ℓ_1-OPTIMAL CONTROL WITH ASYMMETRIC BOUNDS

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Abstract

In this paper a novel way to handle the analysis and design of control systems with bounded asymmetrical signals is presented. Instead of the standard peak-to-peak norm, normally used in ℓ_1-optimal control, this paper propose to use an asymmetric objective functional, which simplifies the consideration of asymmetrically bounded signals. Necessary and sufficient conditions are derived to check if some performance specifications, expressed using this objective functional, are fulfilled. The analysis and the synthesis problems are also formulated with respect to this objective functional: As in the standard ℓ_1-optimal control, this approach leads to a linear programming problem which can be solved by off-the-self tools.

1 Introduction

Usually, real plants are subject to constrained variables. The most frequent constraints are of saturation type, that is, limitations on the magnitude of certain variables, usually the control signal. Hence, the topic of designing control systems that maintains stability (and performance) in the presence of these constraints is a topic of continuing interest. There are several approaches proposed in the literature to solve the this problem; we can cite, not exhaustively, the Positive Invariance concept [1,2], the Small and High Gain concept [9], the l_1 Norm Optimization concept [6], Predictive Control [4] and several other approaches [8,13].

This paper uses the ℓ_1-optimal control theory [6], which has received considerable attention, because it gives simple methods to calculate controllers, for different kinds of constraints (limitations in magnitude, slope, overshoot, undershoot, etc). This approach is based on Constraint Avoidance as defined in [8]: preventing the saturation, the closed-loop system stays in a region of linear behavior. In [17] this problem was first formulated. Some basic results on this theory were presented in [7] for the special case of square systems. Then, the solution of the non square case was given in [5]. These results have shown that the ℓ_1-optimal control problem can be stated as linear programming problem. Since then, many other results and extensions have been presented [3,12]. Furthermore, compared with other optimal techniques to deal with constraints, application to real process have already been published [10,15,16].

This paper studies a novel extension of the ℓ_1-optimal control theory, to directly handle the asymmetrical aspect of bounded signals, following the ideas presented for the Positive Invariance concept in [1]. Furthermore, the analysis problem of closed-loop systems performance is formulated, and the transformation of the synthesis problem of an appropriate controller as a linear programming problem is presented. For simplicity reasons the problem is presented and solved in the SISO case, but can be easily extended to the multivariable case, following the ideas presented in [6].

The paper is organized as follows: Section 2 defines a new asymmetric objective functional which can handle asymmetrical signals. In addition, a necessary and sufficient condition to check if some performance specifications are fulfilled is discussed and proved. In section 3, the formulation of the analysis problem taking into account the previous result is presented, and an illustrative example will also be given. Section 4 is concerned with the design procedure of an optimal controller with respect to the proposed objective functional.

2 Asymmetric objective functional definition

In this section we define the new objective functional which will replace the peak-to-peak-norm normally used in the ℓ_1 analysis.

A drawback of using the peak-to-peak-norm appears when the considered signal $x(k)$ is constrained to evolve in asymmetrical domain (i.e., signal $x(k)$ is such that $-x_{\min} \leq x(k) \leq x_{\max}$, where $x_{\min} \geq 0$; $x_{\max} \geq 0$; $x_{\min} \neq x_{\max}$). In this case the peak-to-peak-norm of the signal $x(t)$ (by definition, its -norm: $\|x(k)\|_\infty = \max_t (|x(k)|) = \max(x_{\max}, x_{\min})$) gives a truncated information on the signal amplitude.

Of course this problem can be solved, in some situations, by redefining the working point. For example, if the control signal is bounded in amplitude, the value of the control signal at the working point can be assigned to the mean value of the constraints ($x_{mean} = x_{\max} - x_{\min}$; $\|x(k)\|_\infty = \max_t (|x(k)|) = \frac{x_{\max}+x_{\min}}{2}$). However, even for this simple situation, this technique is inadequate if the constraints change with time, or the system behaviour changes with the working point. Moreover, it is very cumbersome if additional constraints are imposed (constraints on the control variation, on the output amplitude, etc.).

This paper proposes to solve this drawback by defining a new objective functional (noted $\|x(k)\|_d$) which gives more information on the signal amplitude, including its asymmetricity:

$$\|x(k)\|_d = \begin{bmatrix} \max\left\{0, \max\limits_k (x(k))\right\} \\ \max\left\{0, -\min\limits_k (x(k))\right\} \end{bmatrix}.$$

In our context this objective functional is used to handle the asymmetrical aspect of any bounded signal as the examples discussed below:

- In most practical control systems, due to technological and safety reasons the actuators cannot drive unlimited energy to the controlled plant. This fact can be translated into bounds on control $u(t)$ as $-u_{\min} \leq u(k) \leq u_{\max}$. This can be written as $\|u(k)\|_d \leq U$ where $U^t = [u_{\max}, u_{\min}]$.
- In many industrial systems only a range on the amplitude of disturbance signal may be known. Generally, $w(t)$ is such that $-w_{\min} \leq w(k) \leq w_{\max}$ which can be rewrite as $\|w(k)\|_d \leq W$ where $W^t = [w_{\max}, w_{\min}]$.
- Moreover, there are frequently limitations on the control rate: In many applications these limitations arise as an inherent behavior of the actuators [11]. The control increment between sample times fulfills $-\Delta u_{\min} \leq u(k) - u(k-1) \leq \Delta u_{\max}$, or in more compact form using the shift operator $-\Delta u_{\min} \leq (1-z^{-1})u(k) \leq \Delta u_{\max}$ this can be rewritten as $\left\|(1-z^{-1})u(k)\right\|_d \leq \Delta U$.

Following the usual approach in ℓ_1 Optimization, these and some additional specifications can be written as:

$$\left\|G(z^{-1})u(k)\right\|_d \leq U.$$

Let us now consider $u(t)$ and $y(t)$ as input and output signals of a linear time invariant discrete-time system described by its transfer function $G(z^{-1})$. Necessary and sufficient condition to have $\|y\|_d \leq Y$ for any bounded input $\|u\|_d \leq U$ is given by the following proposition.

Proposition 1. *The output $y(k)$ of a LTI system is constrained in the non symmetrical domain defined by $\|y\|_d \leq Y$ for any input $u(k)$ such that $\|u\|_d \leq U$ if and only if:*

$$\Psi\, U \leq Y \tag{1}$$

where $\Psi = \begin{bmatrix} \sum\limits_{i=1}^{\infty}\Phi_i^+ & \sum\limits_{i=1}^{\infty}\Phi_i^- \\ \sum\limits_{i=1}^{\infty}\Phi_i^- & \sum\limits_{i=1}^{\infty}\Phi_i^+ \end{bmatrix}$; $\begin{array}{l} \Phi_i^+ = \max\{\Phi_i, 0\} \\ \Phi_i^- = \max\{-\Phi_i, 0\} \end{array}$

and $\{\Phi_i \quad i = 1, ..., \infty\}$ denotes the impulse response of the transfer function $G(z^{-1})$ of the system.

Proof. Since the output can be written using the impulse response of the system as

$$y(k) = \sum_{i=1}^{k} \Phi_i\ u(k-i)$$

then, for any bounded input signal such that $\|u\|_d \leq U$ we can write

$$\begin{cases} \max\limits_{\|u\|_d \leq U} \{y(k)\} = \sum\limits_{i=1}^{k}\Phi_i^+\ u_{\max} + \sum\limits_{i=1}^{k}\Phi_i^-\ u_{\min} \\ \min\limits_{\|u\|_d \leq U} \{y(k)\} = -\sum\limits_{i=1}^{k}\Phi_i^-\ u_{\max} - \sum\limits_{i=1}^{k}\Phi_i^+\ u_{\min} \end{cases}.$$

thus, it follows that

$$\begin{cases} -\sum\limits_{i=1}^{k}\Phi_i^-\ u_{\max} - \sum\limits_{i=1}^{k}\Phi_i^+\ u_{\min} \leq y(k) \\ y(k)\ \leq\ \sum\limits_{i=1}^{k}\Phi_i^+\ u_{\max} + \sum\limits_{i=1}^{k}\Phi_i^-\ u_{\min}. \end{cases}$$

Since $y(k)$ can reach these limits, then a necessary and sufficient condition to have -$y_{\min} \leq y(k) \leq y_{\max}$ is given by

$$\begin{cases} \sum\limits_{i=1}^{k}\Phi_i^+\ u_{\max} + \sum\limits_{i=1}^{k}\Phi_i^-\ u_{\min} \leq y_{\max} \\ -y_{\min} \leq -\sum\limits_{i=1}^{k}\Phi_i^-\ u_{\max} - \sum\limits_{i=1}^{k}\Phi_i^+\ u_{\min} \end{cases}.$$

which is equivalent to $\Psi\, U \leq Y$. □

Remark 2. *It is well known that the sums defined in matrix Ψ converge if and only if $G(z^{-1})$ is a stable transfer function.*

Remark 3. *If the constraints are symmetric ($u_{\min} = u_{\max}$ and $y_{\min} = y_{\max}$), the second row in the definition of Φ is redundant, and the standard ℓ_1 optimization problem is obtained:* $\sum\limits_{i=1}^{k} |\Phi_i|\ u_{\max} \leq y_{\max}$

3 Analysis Problem

This section shows how the previous result can be used to check if a given regulator fulfills some performance specifications. For this, let us first consider an illustrative example given in figure (1).

Assume that the input and the disturbance signals are subject to non symmetrical constraints.

$$\begin{array}{lcl} u(k) & \in & \left\{u(k)\quad / \quad \|u(k)\|_d \leq U\ ;\quad U^T = [2, 3]\right\} \\ w(k) & \in & \left\{w(k)\quad / \quad \|w(k)\|_d \leq W\ ;\quad W^T = [1, 4]\right\}. \end{array}$$

The transfer function of the plant and the controller are given by

$$G(z^{-1}) = \frac{z^{-1}(3-z^{-1})}{(2-z^{-1})(4-z^{-1})} \quad \text{and} \quad K(z^{-1}) = \frac{(2-z^{-1})}{(3-z^{-1})}.$$

Our objective is to check if the considered regulator prevents input saturation for any disturbance signal in the predefined

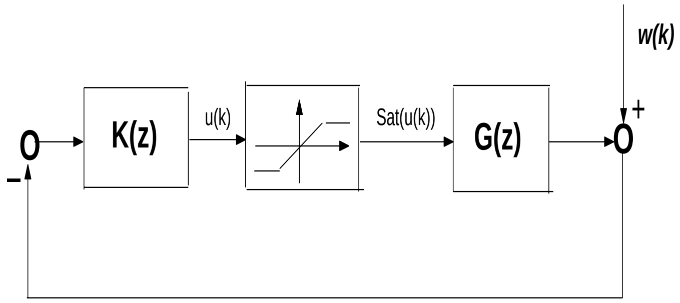

Figure 1: System with constrained input.

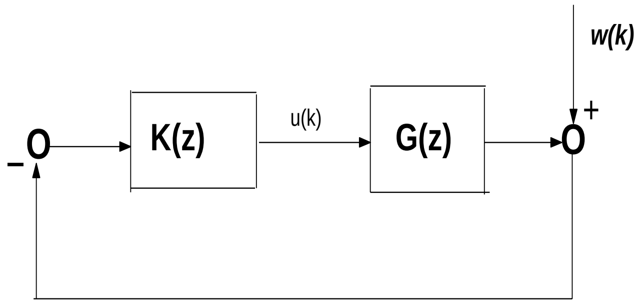

Figure 2: System with output disturbance

domain. For this, denote as $H(z^{-1})$ the input-output transfer function between $u(k)$ and $w(k)$.

$$H(z^{-1}) = \frac{-K(z^{-1})}{1+K(z^{-1})G(z^{-1})} = \frac{-(2-z^{-1})(4-z^{-1})}{4(3-z^{-1})}$$

The inequality (1) given in the last proposition is then tested

$$\begin{bmatrix} 0.2917 & 0.6667 \\ 0.6667 & 0.2917 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

It is easy to check that (1) is satisfied: Thus, for every disturbance signal $w(k)$ such that $\|w(k)\|_d \leq W$ it is possible to ensure that no saturation occurs.

Remark 4. *It is interesting to note that for this same example, using the standard peak-to-peak norm it is impossible to make such judgement. Effectively, using the peak-to-peak norm we have to consider that the control must be as* $\|u(k)\|_\infty \leq (\min(u_{\max}, u_{\min}) = 2)$ *and that the disturbance signal* $w(k)$ *is such that* $\|w(k)\|_\infty \leq (\max(w_{\max}, w_{\min}) = 4)$. *The* ℓ_1-*norm of* $H(z^{-1})$ *is computed,* $\left\|H(z^{-1})\right\|_1 = 0.9583$. *It is possible to see that* $\left\|H(z^{-1})\right\|_1 4 > 2$.

Then, the analysis problem can be summarized in three steps:

- Express the input and output signals using the asymmetric objective functional $\|u(t)\|_d \leq U$ and $\|y(t)\|_d \leq Y$.
- Find the transfer function between input and output signal $y(t) = H(z^{-1})u(t)$.
- Check if $\left\|H(z^{-1})\right\|_d U \leq Y$.

Remark 5. *As it is possible to see in the previous example, the denominations* Input *and* Output *are context dependent.*

4 Synthesis Problem

By analogy with the ℓ_1-optimal control this section gives a new formulation which take into account the asymmetrical aspect of different signals. Note that, for simplicity reasons, the study presented in this paper is restricted to the one block problem, but can be easily extended to the non-square case.

Consider the system given in in figure (2), where $G(z^{-1})$ and $K(z^{-1})$ are respectively the plant and the controller transfer function. We note $H(z^{-1})$ the closed-loop transfer function from the disturbance signal to the output. The disturbance signal is supposed to evolve in the domain given by $\{w \in \mathcal{R} \ / \ \|w\|_d \leq W\}$. In this context we are looking, among all internally stabilizing controllers, for the one which minimizes the effect of the non symmetrical bounded disturbance signal on the output. Using the previous development this can be stated as:

$$\min \varepsilon \qquad \text{such that} \qquad \Psi W \leq \varepsilon\, \mathrm{I}$$

Where Ψ is formed, as in Proposition 1, by the impulse response of the transfer function $H(z^{-1})$, and $\mathrm{I}^T = [1,1]$. This study does not use the Youla Parametrization to describe the set of all stabilizing controllers [9]. Instead, the Optimization problem is expressed directly on the coefficients of the impulse response of the closed-loop system [14]. When the optimal transfer function $H(z^{-1})$ is computed, one can easily extract the regulator transfer function from the relation which define $H(z^{-1})$.

Furthermore, additional constraints (or namely *interpolation constraints)* must be added to the optimization problem to ensure the internal stability of the system. These constraints mean that any unstable pole (resp. zero) of the plant can not be cancelled by any zero (resp. pole) of the controller. Denote as $\{q_k, \ k = 1, ..., n\}$ the unstable zeros of the plant and $\{p_k, \ k = 1, ..., m\}$ its unstable poles, then

$$\begin{cases} K(z^{-1})G(z^{-1})\,|_{z=q_k} & \text{must be equal to 0} \\ K(z^{-1})G(z^{-1})\,|_{z=p_k} & \text{must be } \infty \end{cases} \qquad (2)$$

In terms of the impulse response this can be stated as the following interpolation constraints:

$$\begin{cases} \sum_{i=0}^{\infty} \Phi_i \, q_k^{-i} = \alpha_k & k = 1, ..., n \\ \sum_{i=0}^{\infty} \Phi_i \, p_k^{-i} = \beta_k & k = 1, ..., m \end{cases}$$

where α_k and β_k are the value of the closed-loop transfer function at these points while taking into account the equalities (2). For example, if $H = \frac{KG}{1+KG}$ then $\alpha_k = 0$ and $\beta_k = 1$ [14].

Hence, the optimization problem becomes

$$\min \varepsilon$$
$$\begin{cases} \sum_{i=0}^{\infty} \Phi_i^+ \omega_{max} + \sum_{i=0}^{\infty} \Phi_i^- \omega_{min} \le \varepsilon \\ \sum_{i=0}^{\infty} \Phi_i^- \omega_{min} + \sum_{i=0}^{\infty} \Phi_i^+ \omega_{max} \le \varepsilon \\ \sum_{i=0}^{\infty} \Phi_i^+ q_k^{-i} - \sum_{i=0}^{\infty} \Phi_i^- q_k^{-i} = \alpha_k \qquad k = 1, ..., n \\ \sum_{i=0}^{\infty} \Phi_i^+ p_k^{-i} - \sum_{i=0}^{\infty} \Phi_i^- p_k^{-i} = \beta_k \qquad k = 1, ..., m. \end{cases}$$

In order to obtain a finite dimensional problem, this semi-infinite linear programming problem can be truncated. The dimension of the new linear programming problem noted μ can be calculated from the dual problem. Finally, we have a linear programming problem in 2μ+1 variables which can be solved by any of the commonly available tools.

5 Conclusion

In this paper a new way to handle asymmetrical signals is given. Using a new asymmetric objective functional instead of the usual peak-to-peak norm, necessary and sufficient condition are given to check if some performance specifications are fulfilled. In the next step the analysis and the synthesis problems are also formulated with respect to this new objective functional. As in the ℓ_1-optimal control, this approach leads to a linear programming problem which can be solved by off-the-self tools.

Compared with the linear problem obtained using standard ℓ_1-optimal control, no additional variables are added to the primal problem, but the number of constraints is doubled, to cope with the asymmetricity. Further work must be done to simplify the linear programming problem.

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References:

[1] A. Benzaouia, C. Burgat. ”Regulator problem for linear discrete-time systems with non symmetrical constrained control”, *International Journal of Control*, **Vol. 48**, N^o 6, pp. 2441-2451, (1988).

[2] F. Blanchini. ”Feedback control for linear systems with state and control bounds in the presence of disturbance”, *IEEE Trans. Aut. Control*, **Vol. 35**, pp. 1131-1135, (1990).

[3] F. Blanchini, M. Sznaier. ”Persistent disturbance rejection via static-state feedback”, *IEEE Trans. Autom. Contr.*, **Vol. 40**, pp. 1127-1130, (1995).

[4] D. W. Clarke, C. Mohtadi, P. S. Tuffs. ”Generalised predictive control, Part 1: the Basic algorithm; Part 2: Extension and Interpretations”, *Automatica*, **Vol. 23**, N^o 2, pp. 137-160, (1987).

[5] M. A Dahleh, M. Dahleh. ”Optimal rejection of persistent disturbances, robust stability and mixed sensitivity minimization”, *IEEE Trans. Autom. Contr.*, **Vol. 33**, pp. 722-731, (1988).

[6] M. A. Dahleh, I. J. Diaz-Bobillo. ”Control of uncertain systems: a linear programming approach”, *Englewoods Cliffs, NJ: Prentice Hall*, (1995).

[7] M. A Dahleh, J. B. Pearson. ”ℓ_1-Optimal feedback controllers for MIMO discrete-time systems”, *IEEE Trans. Autom. Contr.*, **Vol. 32**, pp. 314-322, (1987).

[8] D. Henrion, S. Tarbouriech, V. Kucera. ”Control of linear systems subject to input constraints: a polynomial approach”, *Automatica*, **Vol. 36**, pp. 597-604, (2001).

[9] Z. Lin, A. Saberi. ”Semi-global exponential stabilization of linear discrete time systems subject to input saturation via linear feedbacks”, *Systems and Control Letters*, **Vol. 21**, pp. 225-239, (1995).

[10] P. O. Malaterre, M. Khammash. ”ℓ_1 controller design for a high-order 5-pool irrigation canal system”, *Proceedings of the Conference on Decision and Control*, Sydney, Australia, (2000).

[11] F. Mesquine, F. Tadeo, A. Benzaouia. ”Regulator problem for linear systems with constraints on control and its increment”, *Proceedings of the 15th IFAC World Congress*, Barcelona, Spain, (2001).

[12] V. F. Sokolov. ”Adaptive ℓ_1 robust control for SISO systems”, *System and Control Letters*, **Vol. 42**, pp. 379-393, (2001).

[13] A. A. Stoorvogel, A. Saberi, P. Sannuti. ”Performance with regulation constraints”, *Automatica*, **Vol. 36**, pp. 1443-1456, (2000).

[14] F. Tadeo, M. J. Grimble. ”Controller design using linear programming for systems with constraints. Part 1: Tutorial Introduction; Part 2: Controller Design; Part 3: Design Examples”, *IEE Computing and Control Engineering Journal*, **Vol. 12**, pp. 273-276, (2002), **Vol. 13**, pp. 49-52, 89-93, (2002).

[15] F. Tadeo, M. J. Grimble. ”Advanced Control of a Hydrogen Reformer”, *IEE Computing and Control Engineering Journal*, **Vol. 13**, pp. 305-314, (2002).

[16] F. Tadeo, A. Holohan, P. Vega.”ℓ_1-Optimal control of a pH plant”, *Computers and Chemical Engineering*, pp. S459-S466, (1998).

[17] M. Vidyasagar. ”Optimal rejection of persistent bounded disturbances”, *IEEE Trans. Autom. Contr.*, **Vol. 31**, pp. 527-534, (1986).

[18] D .C. Youla, H. A. Jabr, J. J. Bongiorno. ”Modern Wiener-Hopf design of optimal controllers. Part 2: The multivariable case” *IEEE Trans. Autom. Contr.*, **Vol. 21**, pp. 319-338, (1976).