Direct Adaptive Control Design and Synchronization of Chua's Circuits

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Abstract
Direct adaptive control design is presented for linear systems where there is no need to solve any Lyapunov equation simplifying the result introduced in [3]. Using this direct adaptive controller, a numerical example for the synchronization problem of Chua's circuit using one transmitted signal is exhibited too.

1 Introduction

In this paper a direct adaptive control design for linear systems is developed in the similar way than in [3] but we do not require to solve any Lyapunov equation to implement the adaptive controller. This bring us more flexibility and simplicity in the algorithm design. In general, the algorithm design presented in [3], for the state feedback case, require three steps (Corollary 2, [3]): 1) to find a matrix K_s such that $A_s = A + BK_s$ be a Hurwitz matrix, where the matrix A comes from the linear system, 2) to find a matrix R such that (A_s, R) be controllable, and 3) to solve the Lyapunov equation $0 = A_s^T P + PA_s + R$. Certainly, using a computer there is not problem in solving the Lyapunov equation, but for high order systems, it is useful no to solve it [7]. Our propose simplifies this algorithm design into only the first step. The final results obtained was then used successfully in solving the synchronization problem for the Chua's circuit where only one transmitted signal is required, making the final adaptive controller attractive for the design of masking systems (see, for example, [2] and [1]). The restriction in solving the Lyapunov equation could be eliminated by selecting an appropriate Lyapunov function in the same context that in [7].

The organization of the paper is as follows. In section two we introduced the algorithm design used by [3] and we also presented our main result. In section three we utilized our main result to show a solution to the synchronization problem using the Chua's circuit. Finally, section four the conclusions are stated.

2 Problem Statement

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) + d \tag{1}$$

where $x(t) \in \Re^{n_x}$, $u(t) \in \Re^{n_u}$, and $d \in \Re^{n_x}$. We have the next result.

Theorem 1 [3].- *Assume there exists $K_s \in \Re^{n_u \times n_x}$ such that $A_s = A + BK_s$ is asymptotically stable and assume there exists $\phi_s \in \Re^{n_u}$ such that $B\phi_s = d$. Let $R \in \Re^{n_x \times n_x}$ be positive semidefinite and assume (A_s, R) is controllable. Let $P \in \Re^{n_x \times n_x}$ be positive definite solution to the Lyapunov equation $0 = A_s^T P + PA_s + R$. Finally, let $\Gamma \in \Re^{n_u \times n_u}$ and $\Lambda \in \Re^{n_x \times n_x}$ be positive definite, and let $\lambda > 0$. Then (1) with control law*

$$u(t) = K(t)x(t) + \phi(t) \tag{2}$$

where

$$\dot{K}(t) = -\Gamma B^T Px(t)x^T(t)\Lambda \tag{3}$$

$$\dot{\phi}(t) = -B^T Px(t)\lambda \tag{4}$$

yields $Rx(t) \to 0$ as $t \to \infty$.

Note that expressions from (3) to (4) requires to solve the Lyapunov equation $0 = A_s^T P + PA_s + R$.

Define

$$\widehat{K}(t) = K(t) - K_s \tag{5}$$

$$\widehat{\phi}(t) = \phi(t) + \phi_s; \tag{6}$$

so that, (3) to (4), implies

$$\dot{\widehat{K}}(t) = -\Gamma B^T Px(t)x^T(t)\Lambda \tag{7}$$

$$\dot{\widehat{\phi}}(t) = -B^T Px(t)\lambda. \tag{8}$$

Then, the closed-loop system consists of (7) -(8) and

$$\dot{x}(t) = (A_s + B\widehat{K})x(t) + B\widehat{\phi}(t).$$

The proof of Theorem 1 is based on the next Lyapunov function ([3]):

$$V(x, \widehat{K}, \widehat{\phi}) = x^T P x + tr(\Gamma^{-1}\widehat{K}\Lambda^{-1}\widehat{K}^T) + tr(\widehat{\phi}\lambda^{-1}\widehat{\phi}^T) \tag{9}$$

where the time derivative of (9) along of the trajectories of the closed-loop system yields

$$\dot{V}(x, \widehat{K}, \widehat{\phi}) = 2x^T P[(A_s + B\widehat{K})x(t) + B\widehat{\phi}]$$
$$+2tr(\Gamma^{-1}\widehat{K}\Lambda^{-1}\dot{\widehat{K}}^T) + 2tr(\widehat{\phi}\lambda^{-1}\dot{\widehat{\phi}}^T)$$
$$= x^T[A_s^T P + PA_s]x + 2x^T PB\widehat{K}x + 2x^T PB\widehat{\phi}$$
$$+2tr(\Gamma^{-1}\widehat{K}\Lambda^{-1}\dot{\widehat{K}}^T) + 2tr(\widehat{\phi}\lambda^{-1}\dot{\widehat{\phi}}^T)$$
$$= -x^T Rx + 2tr(-\widehat{K}xx^T PB + x^T PB\widehat{K}x)$$
$$+2tr(x^T PB\widehat{\phi} - \widehat{\phi}x^T PB) = -x^T Rx$$

This conclude the proof ■.

Next is our main result.

Theorem 2.- *Assume there exists $K_s \in \Re^{n_u \times n_x}$ such that $A_s = A + BK_s$ is asymptotically stable and assume there exists $\phi_s \in \Re^{n_u}$ such that $B\phi_s = d$. Let $R \in \Re^{n_x \times n_x}$ be positive definite and assume (A_s, R) is controllable. Let $P \in \Re^{n_x \times n_x}$ be positive definite solution to the Lyapunov equation $0 = A_s^T P + PA_s + R$ and assume that $RP^{-1} = P^{-1}R$. Finally, let $\Gamma \in \Re^{n_u \times n_u}$ and $\Lambda \in \Re^{n_x \times n_x}$ be positive definite, and let $\lambda > 0$. Then (1) with control law*

$$u(t) = K(t)x(t) + \phi(t) \tag{10}$$

where

$$\dot{K}(t) = -\Gamma B^T x(t)x^T(t)\Lambda \tag{11}$$

$$\dot{\phi}(t) = -B^T x(t)\lambda \tag{12}$$

yields $Rx(t) \to 0$ as $t \to \infty$.

Proof.- The proof is similar to the proof of Theorem 1 but using now the next Lyapunov function:

$$V(x, \widehat{K}, \widehat{\phi}) = x^T x + tr(\Gamma^{-1}\widehat{K}\Lambda^{-1}\widehat{K}^T) + tr(\widehat{\phi}\lambda^{-1}\widehat{\phi}^T) \tag{13}$$

where its time derivative along of the trajectories of the closed-loop system yields

$$\dot{V}(x, \widehat{K}, \widehat{\phi}) = 2x^T[(A_s + B\widehat{K})x(t) + B\widehat{\phi}]$$
$$+2tr(\Gamma^{-1}\widehat{K}\Lambda^{-1}\dot{\widehat{K}}^T) + 2tr(\widehat{\phi}\lambda^{-1}\dot{\widehat{\phi}}^T)$$
$$= x^T[A_s^T + A_s]x + 2x^T B\widehat{K}x + 2x^T B\widehat{\phi}$$
$$+2tr(\Gamma^{-1}\widehat{K}\Lambda^{-1}\dot{\widehat{K}}^T) + 2tr(\widehat{\phi}\lambda^{-1}\dot{\widehat{\phi}}^T)$$
$$= tr(x^T(PA_sP^{-1} + A_s^T PP^{-1})x) +$$
$$2tr(-\Gamma^{-1}\widehat{K}xx^T B\Gamma + x^T B\widehat{K}x) + 2tr(x^T B\widehat{\phi} - \widehat{\phi}\lambda^{-1}\lambda x^T B)$$
$$= tr(x^T(PA_s + A_s^T P)P^{-1}x) + 2tr(-\widehat{K}xx^T B + x^T B\widehat{K}x)$$
$$+2tr(x^T B\widehat{\phi} - \widehat{\phi}x^T B)$$
$$= tr(x^T(-R)P^{-1}x) = -tr(x^T RP^{-1}x).$$

Note that there exists a term $R^{-1}P$, where the matrices R^{-1} and P are positive definite. The multiplication of two matrices positive definite is not (in general) a definite positive matrix; however , we can select $R = I$ without altering the main proof; i.e., $\dot{V}(x, \widehat{K}, \widehat{\phi}) = -tr(x^T IP^{-1}x) = -x^T P^{-1}x$.■

Note that since A_s is Hurwitz matrix, it is guaranteed that for any given $R = R^T > 0$ there exists a unique symmetric and positive definite matrix P solution to the Lyapunov equation $0 = A_s^T P + PA_s + R$; so, we arrive to the next Corollary.

Corollary 3 *Assume that (A, B) is stabilizable and assume that there exists $\phi_s \in \Re^{n_u}$ such that $B\phi_s = d$. Finally, let $\Gamma \in \Re^{n_u \times n_u}$ and $\Lambda \in \Re^{n_x \times n_x}$ be positive definite, and let $\lambda > 0$. Then (1) with control law*

$$u(t) = K(t)x(t) + \phi(t)$$

where

$$\dot{K}(t) = -\Gamma B^T x(t)x^T(t)\Lambda$$

$$\dot{\phi}(t) = -B^T x(t)\lambda$$

yields $Rx(t) \to 0$ as $t \to \infty$.

In fact, in Theorem 2 we have to compute P to make sure that $RP^{-1} = P^{-1}R > 0$; but, if we use $R = I$, Corollary 3 is indeed a new control law.

3 Synchronization of Chua's circuits

The Chua's circuit consists of two resistors, R_1 and R_2, two capacitors, C_1 and C_2, an inductor L_1, and a nonlinear resistor (see Fig. 1). The piecewise-linear $v_1 - i_C$ characteristic of the nonlinear resistor (the Chua's diode) is as follows ([1])

$$f(v_1) = G_b v_1 + \frac{1}{2}(G_a - G_b)(|v_1 + E| - |v_1 - E|),$$

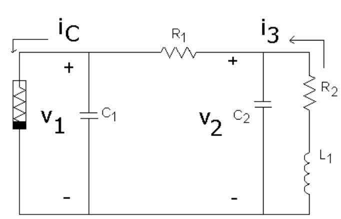

Fig. 1 Chua's circuit.

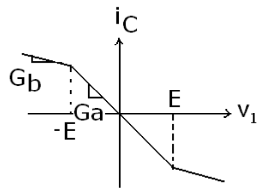

Fig. 2 Nonlinear representation of v_1 versus i_C.

After appropriate state variables representation and parameters transformation, the set of differential equations of the Chua's circuit is give by ([8])

$$\begin{aligned}\dot{x}_1(t) &= \alpha(y_1(t) - x_1(t) - f(x_1(t)) \\ \dot{y}_1(t) &= x_1(t) - y_1(t) + z_1(t) \\ \dot{z}_1(t) &= -\beta y_1(t) - \gamma z_1(t)\end{aligned} \tag{14}$$

and

$$f(x_1(t)) = bx_1(t) + 0.5(a-b)(|x_1(t)+1| - |x_1(t)-1|)$$

where x_1, y_1, and z_1 are state variables, and α, β and γ are positive real constants. The real constants a and b are negative with $a < -1$ and $-1 < b < 0$. The dynamic of the Chua's circuit above described contains three equilibrium points being the origin one of them.

In the context of synchronization, the system (14) is named the *drive* system and a second set given by

$$\begin{aligned}\dot{x}_2(t) &= \alpha(y_2(t) - x_2(t) - f(x_1(t)) + u + d_0 \\ \dot{y}_2(t) &= x_2(t) - y_2(t) + z_2(t) \\ \dot{z}_2(t) &= -\beta y_2(t) - \gamma z_2(t)\end{aligned} \tag{15}$$

is called the *response* system, where d_o can be some disturbance, and u is the external input which will be suitable designed such that

$$\lim_{t\to\infty} (x_2(t) - x_1(t)) = 0$$

$$\lim_{t\to\infty} (y_2(t) - y_1(t)) = 0$$

and

$$\lim_{t\to\infty} (z_2(t) - z_1(t)) = 0.$$

Observe that $x_1(t)$ is the transmitted signal and it appears into the nonlinear term in (15). We want $u = u(x_1(t))$ too. Define the states errors between the response systems and the drive system as

$$\begin{aligned}e_x(t) &= x_2(t) - x_1(t) \\ e_y(t) &= y_2(t) - y_1(t), \\ e_z(t) &= z_2(t) - z_1(t).\end{aligned} \tag{16}$$

Subtracting (14) from (15) and using (16) we have

$$\begin{aligned}\dot{e}_x(t) &= \alpha(e_y(t) - e_x(t)) + u + d_o \\ \dot{e}_y(t) &= e_x(t) - e_y(t) + e_z(t) \\ \dot{e}_z(t) &= -\beta e_y(t) - \gamma e_z(t)\end{aligned} \tag{17}$$

Clearly, the synchronization problem is now equivalent to the stabilization problem for the system (17) based on the adequate design of the external input $u(t)$.

The system described in (17) can be written as follows:

$$\dot{x}(t) = \underbrace{\begin{pmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{pmatrix}}_{A} x(t) + \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{B} u + \underbrace{\begin{pmatrix} d_o \\ 0 \\ 0 \end{pmatrix}}_{d} \tag{18}$$

where $x^T = [\ x_1 \quad x_2 \quad x_3\ ]^T = [\ e_x \quad e_y \quad e_z\ ]^T$. We can use Theorem 2 to design an adaptive control law to solve the synchronization problem.

Selecting the parameters of Chua's circuit as $\alpha = 10$, $\beta = 13.15$, and $\gamma = 0.07727$ with $K_s = [\ 1 \quad 0 \quad 0\ ]$, we obtain that the matrix $A_s = A + BK_s$ is Hurwitz, which the existence to the solution to the Lyapunov equation $0 = A_s^T P + PA_s + R$ is guaranteed for any positive definite and symmetric matrix R. With $\phi_s = -d_o$ the condition $B\phi_s = d$ is fulfilled.

From Theorem 2, with $\lambda = 10$, $\Gamma = 10$, and $\Lambda = I_{3\times 3}$ we obtain

$$\begin{aligned}u &= k_1 e_x + k_2 e_y + k_3 e_z + \phi \\ \dot{k}_1(t) &= -10 e_x^2(t) \\ \dot{k}_2(t) &= -10 e_x(t) e_y(t) \\ \dot{k}_3(t) &= -10 e_x(t) e_z(t) \\ \dot{\phi}(t) &= -10 e_x(t)\end{aligned}$$

From (5), we can see that the adaptive controller tries to adapt the parameters in $K(t)$ to the parameters in K_s, and because one possible value of it is $K_s = [\ k_{s1} \quad 0 \quad 0\ ] = [\ 1 \quad 0 \quad 0\ ] = [\ k_1(t) \quad k_2(t) \quad k_3(t)\ ]$, we can pick up $k_2(t) = 0$ and $k_2(t) = 0$; so, the final controller obtained is

$$u = k_1 e_x + \phi \tag{19}$$

$$\dot{k}_1(t) = -10 e_x^2(t) \tag{20}$$

$$\dot{\phi}(t) = -10 e_x(t) \tag{21}$$

Proposition 4.- *The direct adaptive controller (19)-(21) solves the synchronization problem.*

Proof.- The closed-loop system of the adaptive controller proposed with (18) yields

$$\dot{x}(t) = A_s x(t) + \begin{pmatrix} \widehat{k}_1(t) x_1(t) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \widehat{\phi}(t) \\ 0 \\ 0 \end{pmatrix} \tag{22}$$

where $\widehat{k}_1 = k_1 - k_{s1}$ and $\widehat{\phi} = \phi + \phi_s$. Let us use the next Lyapunov equation

$$V(x, \widehat{k}_1, \widehat{\phi}) = x^T x + \frac{1}{10}\widehat{k}_1^2 + \frac{1}{10}\widehat{\phi}^2$$

where its time derivative along of the trajectories of the system (22) yields

$$\dot{V}(x, \widehat{k}_1, \widehat{\phi}) = 2x^T A_s x + 2x^T \begin{pmatrix} \widehat{k}_1 x_1 \\ 0 \\ 0 \end{pmatrix}$$

$$+2x^T \begin{pmatrix} \widehat{\phi} \\ 0 \\ 0 \end{pmatrix} + \frac{2}{10}\widehat{k}_1\ \dot{\widehat{k}}_1 + \frac{2}{10}\widehat{\phi}\ \dot{\widehat{\phi}}$$

$$= x^T(A_s + A_s^T)x + 2\widehat{k}_1 x_1^2 + 2\widehat{\phi}x_1 + \frac{2}{10}\widehat{k}_1\ \dot{\widehat{k}}_1 + \frac{2}{10}\widehat{\phi}\ \dot{\widehat{\phi}}$$

$$= x^T(A_s + A_s^T)x + 2\widehat{k}_1 x_1^2 + 2\widehat{\phi}x_1 - 2\widehat{k}_1 x_1^2 - 2\widehat{\phi}x_1$$

$$= x^T(A_s + A_s^T)x = tr(x^T(A_s + A_s^T)x)$$

$$= tr(x^T(PA_sP^{-1} + A_s^T PP^{-1})x)$$

$$= tr(x^T(PA_s + A_s^T P)P^{-1}x) = tr(x^T(-R)P^{-1}x)$$

$$= -tr(x^T RP^{-1}x).$$

Following the same lines used to prove Theorem 2 and Corollary 3, we concluded the proof of Proposition 4.■

Simulation results are shown in Figures 3 and 4 using $a = -1.28$, and $b = -0.69$. The initials condition used were $x_1(0) = 1.1$, $y_1(0) = z_1(0) = 0$ for the *drive* system and $x_2(0) = -5$, $y_2(0) = 5$, and $z_2(0) = 0$ for the *response* system. $k_1(0) = 0$, $\phi(0) = 0$, and $d_o = 1$.

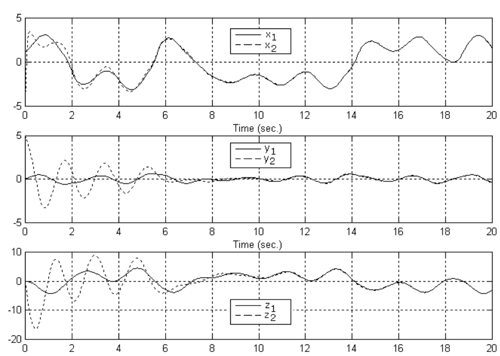

Fig. 3 Simulation results.

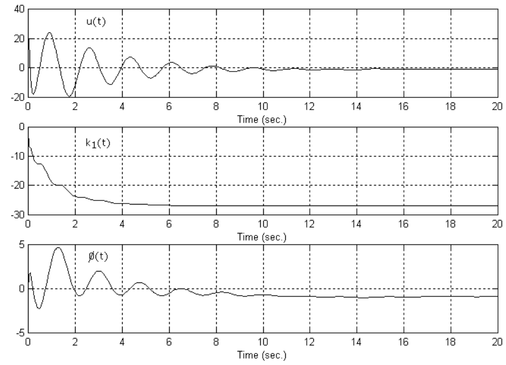

Fig. 4 Simulation results

4 Conclusions

The design of a direct adaptive control is introduced where we do not require to solve any Lyapunov equation for the controller design. Using this result, we could solve the synchronization problem using Chua's circuit and where only one transmitted signal was required. This result was facilitated due to the simplicity of our propose.

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