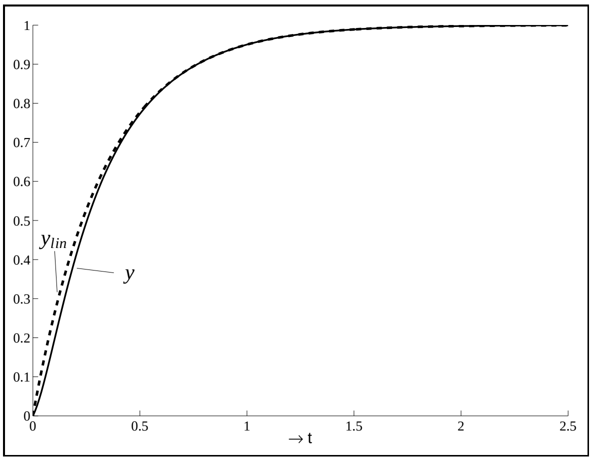

Figure 2: Step response of the closed loop system (solid line). The dashed line shows the linear reference behaviour y_{lin}.

The simulation verifies the convergence of the closed loop input-output behaviour (solid line) to the linear reference behaviour (dashed line)

$$\dot{y}_{lin} + 3y_{lin} = 3w \tag{71}$$

5 Conclusions

In this contribution a new observer design procedure is presented on the basis of the Carleman linearization. By way of a simple example it is shown, that the proposed approach reproduces the exact observer with linear error dynamics if all nonlinearities are polynomial and measurable. An extension of this result to nonpolynomial nonlinearities is possible by introducing nonpolynomial nonlinearities as new states yielding a higher order polynomial system (see [6]). However, it is interesting to investigate the observer design for systems with unmeasurable nonlinearities. This will be addressed in future work using L_2-optimal bilinearization (see [3]), which different from the Carleman linearization yields a system approximation on a prespecified n-dimensional interval in the state space. The proposed observer was also extended to the design of dynamic output feedback controllers for relative degree one and minimum phase systems. The dynamics of the resulting linear reference input-output behaviour and the dynamics of the convergence to the reference behaviour can be assigned independently. An additional advantage of the presented observer and controller design is, that it can be carried out using numerical software packages, since the design procedures only involve the manipulation of constant matrices.

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