SINGULAR STRUCTURE CONVERGENCE FOR LINEAR QUADRATIC PROBLEMS

Juan Yuz*,1, Graham Goodwin*, Arie Feuer† & José De Doná*

*School of Electrical Engineering, *The* University *of* Newcastle, Callaghan NSW 2308, Australia.
[1] `jiyuze@ee.newcastle.edu.au`, Fax: +61-2-4960-1712
†Department of Electrical Engineering, The Technion, Haifa 32000, Israel.

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Abstract

This paper considers the singular structure for sampled data LQ problems of continuous linear systems, and the asymptotic behaviour when the sampling rate is increased. We show that there is a natural convergence between the finite set of singular values of the discrete time problem, and the (infinite) countable set in continuous time.

1 Introduction

There exist many results connecting discrete time models to the underlying continuous time systems. One mechanism for elucidating this connection is to utilize the delta operator as reported, for example, in [4, 10, 12, 11, 15, 17, 3, 9, 16].

The motivation for the current paper is work originally reported in [6] which suggests a mechanism for solving constrained LQ problems for continuous time systems using a singular value decomposition. Related work regarding singular value structures has also been reported in the context of cross directional control [14] and constrained receding horizon control [13] for discrete time systems.

This body of work raises the more general system theoretic question regarding the connection between the singular value structure of discrete time LQ problems and the associated continuous time case. A deeper understanding of this connection could, for example, lead to approximate algorithms for the continuous time problem but which are solved using standard discrete time methods. The existence of a well defined limit as the sampling rate increases could be exploited in high speed applications, using ad-hoc algorithms for constrained systems in MPC strategies.

The remainder of the paper is as follows: In Section 2 we describe the continuous time problem and its sampled data counterpart. In Section 3 we summarize the known results on the singular value for the continuous time case. In Section 4, we explore the connections between the singular values for the associated sampled data problem and the continuous one. Section 5 presents an example and conclusions are discussed in Section 6.

2 Problem Formulation

We study two related fixed horizon linear quadratic control problems, one defined in continuous time and the associated sampled data problem, using zero order hold for the input signal:

2.1 Continuous time

We consider the problem $\mathcal{P}$, defined in the continuous time domain as follows:

1. Continuous time model:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad ; \; x(0) = x_o \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

   where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$.

2. A fixed time horizon $T_f < \infty$.

3. A quadratic cost function J to be minimized:

$$J = J_1 + J_\infty$$
$$\begin{cases} J_1 = \int_0^{T_f} \left(x(t)^T Q x(t) + u(t)^T R u(t) \right) dt \\ J_\infty = x(T_f)^T P \, x(T_f) \end{cases} \tag{3}$$

   where $Q \geq 0$, $R > 0$, and where the final state weighting matrix, P satisfies the corresponding *continuous-time* algebraic Riccati equation and gives the infinite horizon optimal unconstrained cost associated with the cost J_1 when the initial state is $x(T_f)$.

Note that we could also add input and/or state constraints to the problem $\mathcal{P}$, as in [6]. However, these constraints do not affect the thrust of the argument presented here.

2.2 Sampled Data Problem

A natural way to approximate the problem $\mathcal{P}$ would seem to be to use a (small) sampling period Δ together with a zero-order hold approximation:

$$u(t) = u_\Delta(t) = u_k \qquad ; \; k\Delta \leq t < (k+1)\Delta \tag{4}$$

We consider as sampling interval an integer fraction of the fixed time horizon, i.e., $T_f = N\Delta$ for some $N \in \mathbb{N}$.

The discrete time problem $\mathcal{P}_\Delta$ is then defined as follows:

1. Discrete Model:

$$x_{k+1} = A_q x_k + B_q u_k \quad (5)$$
$$y_k = C_q x_k \quad (6)$$

where $k \in \mathbb{Z}$, the initial state is x_o, given in (1), and:

$$A_q = e^{A\Delta}; \quad B_q = \int_0^\Delta e^{A\eta} B\, d\eta; \quad C_q = C \quad (7)$$

2. Fixed discrete time horizon $N = T_f/\Delta$.

3. A quadratic cost function J_Δ to be minimized:

$$J_\Delta = \sum_{k=0}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} Q_q & S_q \\ S_q^T & R_q \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} + x_N^T P_\Delta x_N \quad (8)$$

where:
$$Q_q = \int_0^\Delta e^{A^T t} Q e^{At}\, dt \quad (9)$$
$$S_q = \int_0^\Delta e^{A^T t} Q h(t)\, dt \quad (10)$$
$$R_q = R\Delta + \int_0^\Delta h(t)^T Q h(t)\, dt \quad (11)$$
$$h(t) = \int_0^t e^{A(t-\tau)} B\, d\tau \quad (12)$$

and P_Δ satisfies the *discrete-time* algebraic Riccati equation associated with (8).

Again, input and/or state constraints can be added if relevant.

Remark 1 *Note that from equations (9)–(12) we have that, when* $\Delta \to 0$*:*

$$\frac{Q_q}{\Delta} \to Q; \quad \frac{S_q}{\Delta} \to 0; \quad \frac{R_q}{\Delta} \to R \quad (13)$$

Remark 2 *The choice given by equations (9)–(12) for the matrices in the discrete cost function (8), ensures that:*

$$J(u_\Delta) = J_\Delta(u_k) \quad (14)$$

where the signal $u_\Delta(t)$ *and the sequence* u_k *are related by (4) [1].*

3 Continuous Time Singular Structure

In this section we summarize the main result on the singular value structure of problem $\mathcal{P}$ as presented in [6].

We consider the Hilbert spaces $\mathcal{V} = \mathcal{L}_2(0, T_f; \mathbb{R}^m)$ and $\mathcal{Z} = \mathbb{R}^n \times \mathcal{L}_2(0, T_f; \mathbb{R}^n)$, with inner products:

$$\langle f_1, f_2 \rangle_\mathcal{V} = \int_0^{T_f} f_1(t)^T f_2(t)\, dt \qquad ; f_1, f_2 \in \mathcal{V} \quad (15)$$

$$\langle g_1, g_2 \rangle_\mathcal{Z} = \left(g_1^0\right)^T g_2^0 + \int_0^{T_f} g_1^1(t)^T g_2^1(t)\, dt \ ; \quad g_1 = \begin{bmatrix} g_1^0 \\ g_1^1 \end{bmatrix}, g_2 = \begin{bmatrix} g_2^0 \\ g_2^1 \end{bmatrix} \in \mathcal{Z} \quad (16)$$

We rewrite the input and response signal as:

$$v = v(t) = R^{1/2} u(t) \in \mathcal{V} \quad (17)$$
$$z = \begin{bmatrix} z^0 \\ z^1(t) \end{bmatrix} = \begin{bmatrix} P^{1/2}\, x(T_f) \\ Q^{1/2}\, x(t) \end{bmatrix} \in \mathcal{Z} \quad (18)$$

The cost function (3) and the system dynamics can then be expressed as:

$$J = \|v\|_\mathcal{V}^2 + \|z\|_\mathcal{Z}^2 \quad (19)$$
$$z = \mathcal{F} x_o + \mathcal{G} v \quad (20)$$

where $x_o \in \mathbb{R}^n$, and $\mathcal{F}$ and $\mathcal{G}$ are *linear operators*:

$$\mathcal{F} x_o = \begin{bmatrix} (\mathcal{F}_\Delta x_o)^0 \\ (\mathcal{F}_\Delta x_o)^1(t) \end{bmatrix} = \begin{bmatrix} P^{1/2} e^{AT_f} x_o \\ Q^{1/2} e^{At} x_o \end{bmatrix} \quad (21)$$

$$\mathcal{G} v = \begin{bmatrix} (\mathcal{G} v)^0 \\ (\mathcal{G} v)^1(t) \end{bmatrix} = \begin{bmatrix} P^{1/2} \int_0^{T_f} e^{A(T_f-\xi)} B R^{-1/2} v(\xi)\, d\xi \\ Q^{1/2} \int_0^t e^{A(t-\xi)} B R^{-1/2} v(\xi)\, d\xi \end{bmatrix} \quad (22)$$

The following theorem establishes the singular values of the operator $\mathcal{G}$, which satisfy:

$$\sigma > 0 : \quad \mathcal{G} f = \sigma g \quad , \quad \mathcal{G}^* g = \sigma f \quad (23)$$

where $\mathcal{G}^*$ is the adjoint operator of $\mathcal{G}$ [5].

Theorem 1 *The set of singular values* $\{\sigma_i\}$ *of the linear operator* $\mathcal{G}$ *in (22) are given by the roots of the equation:*

$$\det\left(\begin{bmatrix} -\sigma^{-1} P & I_n \end{bmatrix} e^{M(\sigma) T_f} \begin{bmatrix} 0 \\ I_n \end{bmatrix} \right) = 0 \quad (24)$$

where $\sigma > 0$*, and:*

$$M(\sigma) = \begin{bmatrix} A & \sigma^{-1} B R^{-1} B^T \\ -\sigma^{-1} Q & -A^T \end{bmatrix} \quad (25)$$

The corresponding singular vectors $f_i \in \mathcal{V}$ *and* $g_i \in \mathcal{Z}$*, are given by the following functions:*

$$f_i(\xi) = R^{-1/2} B^T \begin{bmatrix} 0 & I_n \end{bmatrix} e^{M(\sigma_i)\xi} \begin{bmatrix} 0 \\ I_n \end{bmatrix} d_i \quad (26)$$

$$g_i^0 = P^{1/2} \begin{bmatrix} I_n & 0 \end{bmatrix} e^{M(\sigma_i) T_f} \begin{bmatrix} 0 \\ I_n \end{bmatrix} d_i \quad (27)$$

$$g_i^1(\xi) = Q^{1/2} \begin{bmatrix} I_n & 0 \end{bmatrix} e^{M(\sigma_i)\xi} \begin{bmatrix} 0 \\ I_n \end{bmatrix} d_i \quad (28)$$

where $d_i \neq 0$ is a vector such that:

$$\begin{bmatrix} -\sigma^{-1}P & I_n \end{bmatrix} e^{M(\sigma_i)T_f} \begin{bmatrix} 0 \\ I_n \end{bmatrix} d_i = 0 \tag{29}$$

Proof: *see [6].*

4 Discrete Time Singular Structure

We next explore the singular value structure of the discrete (sampled data) problem and show that it converges to that of the underlying continuous problem.

We now consider the Hilbert spaces $\mathcal{V} = l_2(0, N-1; \mathbb{R}^m)$ and $\mathcal{Z} = \mathbb{R}^n \times l_2(0, N-1; \mathbb{R}^n)$, with inner products:

$$\langle f_1, f_2 \rangle_{\mathcal{V}} = \sum_{0}^{N-1} f_1^T f_2 \qquad ; f_1, f_2 \in \mathcal{V} \tag{30}$$

$$\langle g_1, g_2 \rangle_{\mathcal{Z}} = \left(g_1^0\right)^T g_2^0 + \sum_{0}^{N-1} \left(g_1^1\right)^T g_2^1$$
$$; g_1 = \begin{bmatrix} g_1^0 \\ g_1^1 \end{bmatrix}, g_2 = \begin{bmatrix} g_2^0 \\ g_2^1 \end{bmatrix} \in \mathcal{Z} \tag{31}$$

We rewrite the input and response signals as:

$$v = v_k = R_q^{1/2} u_k \in \mathcal{V} \tag{32}$$
$$z = \begin{bmatrix} z^0 \\ z_k^1 \end{bmatrix} = \begin{bmatrix} P_\Delta^{1/2} x_N \\ Q_q^{1/2} x_k \end{bmatrix} \in \mathcal{Z} \tag{33}$$

In the cost function (8) we can neglect the coupled term depending on S_q based on its convergence properties shown in (13). In this case, the cost and the system dynamics can then be expressed as:

$$J_\Delta = \|v\|_{\mathcal{V}}^2 + \|z\|_{\mathcal{Z}}^2 \tag{34}$$
$$z = \mathcal{F}_\Delta x_o + \mathcal{G}_\Delta v \tag{35}$$

where $x_o \in \mathbb{R}^n$, and $\mathcal{F}_\Delta$ and $\mathcal{G}_\Delta$ are *linear operators* defined by:

$$\mathcal{F}_\Delta x_o = \begin{bmatrix} (\mathcal{F}_\Delta x_o)^0 \\ (\mathcal{F}_\Delta x_o)_k^1 \end{bmatrix} = \begin{bmatrix} P_\Delta^{1/2} A_q^N x_o \\ Q_q^{1/2} A_q^k x_o \end{bmatrix} \tag{36}$$

$$\mathcal{G}_\Delta v = \begin{bmatrix} (\mathcal{G}_\Delta v)^0 \\ (\mathcal{G}_\Delta v)_k^1 \end{bmatrix} = \begin{bmatrix} P_\Delta^{1/2} \sum_{l=0}^{N-1} A_q^{N-1-l} B_q R_q^{-1/2} v_l \\ Q_q^{1/2} \sum_{l=0}^{k-1} A_q^{k-1-l} B_q R_q^{-1/2} v_l \end{bmatrix} \tag{37}$$

where $0 \leq k < N$.

Theorem 2 *For a given Δ, the set of singular values $\{\sigma_i\}$ of the linear operator $\mathcal{G}_\Delta$ in (37) are given by the roots of the equation:*

$$\det\left(\begin{bmatrix} -\sigma^{-1}P_\Delta & A_q^T \end{bmatrix} M_\Delta(\sigma)^N \begin{bmatrix} 0 \\ I_n \end{bmatrix} \right) = 0 \tag{38}$$

where $\sigma > 0$, and:

$$M_\Delta(\sigma) = \begin{bmatrix} A_q & \frac{1}{\sigma} B_q R_q^{-1} B_q^T \\ -\frac{1}{\sigma}(A_q^T)^{-1} Q_q A_q & (A_q^T)^{-1}\left(I - \frac{1}{\sigma^2} Q_q B_q R_q^{-1} B_q^T\right) \end{bmatrix} \tag{39}$$

The corresponding singular vectors $f_i \in \mathcal{V}$ and $g_i \in \mathcal{Z}$, are given by the following functions:

$$f_i[k\Delta] = R_q^{-1/2} B_q^T \begin{bmatrix} 0 & I_n \end{bmatrix} M_\Delta(\sigma_i)^k \begin{bmatrix} 0 \\ I_n \end{bmatrix} d_i \tag{40}$$
$$g_i^0 = P_\Delta^{1/2} \begin{bmatrix} I_n & 0 \end{bmatrix} M_\Delta(\sigma_i)^N \begin{bmatrix} 0 \\ I_n \end{bmatrix} d_i \tag{41}$$
$$g_i^1[k\Delta] = Q_q^{1/2} \begin{bmatrix} I_n & 0 \end{bmatrix} M_\Delta(\sigma_i)^k \begin{bmatrix} 0 \\ I_n \end{bmatrix} d_i \tag{42}$$

where $d_i \neq 0$ is a vector such that:

$$\begin{bmatrix} -\sigma^{-1}P_\Delta & A_q^T \end{bmatrix} M_\Delta(\sigma_i)^N \begin{bmatrix} 0 \\ I_n \end{bmatrix} d_i = 0 \tag{43}$$

Proof: *The singular values and singular vectors of the linear operator $\mathcal{G}_\Delta$ are defined by:*

$$\sigma > 0 : \quad \mathcal{G}_\Delta f = \sigma g \quad , \quad \mathcal{G}_\Delta^* g = \sigma f \tag{44}$$

where $f \in \mathcal{V}$, $g \in \mathcal{Z}$, and $\mathcal{G}_\Delta^*$ is the adjoint operator *of $\mathcal{G}_\Delta$ [5], given by:*

$$\mathcal{G}_\Delta^* g = \mathcal{G}_\Delta^* \begin{bmatrix} g^0 \\ g_k^1 \end{bmatrix} = R_q^{-1/2} B_q^T$$
$$\times \left[(A_q^T)^{N-1-l} P_\Delta^{1/2} g^0 + \sum_{k=l+1}^{N-1} (A_q^T)^{k-1-l} Q_q^{1/2} g_k^1 \right] \tag{45}$$

We define the variables:

$$p_k = \sum_{l=0}^{k-1} A_q^{k-1-l} B_q R_q^{-1/2} f_l \tag{46}$$

$$q_l = (A_q^T)^{N-1-l} P_\Delta^{1/2} g^0 + \sum_{k=l+1}^{N-1} (A_q^T)^{k-1-l} Q_q^{1/2} g_k^1 \tag{47}$$

Using (44) it is readily seen that these functions satisfy:

$$\begin{bmatrix} p_{j+1} \\ q_{j+1} \end{bmatrix} = M_\Delta(\sigma) \begin{bmatrix} p_j \\ q_j \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} p_j \\ q_j \end{bmatrix} = M_\Delta(\sigma)^j \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} \tag{48}$$

Also the following relations hold:

$$\begin{bmatrix} -\sigma^{-1}P_\Delta & A_q^T \end{bmatrix} \begin{bmatrix} p_N \\ q_N \end{bmatrix} = 0 \quad , \quad \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} q_0 \tag{49}$$

Using (49) in the solution of (48), with $j = N$, we have:

$$\begin{bmatrix} -\sigma^{-1}P_\Delta & A_q^T \end{bmatrix} M_\Delta(\sigma)^N \begin{bmatrix} 0 \\ I \end{bmatrix} q_0 = 0 \tag{50}$$

However, a necessary condition to have a non trivial solution, different from $(f, g) = 0$, is $q_0 \neq 0$. Therefore condition (38) is a necessary condition. Similarly, sufficiency can be easily verified.

The singular vectors can be obtained using the definitions of $\mathcal{G}_\Delta$ in (37), $\mathcal{G}_\Delta^*$ in (45), (46) and (47):

$$f_k = R_q^{-1/2} B_q^T q_k\,; \quad g^0 = P_\Delta^{1/2} p_N\,; \quad g_k^1 = Q_q^{1/2} p_k \tag{51}$$

Denoting the vector $q_0 = d \neq 0$, equations (40)–(42), are obtained from:

$$\begin{bmatrix} p_k \\ q_k \end{bmatrix} = M_\Delta(\sigma)^k \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} = M_\Delta(\sigma)^k \begin{bmatrix} 0 \\ I \end{bmatrix} q_0 \tag{52}$$

$\nabla\nabla\nabla$

The singular values of the operator $\mathcal{G}_\Delta$ correspond to the (squared) eigenvalues of the compact self-adjoint operator $\mathcal{G}_\Delta^*\mathcal{G}_\Delta$. Also, the following properties hold [7]: *(i)* the singular vectors $\{f_i\}$ are orthogonal, and *(ii)* the singular values $\{\sigma_i\}$ are real and positive, being $\sigma = 0$ the only possible point of accumulation:

$$\sigma_1 \geq \ldots \geq \sigma_N > 0 \tag{53}$$

Theorem 3 *Given a sampling period $\Delta = T_f/N$ the operator $\mathcal{G}_\Delta$ has exactly Nm singular values.*

Proof: *Following the same lines as in [6] it can be proved that $\sigma = 0$ is not an eigenvalue of the operator $\mathcal{G}_\Delta^*\mathcal{G}_\Delta$. Then the set of (orthogonal) eigenvectors form a complete basis of $\mathcal{V} = l_2(0, N-1; \mathbb{R}^m)$, which has dimension Nm. Therefore, operator $\mathcal{G}_\Delta$ has exactly Nm singular values and vectors.* $\nabla\nabla\nabla$

We next explore the limiting result for matrix $M_\Delta(\sigma)^N$, in equation (38), as $\Delta \to 0$.

Lemma 1 *The matrix $M_\Delta(\sigma)^N$ has well defined limit when Δ goes to zero. Specifically:*

$$\lim_{\Delta\to 0} M_\Delta(\sigma)^N = e^{M(\sigma)T_f} \tag{54}$$

where:

$$M(\sigma) = \begin{bmatrix} A & \sigma^{-1}BR^{-1}B^T \\ -\sigma^{-1}Q & -A^T \end{bmatrix} \tag{55}$$

Proof: *We write $M_\Delta(\sigma) = M_\Delta$, and then we consider the following expression:*

$$\begin{aligned} \log M_\Delta^N &= N \cdot \log\left(I + (M_\Delta - I)\right) \\ &= \frac{T_f}{\Delta}\left[(M_\Delta - I) - \frac{(M_\Delta - I)^2}{2} + \ldots\right] \end{aligned} \tag{56}$$

Using the delta domain state space matrices $A_\delta = (A_q - I)/\Delta$ and $B_\delta = B_q/\Delta$ [11], we have:

$$M_\Delta - I = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \tag{57}$$

where:

$$L_{11} = A_q - I = A_\delta \tag{58}$$

$$L_{12} = \frac{1}{\sigma}B_q R_q^{-1} B_q^T = \frac{1}{\sigma}B_\delta \left(\frac{R_q}{\Delta}\right)^{-1} B_\delta^T \tag{59}$$

$$\begin{aligned} L_{21} &= -\frac{1}{\sigma}(A_q^T)^{-1} Q_q A_q \\ &= -\frac{1}{\sigma}(I + \Delta A_\delta^T)^{-1}\frac{Q_q}{\Delta}(I + \Delta A_\delta) \end{aligned} \tag{60}$$

$$\begin{aligned} L_{22} &= (A_q^T)^{-1}\left(I - \frac{1}{\sigma^2}Q_q B_q R_q^{-1} B_q^T - A_q^T\right) \\ &= -(I + \Delta A_\delta^T)^{-1}\left(A_\delta^T + \frac{\Delta}{\sigma^2}\frac{Q_q}{\Delta}B_\delta\left(\frac{R_q}{\Delta}\right)^{-1} B_\delta^T\right) \end{aligned} \tag{61}$$

Using (13), we can thus see that this submatrices converge, as $\Delta \to 0$, to the corresponding submatrices of $M(\sigma)$ in (55). Therefore we finally have:

$$\lim_{\Delta\to 0} \ln\left(M_\Delta(\sigma)^N\right) = T_f \cdot M(\sigma) \tag{62}$$

and (54) is obtained applying the exponential, which is continuous function. $\nabla\nabla\nabla$

Based on the previous result, and the convergence properties of matrices P_Δ and A_q when the sampling rate grows to infinity, we can notice that equation (38) is transformed to (24), as $\Delta \to 0$.

Finally, we show that the finite set of singular values of the discrete problem converge to a countable subset of singular values for the continuous problem in a well defined fashion.

Theorem 4 *When the sampling period Δ goes to zero, the singular values of the operator $\mathcal{G}_\Delta$, for the discrete time problem, converge to a subset of the singular values of the operator $\mathcal{G}$, for the continuous time problem.*

Proof: *The singular values of the operator $\mathcal{G}_\Delta$ are the square root of the eigenvalues of the self-adjoint operator $\mathcal{G}_\Delta^*\mathcal{G}_\Delta$. The last statement is true for every operator, so the same holds for the continuous time operators $\mathcal{G}$ defined in (22) and $\mathcal{G}^*\mathcal{G}$.*

The proof uses the fact that the discrete self-adjoint operator:

$$\mathcal{G}_\Delta^*\mathcal{G}_\Delta v = R_q^{-1/2}B_q^T\left[(A_q^T)^{N-1-l}P_\Delta\sum_{k=0}^{N-1}A_q^{N-1-k}B_qR_q^{-1/2}v_k\right.$$
$$\left.+\sum_{j=l+1}^{N-1}(A_q^T)^{j-1-l}Q_q\sum_{k=0}^{j-1}A_q^{j-1-k}B_qR_q^{-1/2}v_k\right] \quad (63)$$

can be rewritten as:

$$\mathcal{G}_\Delta^*\mathcal{G}_\Delta v = \left(\frac{R_q}{\Delta}\right)^{-\frac{1}{2}}\frac{B_q^T}{\Delta}$$
$$\times\left[e^{A^T\Delta(N-1-l)}P_\Delta\sum_{k=0}^{N-1}e^{A\Delta(N-1-k)}\frac{B_q}{\Delta}\left(\frac{R_q}{\Delta}\right)^{-\frac{1}{2}}v_k\Delta\right.$$
$$\left.+\sum_{j=l+1}^{N-1}e^{A^T\Delta(j-1-l)}\frac{Q_q}{\Delta}\Delta\sum_{k=0}^{j-1}e^{A\Delta(j-1-k)}\frac{B_q}{\Delta}\left(\frac{R_q}{\Delta}\right)^{-\frac{1}{2}}v_k\Delta\right] \quad (64)$$

When $\Delta\to 0$, this operator converges pointwise *exactly to its continuous time counterpart:*

$$\mathcal{G}^*\mathcal{G}v = R^{-1/2}B^T\left[e^{A^T(T_f-\beta)}P\int_0^{T_f}e^{A(T_f-\xi)}BR^{-1/2}v(\xi)\,d\xi\right.$$
$$\left.+\int_\beta^{T_f}e^{A^T(\tau-\beta)}Q\int_0^\tau e^{A(\tau-\xi)}BR^{-1/2}v(\xi)\,d\xi\,d\tau\right] \quad (65)$$

where we used the convergence properties, when $\Delta\to 0$, of:

$$\frac{B_q}{\Delta}\to B\,,\quad \frac{Q_q}{\Delta}\to Q\,,\quad \frac{R_q}{\Delta}\to R\,,\quad P_\Delta\to P \quad (66)$$

and:

$$N\Delta = T_f\,,\quad (l+1)\Delta=\beta\,,\quad j\Delta=\tau\,,\quad k\Delta=\xi \quad (67)$$

According to Theorem 1.6 in [8], given a sequence of bounded functions $\{\mathcal{G}_{\Delta_n}^*\mathcal{G}_{\Delta_n}v\}$ in $\mathcal{L}_2$ which converges pointwise to $\mathcal{G}^*\mathcal{G}v$, then $\mathcal{G}^*\mathcal{G}v$ is in $\mathcal{L}_2$ and $\{\mathcal{G}_{\Delta_n}^*\mathcal{G}_{\Delta_n}v\}$ is $\mathcal{L}_2$-convergent to $\mathcal{G}^*\mathcal{G}v$, i.e., the discrete time operator converges in norm to the continuous time one.

Then, following the same arguments used in [2], we prove that every limiting point of the set of eigenvalues of $\mathcal{G}_\Delta^*\mathcal{G}_\Delta$ (singular values of $\mathcal{G}_\Delta$) converges to an eigenvalue of $\mathcal{G}^*\mathcal{G}$. Let us suppose that $\lambda\neq 0$ is a limit point of a sequence of eigenvalues λ_Δ of $\mathcal{G}_\Delta^*\mathcal{G}_\Delta$, hence, $(\lambda_\Delta I-\mathcal{G}_\Delta^*\mathcal{G}_\Delta)$ converges in norm to $(\lambda I-\mathcal{G}^*\mathcal{G})$. Since the set of invertible operators is open, if $(\lambda_\Delta I-\mathcal{G}_\Delta^*\mathcal{G}_\Delta)$ is not invertible (i.e., λ_Δ belongs to the spectrum of the self-adjoint operator) for all Δ then $(\lambda I-\mathcal{G}^*\mathcal{G})$ is not invertible. This establishes that λ belongs to the spectrum of the compact self-adjoint operator $\mathcal{G}^*\mathcal{G}$, different from zero, so it is one of its eigenvalues. $\nabla\nabla\nabla$

5 Example

We illustrate the previous results for a simple example:

Example 1 *Consider the double integrator system:*

$$\dot{x} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}x + \begin{bmatrix}0\\1\end{bmatrix}u \quad (68)$$
$$y = \begin{bmatrix}1 & 0\end{bmatrix}x \quad (69)$$

with continuous cost function (3), where $T_f=5$, $Q=C^TC$, $R=0.1$, and P is the solution of the algebraic Riccati equation associated to J_1 in (3).

The discrete time model matrices, from (7), are:

$$A_q=\begin{bmatrix}1 & \Delta\\ 0 & 1\end{bmatrix}\;;\quad B_q=\begin{bmatrix}\Delta^2/2\\ \Delta\end{bmatrix}\;;\quad C_q=\begin{bmatrix}1 & 0\end{bmatrix} \quad (70)$$

The matrices for the sampled data cost function (8), are:

$$Q_q=\Delta\begin{bmatrix}1 & \Delta/2\\ \Delta/2 & \Delta^2/3\end{bmatrix} \quad (71)$$
$$R_q=\Delta\left(\Delta^4/20+0.1\right) \quad (72)$$
$$S_q=\Delta\begin{bmatrix}\Delta^2/6\\ \Delta^3/8\end{bmatrix} \quad (73)$$

and P_Δ is obtained from the discrete time algebra Riccati equation associated to (8).

We define $f(\sigma)$ to be the function on the left hand side of equation (24), whose positive roots are the singular values of $\mathcal{G}$, and analogously, we use $f_\Delta(\sigma)$ to define function on the left hand side of (38), whose positive roots are the singular values of $\mathcal{G}_\Delta$. Figure 1 shows the functions $f(\sigma)$ and $f_\Delta(\sigma)$ for three different values of $\Delta=T_f/N$, where we can see that the finite set of roots (singular values) obtained in discrete time approach the countable set of roots in the continuous time set. Note that the zero crossings by $f(\sigma)$ and $f_\Delta(\sigma)$ correspond to the singular values.

To illustrate the results in Section 4, Figure 2 shows the evolution of the first 10 singular values for the discrete problem, $\mathcal{P}_\Delta$, compared with the singular values obtained in [6] for the continuous time problem. Note that given any N and the corresponding sampling period Δ, the operator $\mathcal{G}_\Delta$ has only N singular values given by equation (38), as stated in Theorem 3. In the figure we can see that the discrete-time singular values quickly approach their continuous-time counterpart. For example, with $N=4$ the four singular values obtained are within 10% of the continuous time ones. $\nabla\nabla\nabla$

6 Conclusions

This paper has explored the connections between the singular values of a continuous time linear quadratic control

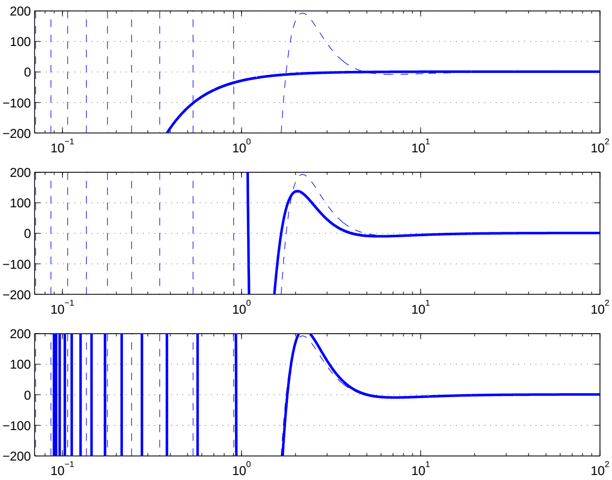

Figure 1: Plot of $f_\Delta(\sigma)$ (solid) and $f(\sigma)$ (dashed) for $N = 1, 4$ and 16.

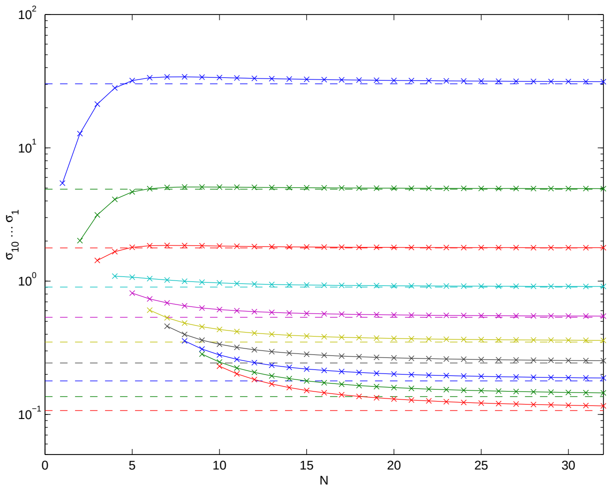

Figure 2: Convergence of the singular values of $\mathcal{G}_\Delta$

problem and its sampled data counterpart. Specifically, it has been shown that the sampled data singular values converge to the continuous time singular values.

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