LQ OPTIMAL CONTROL PROBLEM IN A BEHAVIORAL SETTING: NEW PERSPECTIVES ON THE PROBLEM STATEMENT AND SOLUTION

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Abstract

This paper aims at providing some new insights into the linear quadratic optimization problem in the behavioral approach. A new problem statement is given and some comparisons with the standard LQ problem in the state space approach are provided. The problem solution is analyzed in detailed terms so that all optimal trajectories can be easily computed from the initial data. Noteworthy, the set of optimal solutions proves to be a behavior, too. The asymptotic stability of the optimal behavior as well as the finiteness of the cost function are finally discussed.

1 Introduction

The behavioral approach represents a modern and powerful framework where system and control problems can be stated and solved in a very general form [4]. Within this setting, a dynamical system is completely described by its *behavior*, namely the set of all possible evolutions of its system variables, and hence any optimization problem is naturally stated as the problem of selecting, within this set of trajectories, "the best ones", for they either minimize a given cost function or maximize a certain gain function.

The linear quadratic optimal control problem has received several different statements (and, consequently, solutions) within the behavioral approach. All of them are quite different one from the other and by no means equivalent. Indeed, in [2], two different optimization problems in the behavioral setting are first stated (by Nieuwenhuis), and then proved to be two special instances of a more general "concave problem". The explicit solution of the LQ-problem, however, makes use of the standard state-space theory.

In [6], Jan Willems addresses the LQ problem for controllable behaviors, by assuming that the cost function is expressed by means of a quadratic differential form. Under these assumptions, the optimal behavior is defined as the set of all behavior trajectories whose cost cannot be decreased by means of a finite support perturbation (namely by adding a finite support behavior trajectory). The solution leads to a set of optimal solutions which represents a behavior. However, no mentioning of initial conditions or local constraints is made in the problem statement.

Weiland and Stoorvogel [5] afford and solve the LQ control problem in a purely behavioral framework, without referring to an explicit cost function but deeply resorting to the idea of control as interconnection. Indeed, the LQ problem is stated as the problem of finding a controller such that the resulting controlled system satisfies two control objectives, expressed (in simplified terms) as an upper and a lower bound, respectively. The final solution resorts, again, to a state-space realization.

Finally, Ferrante and Zampieri [1] adopt a problem statement which involves a quadratic cost function and a local constraint on the behavior trajectories. Basing on state-space realizations, they obtain a set of optimal solutions which, in general, is not a shift invariant set of trajectories, and hence is not a behavior.

In this paper, we aim at providing a different and simple statement for the LQ problem in the behavioral framework, by considering linear, time-invariant and complete discrete-time systems, described as the kernels of some polynomial matrix operators, and by assuming a quadratic cost function (the norm of a suitable sequence which is related to the system trajectory by means of some matrix shift operator). Initial conditions are assigned in such a way that the autonomous part of any behavior trajectory which fulfills such conditions is uniquely determined. The optimization problem thus becomes the problem of choosing the controllable part of the trajectory in order to minimize the assigned cost function.

Two main goals inspired our problem formulation: first of all, providing what seems to be a natural extension of the standard LQ theory for state-space models which is, at the same time, purely behavioral, by this meaning that the problem solution does not involve any state-space representation. On the second hand, we aim at endowing the set of optimal solutions of some nice properties, as in the state-space setting, like linearity, shift-invariance and completeness (namely the optimal solutions constitute a behavior), autonomy and asymptotic stability.

Other issues, concerned with the existence of at least one asymptotically convergent solution among all optimal solutions which satisfy the assigned set of initial conditions, are also addressed.

2 Preliminaries

Consider a discrete-time finite dimensional dynamical system $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ whose behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}}$ is defined as the kernel of some Laurent polynomial (L-polynomial, for short) matrix shift operator [4], i.e. there exists some L-polynomial matrix with q columns and say p rows, $R(\xi) \in \mathbb{R}^{p\times q}[\xi, \xi^{-1}]$, such that

$$\mathfrak{B} = \ker(R(\sigma)) := \left\{ \mathbf{w} \in (\mathbb{R}^q)^{\mathbb{Z}} \mid R(\sigma)\mathbf{w} = 0 \right\}.$$

This is known as an autoregressive (AR) representation of the behavior. Every discrete-time system which admits a kernel representation is linear, shift invariant and complete [4] and the converse holds true. It entails no loss of generality assuming that R is a polynomial matrix in the indeterminate ξ. Moreover, we can also assume that $R(0)$ is of full row rank. As every system $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ is uniquely identified by its behavior $\mathfrak{B}$, in the sequel we will refer directly to $\mathfrak{B}$ instead of Σ.

Autonomy and controllability notions are well-known [4]. We just remind here a few equivalent conditions. A behavior $\mathfrak{B} = \ker(R(\sigma))$, with $R(\xi) \in \mathbb{R}[\xi]^{p\times q}$, is autonomous if and only if R is of full column rank q. If so, $\mathfrak{B}$ can be described as the kernel of a nonsingular square matrix, which is uniquely determined up to a left unimodular factor. So, if R is square, $\det R$ (which is, of course, independent of the specific square representation, except for a multiplicative nonzero monomial) is the **characteristic L-polynomial of** $\mathfrak{B}$ [4].

An autonomous behavior $\mathfrak{B}$ is said to be **asymptotically stable** if for every $\mathbf{w} \in \mathfrak{B}$ we have $\lim_{t\to+\infty} \mathbf{w}(t) = 0$. This happens if and only if the characteristic L-polynomial of $\mathfrak{B}$ is Schur, namely all its zeroes[1] are included in the open unit disk $\{\xi \in \mathbb{C} \setminus \{0\} : |\xi| < 1\}$.

Controllability represents the possibility of steering any past behavior trajectory into any future behavior trajectory, provided that one leaves time enough for adjustments. A behavior $\mathfrak{B}$ is controllable if and only if it can be described as the kernel of a left prime L-polynomial matrix. Moreover, a controllable behavior $\mathfrak{B}$ admits an image description. This amounts to saying that there exist $m \in \mathbb{N}$ and an L-polynomial matrix $M \in \mathbb{R}[\xi, \xi^{-1}]^{q\times m}$ such that $\mathbf{w} \in \mathfrak{B}$ if and only if $\mathbf{w} = M(\sigma)\mathbf{u}$, for some $\mathbf{u} \in (\mathbb{R}^m)^{\mathbb{Z}}$. The set of discrete-time trajectories thus obtained is denoted by $\mathrm{im}(M(\sigma))$.

As a final result, we quote the **decomposition theorem** (see [4], Theorem 5.2.14, for the continuous time case) stating that every AR behavior $\mathfrak{B}$ can always be expressed as the direct sum of its **controllable part** $\mathfrak{B}_c$ (the largest controllable behavior included in $\mathfrak{B}$) and of some (not unique) autonomous behavior $\mathfrak{B}_a$:

$$\mathfrak{B} = \mathfrak{B}_a \oplus \mathfrak{B}_c. \tag{1}$$

As a result, every behavior trajectory $\mathbf{w}$ can always be described as the sum $\mathbf{w} = \mathbf{w}_a + \mathbf{w}_c$ of two trajectories $\mathbf{w}_a \in \mathfrak{B}_a$ and $\mathbf{w}_c \in \mathfrak{B}_c$. Both trajectories, however, depend on the specific choice of $\mathfrak{B}_a$ and are uniquely determined only once the autonomous behavior $\mathfrak{B}_a$ involved in the behavior decomposition is specified.

Despite the non-uniqueness of $\mathfrak{B}_a$, there are some common features among the different autonomous behaviors involved in the direct sum decompositions. Indeed, if $R(\xi) \in \mathbb{R}^{p\times q}[\xi]$ is a full row rank matrix involved in the kernel description of $\mathfrak{B}$, it can always factorize as $R(\xi) = \Delta(\xi)\bar{R}(\xi)$, for suitable (Laurent) polynomial matrices $\Delta(\xi) \in \mathbb{R}^{p\times p}[\xi]$ nonsingular and $\bar{R}(\xi) \in \mathbb{R}^{p\times q}[\xi]$ left prime as a polynomial matrix. Moreover, $\mathfrak{B}_c = \ker(\bar{R}(\sigma))$.

Due to the left primeness of $\bar{R}(\xi)$, a unimodular matrix $V(\xi) \in \mathbb{R}^{q\times q}[\xi]$ can be found such that $\bar{R}(\sigma)V(\sigma) = [\ I_p \mid 0\ ]$. It entails no loss of generality partitioning V as $V(\xi) := [L(\xi)|M(\xi)]$, with $L(\xi) \in \mathbb{R}^{q\times p}[\xi]$ and $M(\xi) \in \mathbb{R}^{q\times(q-p)}[\xi]$. But then every trajectory $\mathbf{w}$ of the behavior $\mathfrak{B}$ can be equivalently rewritten as

$$\mathbf{w}(t) = (L(\sigma)\mathbf{v}_a)(t) + (M(\sigma)\mathbf{v}_c)(t), \tag{2}$$

where $\mathbf{v}_a$ belongs to the autonomous behavior $\ker(\Delta(\sigma))$, meanwhile $\mathbf{v}_c$ is arbitrary in $(\mathbb{R}^{(q-p)})^{\mathbb{Z}}$. Both trajectories are uniquely determined once $\mathbf{w}$ is. It is easily seen that $\mathbf{v}_a$ can be immediately obtained as $\mathbf{v}_a = \bar{R}(\sigma)\mathbf{w}$.

The following simple lemma states that, independently of the specific autonomous/controllable decomposition adopted for $\mathfrak{B}$, by constraining the variable $\mathbf{v}_a = \bar{R}(\sigma)\mathbf{w}$ we constrain the autonomous part of any trajectory $\mathbf{w}$ such that $\mathbf{v}_a = \bar{R}(\sigma)\mathbf{w}$.

LEMMA 2.1 [3] *Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be two arbitrary trajectories of $\mathfrak{B}$ and let $\mathfrak{B}_a$ be any autonomous behavior such that $\mathfrak{B} = \mathfrak{B}_a \oplus \mathfrak{B}_c$. Assume, then, that $\mathbf{w}_1$ and $\mathbf{w}_2$ decompose as $\mathbf{w}_i = \mathbf{w}_{ai} + \mathbf{w}_{ci}$, with $\mathbf{w}_{ai} \in \mathfrak{B}_a$ and $\mathbf{w}_{ci} \in \mathfrak{B}_c$, $i = 1, 2$. Condition $\bar{R}(\sigma)\mathbf{w}_1 = \bar{R}(\sigma)\mathbf{w}_2$ ensures $\mathbf{w}_{a1} = \mathbf{w}_{a2}$.*

Notice that in the factorization $R(\xi) = \Delta(\xi)\bar{R}(\xi)$, with $\bar{R}$ left prime and Δ nonsingular square, we can also assume that Δ is a diagonal matrix. In fact, by resorting to the Smith form, we can factorize Δ as

$$\Delta(\xi) = U(\xi) \cdot \mathrm{diag}\{\delta_1(\xi), \delta_2(\xi), \dots, \delta_p(\xi)\} \cdot V(\xi),$$

where U and V are unimodular L-polynomial matrices and the monic L-polynomials δ_i's (which can always be assumed as elements of $\mathbb{R}[\xi]$ and endowed with a nonzero constant term) are mutually related by the divisibility chain condition $\delta_p(\xi) \mid \dots \mid \delta_1(\xi)$. Of course,

$$\ker(R(\sigma)) = \ker\big(\mathrm{diag}\{\delta_1(\sigma), \delta_2(\sigma), \dots, \delta_p(\sigma)\} \cdot V(\sigma)\bar{R}(\sigma)\big),$$

and $V\bar{R}$ is left prime, too, as a polynomial matrix. So, we can replace Δ with its Smith form and $\bar{R}$ with $V\bar{R}$.

[1] Notice that we consider only zeroes which are different from 0, since we are dealing with L-polynomials.

The diagonal structure of Δ better enlightens the complexity of the autonomous behavior $\ker(\Delta(\sigma))$. Indeed, once the ith entry v_{ai} of a trajectory $\mathbf{v}_a \in \ker(\Delta(\sigma))$ is known over the discrete time interval $[0, \deg \delta_i - 1]$, it is uniquely identified. In other terms, once we constrain a trajectory $\mathbf{v}_a \in \ker(\Delta(\sigma))$ to satisfy a set of conditions

$$v_{ai}(t) = b_i(t), \qquad \begin{array}{l} t = 0, 1, \dots, \deg \delta_i - 1 \\ i = 1, 2, \dots, p, \end{array}$$

for arbitrary choices of $\{b_i(t)\}_{\substack{t=0,1,\dots,\deg \delta_i - 1 \\ i=1,2,\dots,p}}$, the trajectory $\mathbf{v}_a$ is uniquely determined.

So, once we constrain a trajectory $\mathbf{w} \in \mathfrak{B}$ to satisfy the set of conditions

$$[(\bar{R}(\sigma)\mathbf{w})(t)]_i := \mathbf{e}_i^T (\bar{R}(\sigma)\mathbf{w})(t) = b_i(t), \quad \begin{array}{l} t = 0, 1, .., \deg \delta_i - 1 \\ i = 1, 2, .., p, \end{array}$$

$\mathbf{e}_i$ the ith canonical vector and $\{b_i(t)\}_{\substack{t=0,1,\dots,\deg \delta_i - 1 \\ i=1,2,\dots,p}}$ arbitrarily chosen, we uniquely identify the trajectory $\mathbf{v}_a := \bar{R}(\sigma)\mathbf{w}$ and hence the autonomous part $\mathbf{w}_a$ of $\mathbf{w}$ (depending on the autonomous/controllable decomposition assumed for $\mathfrak{B}$). We will make use of this type of constraints in order to state the linear quadratic optimization problem in a behavioral setting.

A behavior $\mathfrak{B}$ is said to be **stabilizable** if for every trajectory $\mathbf{w} \in \mathfrak{B}$ and for every $\bar{t} \in \mathbb{Z}$ there exists some trajectory $\mathbf{w}' \in \mathfrak{B}$ such that $\mathbf{w}'(t) = \mathbf{w}(t)$, $\forall\ t \leq \bar{t}$, and $\lim_{t \to +\infty} \mathbf{w}'(t) = 0$. We have that $\mathfrak{B} = \ker(R(\sigma))$ is stabilizable if and only if $\operatorname{rank}(R(\lambda)) = p$, $\forall \lambda \in \mathbb{C}, |\lambda| \geq 1$, or, equivalently, if and only if $\det \Delta$ is Schur or if and only if in every autonomous/controllable decomposition (1) the autonomous behavior $\mathfrak{B}_a$ is asymptotically stable.

3 Problem statement and comparisons with the standard LQ control

As discussed in the Introduction, the linear quadratic optimal control problem has received several different statements and solutions within the behavioral framework [1, 2, 5, 6]. In this paper we choose a problem statement which is close to the one adopted in [1]. However, we introduce a different type of "initial conditions". As underlined in the Introduction, we aimed at finding a statement which provides a generalization of the LQ problem, as it is traditionally stated for classic state-space models, and at obtaining a set of optimal solutions, each of them corresponding to a given set of initial conditions, which is an autonomous behavior.

Problem Statement: *Given two polynomial matrices $R_2(\xi) \in \mathbb{R}^{p_2 \times q}[\xi]$ and $R(\xi) \in \mathbb{R}^{p \times q}[\xi]$, with R factorizing as in (2) with $\Delta := \operatorname{diag}\{\delta_1(\xi), \delta_2(\xi), \dots, \delta_p(\xi)\} \in \mathbb{R}[\xi]^{p \times p}$, $\delta_i(0) \neq 0\ \forall\, i$, and $\bar{R}(\xi)$ left prime as a polynomial matrix, and a set of initial conditions $\{b_i(t)\}_{\substack{t=0,1,\dots,\deg \delta_i - 1 \\ i=1,2,\dots,p}}$, find the set of trajectories $\mathbf{w} \in (\mathbb{R}^q)^{\mathbb{Z}}$ which minimize the cost function*

$$J = \sum_{t=0}^{+\infty} \|(R_2(\sigma)\mathbf{w})(t)\|^2 \tag{3}$$

subject to the following constraints:

$$R(\sigma)\mathbf{w} = 0 \tag{4}$$

$$v_{ai}(t) := [(\bar{R}(\sigma)\mathbf{w})(t)]_i = b_i(t), \quad i = 1, \dots, p, \tag{5}$$
$$t = 0, \dots, \deg\ \delta_i - 1.$$

According to the comments at the end of the previous section, the trajectory

$$\mathbf{v}_a := \bar{R}(\sigma)\mathbf{w} \tag{6}$$

belongs to the autonomous behavior $\ker(\Delta(\sigma))$, is uniquely determined by the "initial conditions" (5) and uniquely determines, once a specific autonomous/controllable decomposition of $\mathfrak{B}$ is chosen, the autonomous part of any trajectory $\mathbf{w} \in \mathfrak{B} = \ker(R(\sigma))$ satisfying (6). This ensures that every $\mathbf{w} \in \ker(R(\sigma))$, in particular each optimal solution $\mathbf{w}^* \in \ker(R(\sigma))$, can be expressed, according to (2), as $\mathbf{w} = L(\sigma)\mathbf{v}_a + M(\sigma)\mathbf{v}_c$, with $\mathbf{v}_a$ a fixed trajectory in $\ker(\Delta(\sigma))$. Since $\mathbf{v}_a$ is the same one involved in the expression of the optimal trajectory $\mathbf{w}^*$ and the optimization problem is independent of it, in the following we will denote it by $\mathbf{v}_a^*$.

One may wonder in which sense the above problem statement provides a generalization of the standard LQ problem for state-space models:

$$\begin{array}{ll} \min\limits_{\mathbf{u}} & \sum\limits_{t=0}^{+\infty} [\mathbf{x}^T(t) Q \mathbf{x}(t) + \mathbf{u}^T(t) R \mathbf{u}(t)] \\ \text{s.t.} & \mathbf{x}(t+1) = F\mathbf{x}(t) + G\mathbf{u}(t), \qquad t \in \mathbb{Z}_+, \\ & \mathbf{x}(0) = \mathbf{x}_0, \end{array}$$

where $\mathbf{x}$ represents the state sequence, $\mathbf{u}$ the input sequence, Q and R are symmetric positive semidefinite matrices (R, in particular, is supposed to be positive definite). Indeed, if we define as system variable $\mathbf{w} := \begin{bmatrix} \mathbf{x} & \mathbf{u} \end{bmatrix}^T$, then the quadratic index J may be rewritten as

$$J = \sum_{t=0}^{+\infty} \left\| \begin{bmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} \right\|^2,$$

where we have denoted by $Q^{1/2}$ (by $R^{1/2}$) any matrix such that $Q = (Q^{1/2})^T Q^{1/2}$ ($R = (R^{1/2})^T R^{1/2}$, respectively). Thus J can be represented as in (3) upon setting

$$R_2(\xi) := \begin{bmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix}.$$

On the other hand, the state-space equation can also be formalized as follows:

$$\begin{bmatrix} F - \sigma I & G \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = 0,$$

and hence (4) holds for $R(\xi) := \begin{bmatrix} F - \xi I & G \end{bmatrix}$. Moreover, it is well-known that the optimal solution is obtained by resorting to a state feedback and hence leads to an autonomous state-space model (as a matter of fact, the optimal solution is obtained by resorting to a time varying state feedback matrix, but our comparison is with the time invariant asymptotic solution of the standard LQ problem). Indeed, the (sub)optimal (time-invariant) solution $\begin{bmatrix} \mathbf{x}^* \\ \mathbf{u}^* \end{bmatrix}$ satisfies

$$\begin{bmatrix} F-\sigma I & G \\ -K^* & I \end{bmatrix} \begin{bmatrix} \mathbf{x}^* \\ \mathbf{u}^* \end{bmatrix} = 0,$$

(for a suitable K^* and starting from the assigned initial condition $\mathbf{x}_0$) and hence the set of optimal solutions (corresponding to all possible choices of the initial condition) represents an autonomous behavior. This motivated our choice of imposing to our set of solutions the structure of an autonomous behavior.

Also, under the stabilizability assumption on the pair (F, G) (equivalently, on the corresponding behavior $\mathfrak{B}$), the matrix K^* (obtained by solving a suitable algebraic Riccati equation) makes the matrix $F + GK^*$ asymptotically stable (makes the optimal behavior autonomous and asymptotically stable) if and only if the pair $(F, Q^{1/2})$ is detectable.

Finally, we want to comment on our choice of the initial conditions (5). In the standard state-space setting, the initial conditions are given by specifying the initial state at time 0, i.e. $\mathbf{x}(0) = \mathbf{x}_0$. Once the initial state is given, the free state evolution is uniquely determined and we have to select the input sequence and, consequently, the forced evolution, in order to minimize the given index. Since the free/forced evolution decomposition is naturally replaced by the autonomous/controllable decomposition in the behavioral setting, the choice of constraining the autonomous part of the optimal behavior trajectories seemed to be the most natural extension of the classical initial condition.

From a purely mathematical point of view, we want to show how, by constraining the autonomous part of the behavior $\ker([\,F-\sigma I \quad G\,])$, we actually impose a constraint not on the whole initial condition $\mathbf{x}(0)$ but just on that portion of $\mathbf{x}(0)$ which corresponds to the uncontrollable part of the dynamical system. Therefore the constraint on the autonomous part we have introduced here is weaker with respect to the one traditionally imposed in the state-space setting. If we suppose that the state-space model is in Kalman reachability form:

$$F = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix} \quad G = \begin{bmatrix} G_1 \\ 0 \end{bmatrix}, \quad (F_{11}, G_1) \text{ a reachable pair},$$

then the behavior $\mathfrak{B}$ of the state-space model can be described as

$$\ker\left(\begin{bmatrix} F_{11}-\sigma I & F_{12} & G_1 \\ 0 & F_{22}-\sigma I & 0 \end{bmatrix}\right),$$

where we have assumed that the system trajectories are partitioned into three (vector) components: $\mathbf{w} = [\,\mathbf{x}_1^T \quad \mathbf{x}_2^T \quad \mathbf{u}^T\,]^T$, $\mathbf{x}_1$ the state component corresponding to the reachable subsystem, $\mathbf{x}_2$ the state component corresponding to the unreachable subsystem, $\mathbf{u}$ the input sequence. Also, it is not hard to see that the "controllable part" of $\mathfrak{B}$ is

$$\mathfrak{B}_c = \ker\left(\begin{bmatrix} F_{11}-\sigma I & F_{12} & G_1 \\ 0 & I & 0 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} F_{11}-\sigma I & 0 & G_1 \\ 0 & I & 0 \end{bmatrix}\right),$$

while a possible choice of an autonomous behavior $\mathfrak{B}_a$ such that $\mathfrak{B} = \mathfrak{B}_a \oplus \mathfrak{B}_c$ is

$$\mathfrak{B}_a = \ker\left(\begin{bmatrix} I & 0 & 0 \\ 0 & F_{22}-\sigma I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} F_{11}-\sigma I & F_{12} & G_1 \\ 0 & I & 0 \\ -K_1 & 0 & I \end{bmatrix}\right),$$

where K_1 is any real matrix such that $F_{11} + G_1K_1$ is nilpotent (i.e., $\det(F_{11} + G_1K_1 - \xi I) = c \cdot \xi^h$ for some $c \in \mathbb{R} \setminus \{0\}$ and some $h \in \mathbb{N}$), thus ensuring that $F_{11} + G_1K_1 - \xi I$ and hence

$$\begin{bmatrix} F_{11}-\xi I & F_{12} & G_1 \\ 0 & I & 0 \\ -K_1 & 0 & I \end{bmatrix}$$

are both unimodular L-polynomial matrices. The trajectories in $\mathfrak{B}_c$ can be expressed as

$$\mathbf{w}_c = \begin{bmatrix} \mathbf{x}_1 \\ 0 \\ \mathbf{u} \end{bmatrix}, \qquad \text{with } \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{u} \end{bmatrix} \in \ker\left([\,F_{11}-\sigma I \quad G_1\,]\right),$$

while the trajectories in $\mathfrak{B}_a$ can be expressed as

$$\mathbf{w}_a = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{u} \end{bmatrix}, \quad \text{with} \quad \begin{array}{c} \mathbf{x}_2 \in \ker(F_{22}-\sigma I), \\ (F_{11} + G_1K_1 - \sigma I)\mathbf{x}_1 = -F_{12}\mathbf{x}_2, \\ \mathbf{u} = K_1\mathbf{x}_1. \end{array}$$

Notice that $\mathbf{x}_2$ belongs to an autonomous behavior meanwhile, due to the unimodularity of $F_{11} + G_1K_1 - \xi I$, the reachable component $\mathbf{x}_1$ (and hence $\mathbf{u}$) is uniquely determined from $\mathbf{x}_2$. This shows that, with respect to this specific direct sum decomposition, constraining $\mathbf{x}_2(0)$ is equivalent to constraining $\mathbf{x}_2(t), t \in \mathbb{Z}$, and hence the whole $\mathbf{w}_a \in \mathfrak{B}_a$ in any trajectory $\mathbf{w} = \mathbf{w}_a + \mathbf{w}_c$ of $\mathfrak{B}$. If we constrain the whole initial condition $\mathbf{x}(0)$, namely both $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$, we introduce a stronger constraint, since we both identify the autonomous part of the trajectory and give some info on the controllable part.

4 Problem solution

We are, now, in a position to derive a kernel description of the set of all optimal trajectories of our optimization problem, as the initial conditions vary over the set of all possible $\sum_{i=1}^{p} \deg \delta_i$-tuples of real numbers. Let $\mathbf{v}_a^*$ be the (uniquely determined) trajectory in $\ker(\Delta(\sigma))$ which satisfies the initial conditions (5). Then, by making use of the arguments previously adopted, we get that a trajectory $\mathbf{w}^* \in \mathfrak{B}$ is optimal if and only if the following conditions holds:

$$\begin{cases} \mathbf{w}^* = L(\sigma)\mathbf{v}_a^* + M(\sigma)\mathbf{v}_c^*, & \exists\, \mathbf{v}_c^* \in (\mathbb{R}^{q-p})^{\mathbb{Z}}, \\ \sum_{t=0}^{+\infty} \|(R_2(\sigma)(\mathbf{w}^* + M(\sigma)\mathbf{v}_c))(t)\|^2 \geq \sum_{t=0}^{+\infty} \|(R_2(\sigma)\mathbf{w}^*)(t)\|^2 & \forall\, \mathbf{v}_c \in (\mathbb{R}^{q-p})^{\mathbb{Z}}. \end{cases}$$

By the linearity of the operator $R_2(\sigma)$, the last equation holds true if and only if [6]

$$\begin{cases} \sum_{t=0}^{+\infty} < (R_2(\sigma)M(\sigma)\mathbf{v}_c)(t), (R_2(\sigma)\mathbf{w}^*)(t) > = 0, \\ \sum_{t=0}^{+\infty} \|(R_2(\sigma)M(\sigma)\mathbf{v}_c)(t)\|^2 \geq 0, \end{cases} \quad \forall \mathbf{v}_c \in (\mathbb{R}^{q-p})^{\mathbb{Z}}$$

where $< \mathbf{t}_1, \mathbf{t}_2 >$ denotes the internal product of the vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ (in $\mathbb{R}^{p_2}$). The second relation is obviously satisfied for every choice of $\mathbf{v}_c$, meanwhile the first one is satisfied iff

$$\mathbf{w}^* \in \ker \left(M^T(\sigma^{-1}) R_2^T(\sigma^{-1}) R_2(\sigma) \right). \qquad (7)$$

So, it turns out that the optimal trajectories (as the initial conditions (5) vary in the set of all possible $\sum_{i=1}^{p} \deg \delta_i$-tuples) are those and those only which satisfy both $R(\sigma)\mathbf{w}^* = 0$ and $M^T(\sigma^{-1}) R_2^T(\sigma^{-1}) R_2(\sigma)\mathbf{w}^* = 0$. This ensures that the set of optimal trajectories is a behavior, denoted in the sequel by the symbol $\mathfrak{B}^*$, that we can identify with

$$\mathfrak{B}^* = \ker \left(\begin{bmatrix} R(\sigma) \\ M^T(\sigma^{-1}) R_2^T(\sigma^{-1}) R_2(\sigma) \end{bmatrix} \right).$$

If we set

$$R^*(\xi) := \begin{bmatrix} R(\xi) \\ M^T(\xi^{-1}) R_2^T(\xi^{-1}) R_2(\xi) \end{bmatrix} \in \mathbb{R}[\xi, \xi^{-1}]^{q \times q},$$

then $\mathfrak{B}^* = \ker(R^*(\sigma))$. Moreover, $\mathfrak{B}^*$ is autonomous if and only if R^* is of full column rank and hence, being square, nonsingular.

Of course, the above description of $\mathfrak{B}^*$ is not the most suitable one in order to obtain the specific optimal trajectory (or trajectories) corresponding to the assigned set of initial conditions. To this end, we may observe that if $\mathbf{v}_a^*$ is the trajectory in $\ker(\Delta(\sigma))$, uniquely determined by the initial conditions (5), then, by resorting to the previous identities, and the fact that an optimal solution $\mathbf{w}^*$ can be expressed as $\mathbf{w}^* = L(\sigma)\mathbf{v}_a^* + M(\sigma)\mathbf{v}_c^*$, we get

$$\begin{cases} \Delta(\sigma)\mathbf{v}_a^* = 0, \\ M^T(\sigma^{-1}) R_2^T(\sigma^{-1}) R_2(\sigma) L(\sigma)\mathbf{v}_a^* + \\ + M^T(\sigma^{-1}) R_2^T(\sigma^{-1}) R_2(\sigma) M(\sigma)\mathbf{v}_c^* = 0. \end{cases} \qquad (8)$$

The previous relations can be used in order to obtain all possible trajectories $\mathbf{v}_c^*$ which lead, together with $\mathbf{v}_a^*$, to an optimal trajectory $\mathbf{w}^*$. Also, (8) can be thought as the set of behavioral equations of a new linear time invariant system which is just

$$\mathfrak{B}_{a,c}^* := \left\{ \begin{bmatrix} \mathbf{v}_a^* \\ \mathbf{v}_c^* \end{bmatrix} : (8) \text{ holds} \right\} \qquad (9)$$
$$= \ker \left(\begin{bmatrix} \Delta(\sigma) & 0 \\ M^T(\sigma^{-1}) R_2^T(\sigma^{-1}) R_2(\sigma) L(\sigma) & M^T(\sigma^{-1}) R_2^T(\sigma^{-1}) R_2(\sigma) M(\sigma) \end{bmatrix} \right)$$

Obviously, $\mathfrak{B}^*$ and $\mathfrak{B}_{a,c}^*$ are bijectively related by $\mathfrak{B}^* = [\, L(\sigma) \quad M(\sigma) \,] \mathfrak{B}_{a,c}^*$, and, due to the unimodularity of $[\, L(\xi) \quad M(\xi) \,]$, the internal properties of the two behaviors are just the same. In particular, $\mathfrak{B}^*$ is autonomous if and only if $\mathfrak{B}_{a,c}^*$ is and this is equivalent to saying that the square matrix $M^T(\xi^{-1}) R_2^T(\xi^{-1}) R_2(\xi) M(\xi)$ is nonsingular.

By resorting to $\mathfrak{B}_{a,c}^*$ and to (8), it is now immediate to evaluate the cost function along an optimal trajectory $\mathbf{w}^* = L(\sigma)\mathbf{v}_a^* + M(\sigma)\mathbf{v}_c^* \in \mathfrak{B}^*$. We get, in fact,

$$\begin{aligned} J^* &= \sum_{t=0}^{+\infty} \|(R_2(\sigma)L(\sigma)\mathbf{v}_a^*)(t)\|^2 + \sum_{t=0}^{+\infty} \|(R_2(\sigma)M(\sigma)\mathbf{v}_c^*)(t)\|^2 \\ &+ \ 2\sum_{t=0}^{+\infty} < (R_2(\sigma)M(\sigma)\mathbf{v}_c^*)(t), (R_2(\sigma)L(\sigma)\mathbf{v}_a^*)(t) > \\ &= \sum_{t=0}^{+\infty} \|(R_2(\sigma)L(\sigma)\mathbf{v}_a^*)(t)\|^2 - \sum_{t=0}^{+\infty} \|(R_2(\sigma)M(\sigma)\mathbf{v}_c^*)(t)\|^2. \end{aligned}$$

Obviously, two distinct optimal trajectories corresponding to the same set of initial conditions (5) lead to the same value of J^*, as is immediately apparent from (9). Therefore J^* is just a function of $\mathbf{v}_a^*$, and hence of the initial conditions (5), and it is independent of $\mathbf{v}_c^*$.

5 Asymptotic stability of the optimal behavior

Since we aimed at obtaining as the set $\mathfrak{B}^*$ of all optimal trajectories an autonomous behavior, from now on we will steadily assume that $M^T(\xi^{-1}) R_2^T(\xi^{-1}) R_2(\xi) M(\xi)$ is nonsingular square. We want now to determine under what conditions the optimal behavior $\mathfrak{B}^*$ is asymptotically stable. To this end it will be useful to resort, again, to the behavior $\mathfrak{B}_{a,c}^*$. Indeed, it is immediately seen that $\mathfrak{B}^*$ is asymptotically stable if and only if $\mathfrak{B}_{a,c}^*$ is. As a consequence, once we resort to the kernel description of $\mathfrak{B}_{a,c}^*$ given in (9), we get that the optimal behavior $\mathfrak{B}^*$ is autonomous and asymptotically stable if and only if the following two conditions hold:

- $\det \Delta$ is Schur;
- $M^T(\xi^{-1}) R_2^T(\xi^{-1}) R_2(\xi) M(\xi)$ is nonsingular square and its determinant is Schur.

The former condition amounts to assuming that $\mathfrak{B}$ is stabilizable, meanwhile the latter condition is satisfied if and only if $M^T(\xi^{-1}) R_2^T(\xi^{-1})\ R_2(\xi) M(\xi)$ is unimodular, since when λ is a zero of $\det\Big(M^T(\xi^{-1}) R_2^T(\xi^{-1}) R_2(\xi)\ M(\xi) \Big)$, then also λ^{-1} is. On the other hand, it is not hard to see that the unimodularity of such a matrix corresponds to the right primeness condition on $R_2(\xi)M(\xi)$. Since $\mathfrak{B}_c = \ker(\bar{R}(\sigma)) = \mathrm{im}(M(\sigma))$ [4], the following result immediately holds.

PROPOSITION 5.1 *The optimal behavior $\mathfrak{B}^*$ is asymptotically stable and autonomous if and only if*

- $\mathfrak{B}$ *is stabilizable and*
- $\ker(R_2(\sigma)) \cap \mathfrak{B}_c = \{0\}$.

Notice that, due to the previous proposition, when $\mathfrak{B}^*$ is asymptotically stable then the optimal solution corresponding to an assigned set of initial conditions (5) is necessarily unique. Indeed, corresponding to every $\mathbf{v}_a^* \in \ker(\Delta(\sigma))$ we can determine, by means of (8), a unique trajectory $\mathbf{v}_c^*$ such that $\begin{bmatrix} \mathbf{v}_a^* \\ \mathbf{v}_c^* \end{bmatrix} \in \mathfrak{B}_{a,c}^*$ and hence a unique trajectory $\mathbf{w}^* = L(\sigma)\mathbf{v}_a^* + M(\sigma)\mathbf{v}_c^* \in \mathfrak{B}^*$ such $\mathbf{v}_a^* = \bar{R}(\sigma)\mathbf{w}^*$.

Asymptotic stability is a very useful property. However, we can obtain good system performances (and, in particular, finite values of the optimal cost index J^* for every choice of the initial conditions (5)) under quite weaker conditions. Indeed, we can be simply interested in obtaining, for every choice of the initial conditions (5), **at least one** optimal solution $\mathbf{w}^*$ which fits the initial conditions constraint and asymptotically converges to zero. The following proposition addresses and solves just this problem.

PROPOSITION 5.2 *Suppose that the optimal behavior $\mathfrak{B}^*$ is autonomous, namely $M^T(\xi^{-1})\ R_2^T(\xi^{-1})R_2(\xi)M(\xi)$ is nonsingular square. The following facts are equivalent:*

i) for every set of initial conditions (5) and hence for every assigned $\mathbf{v}_a^* \in \ker(\Delta(\sigma))$, there exists $\mathbf{w}^* \in \mathfrak{B}^*$ which satisfies $\mathbf{v}_a^* = \bar{R}(\sigma)\mathbf{w}^*$ and converges to 0, as $t \to +\infty$;

ii) $\mathfrak{B}$ is stabilizable, i.e. $\det\Delta$ is Schur.

Proof. i) $\Rightarrow$ ii) Let $\mathbf{w}^*$ be an arbitrary trajectory in $\mathfrak{B}^*$ so that $\mathbf{w}^*(t) = \Big(L(\sigma)\mathbf{v}_a^* + M(\sigma)\mathbf{v}_c^*\Big)(t)$ for a suitable (and uniquely determined) $\begin{bmatrix}\mathbf{v}_a^*\\ \mathbf{v}_c^*\end{bmatrix} \in \mathfrak{B}_{a,c}^*$. $\mathbf{w}^*(t)$ converges to zero (as t goes to $+\infty$) if and only if both $\mathbf{v}_a^*(t)$ and $\mathbf{v}_c^*(t)$ do, namely $\mathfrak{B}_{a,c}^*$ is asymptotically stable. By (9), the sequence $\mathbf{v}_a^*$ belongs to $\ker(\Delta(\sigma))$, thus $\mathbf{v}_a^*$ converges to zero for every choice of the initial conditions (if and) only if $\det\Delta(\xi)$ is a Schur polynomial. Therefore, a necessary condition for the asymptotic stability of $\mathfrak{B}^*$ is that $\det\Delta(\xi)$ is Schur.

ii) $\Rightarrow$ i) Assume, now, that $\ker(\Delta(\sigma))$ is an asymptotically stable autonomous behavior. Once $\mathbf{v}_a^* \in \ker(\Delta(\sigma))$ (and hence converging to zero) has been assigned, the set of all $\mathbf{v}_c^*$ such that $\begin{bmatrix}\mathbf{v}_a^*\\ \mathbf{v}_c^*\end{bmatrix} \in \mathfrak{B}_{a,c}^*$ are determined by solving the equation

$$\begin{aligned} M^T(\sigma^{-1})\, R_2^T(\sigma^{-1})R_2(\sigma)L(\sigma)\mathbf{v}_a^* &= \\ = \ -M^T(\sigma^{-1})R_2^T(\sigma^{-1})R_2(\sigma)M(\sigma)\mathbf{v}_c^* & \end{aligned} \quad (10)$$

and hence all such $\mathbf{v}_c^*$ differ in an element of $\ker(M^T(\sigma^{-1})R_2^T(\sigma^{-1})R_2(\sigma)M(\sigma))$. It entails no loss of generality assuming that $M^T(\xi^{-1})R_2^T(\xi^{-1})R_2(\xi)M(\xi)$ is in Hermite form. This allows us to reduce our multivariable problem to a family of scalar problems. Indeed, once we set

$$\left(M^T(\sigma^{-1})R_2^T(\sigma^{-1})R_2(\sigma)L(\sigma)\mathbf{v}_a^*\right)(t) =: \begin{bmatrix} b_1(t)\\ b_2(t)\\ \vdots\\ b_{q-p}(t)\end{bmatrix},$$

$$-M^T(\xi^{-1})R_2^T(\xi^{-1})R_2(\xi)M(\xi) =: \begin{bmatrix} a_{1,1}(\xi) & \dots & a_{1,q-p}(\xi)\\ & \ddots & \vdots\\ & & a_{q-p,q-p}(\xi)\end{bmatrix},$$

equation (10) becomes

$$\begin{bmatrix} b_1(t)\\ b_2(t)\\ \vdots\\ b_{q-p}(t)\end{bmatrix} = \left(\begin{bmatrix} a_{1,1}(\sigma) & a_{1,2}(\sigma) & \dots & a_{1,q-p}(\sigma)\\ & a_{2,2}(\sigma) & \dots & a_{2,q-p}(\sigma)\\ & & \ddots & \vdots\\ & & & a_{q-p,q-p}(\sigma)\end{bmatrix}\mathbf{v}_c^*\right)(t).$$

Since $\mathbf{v}_a^*$ is a suitable linear combination of asymptotically stable exponential functions $t^k\lambda_i^t$, with $|\lambda_i| < 1$, each $b_j(t)$ is, in turn, a combination of asymptotically stable exponential functions. So, if we assume that $b_{q-p}(t)$ is expressed, for instance, as $v_{q-p}(t) = \sum_{i=1}^r \sum_{k=0}^{m_i-1} c_{ik}t^k\lambda_i^t$, then, by resorting to the theory of standard difference equations,we can immediately state that there exists an exponentially stable trajectory $v_{c,q-p}^*(t)$ s.t.

$$b_{q-p}(t) = (a_{q-p,q-p}(\sigma)v_{c,q-p}^*)(t).$$

Indeed such a solution $v_{c,q-p}^*(t)$ can be explicitly computed starting from the expression of $b_{q-p}(t)$ and it involves only exponential modes corresponding to the same λ_i's:

$$v_{c,q-p}^*(t) = \sum_{i=1}^r \sum_{k=0}^{m_i-1} A_{ik}t^{k+r_i}\lambda_i^t,$$

where r_i is a nonnegative integer which represents the multiplicity (possibly 0) of λ_i as a zero of $a_{q-p,q-p}(\xi)$. But then, the same argument applies to the difference equation

$$\begin{aligned} b_{q-p-1}(t) - (a_{q-p-1,q-p}(\sigma)v_{c,q-p}^*)(t) &= \\ = (a_{q-p-1,q-p-1}(\sigma)v_{c,q-p-1}^*)(t) & \end{aligned}$$

where, now, $b_{q-p-1}(t) - (a_{q-p-1,q-p}(\sigma)v_{c,q-p}^*)(t)$ is a combination of asymptotically stable exponential functions and $v_{c,q-p-1}^*$ is the solution, which can always be chosen exponentially stable. By recursively applying this argument, we can obtain the desired converging $\mathbf{v}_c^*$ and hence a converging optimal solution $\mathbf{w}^*$ satisfying the assigned initial conditions. ■

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