FURTHER RESULTS ON THE EXISTENCE OF A CONTINUOUS STORAGE FUNCTION FOR NONLINEAR DISSIPATIVE SYSTEMS

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Abstract

The problem of existence of continuous storage function for dissipative nonlinear systems is considered. It is shown that, if a nonlinear system is dissipative in the state x_*, then, under certain assumptions, a continuous storage function can be constructed on a set of points accessible from x_* by concatenation of a finite number of forward and backward motions of the system. Most of these assumptions are weaker than certain controllability-type properties and can be checked by the similar tests.

1 Introduction

The theory of dissipative systems was established in the pioneering work of Willems [14] and subsequently developed in the series of papers by Hill and Moylan [4, 3, 5]. Since then this theory plays an increasing role in nonlinear control and, in particular, has found applications in nonlinear H_∞ control [7, 13], control of mechanical and port-controlled Hamiltonian systems [8], ISS systems and related notions [12], and other areas. For general references, see [13, 8]. The system is called dissipative if a certain integral functional is nonnegative along the trajectories. One of the most important results in this theory states that the dissipativity property is equivalent to the existence of a nonnegative function, called storage function, which is defined on the state space of the system and satisfies a so called dissipation inequality. This result provides a connection between the theory of dissipative systems and a variety of nonlinear control problems, since the storage function may serve as (or can be used for construction of) a Lyapunov function corresponding to a given control problem. In general, however, a storage function is discontinuous, while the Lyapunov function is usually required to satisfy some regularity assumptions. For example, essentially stronger results in nonlinear H_∞ control theory can be obtained under the assumption that the corresponding storage function is continuously differentiable [7]; recursive design procedures like backstepping require the corresponding Lyapunov function to be sufficiently smooth; the possibility of stabilization by feedback depends on existence of a continuous control Lyapunov function [1], etc. Thus the problem of finding conditions for existence of more regular (continuous, smooth, etc) storage functions is of essential interest. However, this problem has received scant attention in the literature. For dissipative systems with supply rate of general form such a problem was addressed in [5, 6, 10]. In particular, in [6] it is proved that any dissipative system has a lower semicontinuous storage function. Hill & Moylan [5] stated that any storage function is continuous, if the dissipative system has the local w-uniform reachability property in all points of the state space. A more refined version of this result has been presented in [10]: if the system is dissipative in one point x_* and satisfies the local w-uniform reachability assumption in the same point x_*, then there exists a continuous storage function defined on the set $\mathcal{R}(x_*)$ of points reachable from x_*.

The main purpose of this paper is to provide conditions for the existence of a continuous storage function defined on a set larger than $\mathcal{R}(x_*)$. In fact, we show that, under some assumptions, a continuous storage function can be constructed on a set of points accessible from x_* by concatenation of a finite number of forward and backward motions of the system. It is also shown that most of our assumptions are weaker than some well-studied controllability type properties of the system and can be checked in a similar manner.

The structure of the paper is as follows. In section 2 the necessary definitions are given and some preliminary results are provided, in particular, the relations between our assumptions and the corresponding controllability-type properties of the system are studied. The main results together with the proofs are presented in section 3. Finally section 4 contains some concluding remarks.

2 Preliminaries

Consider a nonlinear control system

$$\dot{x} = F(x, u). \tag{1}$$

Here $x \in X \subset R^n$ is state, $u \in U \subset R^m$ is input, X and U are open and connected sets, U contains 0, $F(x, u)$ is assumed to have following properties: for every fixed u the function $F(\cdot, u)$ is of class C^1 (continuously differentiable), and both $F(x,u)$ and $\frac{\partial F}{\partial x}(x,u)$ are continuous on x, u. Under these conditions for every measurable essentially bounded control $u\colon [0,s) \to U$, where $s > 0$, and every initial condition $x_0 \in X$ the corresponding solution $x(t) = \phi(t, x_0, u)$ of the system (1) exists at least for all $t \in [0, t_0]$, where $t_0 \in (0, s]$. A measurable essentially bounded control ω: $[0, s) \to U$ is said to be admissible for initial condition x_0 ($u \in \mathcal{U}_{x_0}^{[0,s)}$) if the corresponding solution $x(t) = \phi(t, x_0, u)$ is defined for all $t \in [0, s]$.

Let U be a subset of X. We will say the set U is forward invariant with respect to trajectories of the system (1) if for any $x_0 \in U$, $u \in \mathcal{U}_{x_0}^{[0,t)}$, $t \geq 0$ the corresponding solution satisfies $\phi(s, x_0, u) \in U$ for all $s \in [0, t]$. The set U is said to be backward invariant with respect to trajectories of (1) if it is forward invariant with respect to trajectories of the time-reversed system

$$\dot{x} = -F(x, u).$$

Finally, the set is said to be invariant if it is both forward and backward invariant.

Again, let U be a subset of X. The state $\xi \in U$ is said to be reachable from $\eta \in U$ (equivalently, $\eta \in U$ is controllable to $\xi \in U$) without leaving U if there exists $t \geq 0$ and $u \in \mathcal{U}_{\eta}^{[0,t)}$ such that $\phi(s, \eta, u) \in U$ for all $s \in [0, t]$, and $\phi(t, \eta, u) = \xi$. If $U = X$, we will simply said the state ξ is reachable from η (η is controllable to ξ). The set of all points reachable from ξ (controllable to η) without leaving U is denoted by $\mathcal{R}_U(\xi)$ ($\mathcal{C}_U(\eta)$), or, if $U = X$, simply by $\mathcal{R}(\xi)$ ($\mathcal{C}(\eta)$).

Finally, given a set U, then it's interior will be denoted by $\operatorname{int} U$, it's closure by $\bar{U}$, and it's boundary by ∂U.

2.1 Dissipativity, virtual storage and storage functions

Let $w(x,u)$ be a scalar function continuous on $X \times U$ in both arguments. Following [14, 5] we will say that the system (1) in the state $x_0 \in X$ is dissipative with respect to supply rate $w(x,u)$ if there exists $\beta \geq 0$ such that for each $t \geq 0$ and each $u \in \mathcal{U}_{x_0}^{[0,t)}$,

$$\int_0^t w\left(\phi(s, x_0, u), u(s)\right) ds + \beta \geq 0.$$

The following notion are closely related with the dissipativity property of the system.

Definition 1. The function $V : X \to R$ is said to be a virtual storage function for the system (1) with supply rate $w(x,u)$ if for every $x_0 \in X$, every $t \geq 0$ and every $u \in \mathcal{U}_{x_0}^{[0,t)}$

$$V\left(\phi(t, x_0, u)\right) - V\left(x_0\right) \leq \int_0^t w\left(\phi(s, x_0, u), u(s)\right) ds. \quad (2)$$

If, in addition, $V(x) \geq 0$ for all $x \in X$, then V is called a storage function.

It is a well-known fact (see [5]) that the dissipativity property of the system is equivalent to the existence of a storage function, while so-called cyclo-dissipativity (i.e. dissipativity along the pieces of trajectories where the initial and the final states coincide) is equivalent to the existence of a virtual storage function. In general, however, the storage function is discontinuous. To formulate conditions for the existence of a continuous storage function, we need the following notion.

2.2 w-uniform reachability

Again, let $w(x,u)$ be a scalar function continuous on $X \times U$ in both arguments.

Definition 2. [5, 10]The system (1) is said to be locally w-uniformly reachable in the state x_* if there exists a neighborhood Ω of x_* and a class-$\mathcal{K}$ function ρ such that for each $x \in \Omega$ there exist $t \geq 0$ and $u \in \mathcal{U}_{x_*}^{[0,t)}$ such that $x = \phi(t, x_*, u)$ and

$$\left| \int_0^t w\left(\phi(s, x_*, u), u(s)\right) ds \right| \leq \rho\left(|x - x_*|\right).$$

It is shown in [5] that under the assumptions of dissipativity in every state and local uniform reachability in every state, every storage function is continuous. On the other hand, we have the following result.

Theorem 1. [10] Suppose the system (1) is dissipative in the state $x_* \in X$ with respect to supply rate $w(x,u)$ and uniformly w-reachable in the same state x_*. Then there exists a continuous storage function defined on the set $\mathcal{R}(x_*)$.

The purpose of this paper is to present conditions under which a continuous storage function can be defined on a set larger than $\mathcal{R}(x_*)$.

Remark 1. In some cases the local w-uniform reachability property follows from the local controllability and, therefore, can be checked using controllability-type tests [10]. Below we formulate a simple statement of this type which generalizes one given in [10]. To this end, denote

$$\begin{aligned} \mathcal{R}_{T,L}(x_*) := \{ \phi\left(t, x_*, u\right) | \ t \in [0, T], \\ u \in \mathcal{U}_{x_*}^{[0,t)}, \ esssup_{s \in [0,t]} |u(s)| \leq L \Big\}. \end{aligned} \quad (3)$$

In words, $\mathcal{R}_{T,L}(x_*)$ is the set of states reachable from x_* in time less than or equal to T by an admissible control essentially bounded by L. The system (1) in the state $x_* \in X$ is said to be small-time locally controllable by uniformly bounded control (STLC-UBC) if there exists $L < \infty$ such that for any $T > 0$

$$x_* \in \operatorname{int} \mathcal{R}_{T,L}(x_*).$$

In other words, the system is STLC-UBC if $\mathcal{R}_{T,L}(x_*)$ contains an open neighborhood of x_*. The notion of STLC-UBC is a version of small-time local controllability notion extensively studied in the literature(see [9, 11] and the bibliography therein).

Proposition 1. The system (1) is locally w-uniformly reachable in the state x_* if it is STLC-UBC in x_*.

Proof. Take an arbitrary $\delta > 0$ and consider the closed ball $\bar{B}_\delta(x_*) := \{x \in X : \; |x - x_*| \le \delta\}$. By assumption, the right-hand side $F(x,u)$ as well as supply rate $w(x,u)$ are continuous in both arguments, therefore we have

$$\sup_{\substack{x \in \bar{B}_\delta(x_*) \\ |u| \le L}} |F(x,u)| = F^* < \infty, \tag{4}$$

and

$$\sup_{\substack{x \in \bar{B}_\delta(x_*) \\ |u| \le L}} |w(x,u)| = D^* < \infty. \tag{5}$$

Put $T^* = \delta/F^*$. From (4) we see that

$$\mathcal{R}_{T,L}(x_*) \subset \bar{B}_\delta(x_*) \quad \text{for all } T \in [0, T^*]. \tag{6}$$

Now for each $T \in [0,T^*]$ define $\beta^*(T)$ as the supremum of all posiible $\beta \ge 0$ such that

$$\{x \in X : \; |x - x_*| < \beta\} \subset \operatorname{int} \mathcal{R}_{T,L}(x_*).$$

Clearly, $\beta^*(0) = 0$, $\beta^*(\cdot)$ is nondecreasing, and, since the system is STLC-UBD, we have $\beta^*(t) > 0$ for all $t \in (0, T^*)$. Further, take any continuous function β: $[0,T^*] \to R^+$ strictly increasing and satisfying $\beta(t) \le \beta^*(t)$ for all $t \in [0,T^*]$. It is easy to see that such a function $\beta(\cdot)$ always exists . Due to the properties of $\beta(\cdot)$, the inverse function $\beta^{-1}(s)$ is well defined for $s \in [0, \beta(T^*)]$, satisfies $\beta^{-1}(0) = 0$, and is strictly increasing. By construction of $\beta^{-1}(\cdot)$, we see that $|x - x_*| \le \beta(T^*)$ implies $x \in \mathcal{R}_{\beta^{-1}(|x-x_*|),L}(x_*)$, i.e. there exists $t \in \left[0, \beta^{-1}(|x - x_*|)\right]$, and a control $u \in \mathcal{U}_{x_*}^{[0,t)}$ satisfying $\|u_{[0,t)}\|_\infty \le L$ such that $\phi(t, x_*, u) = x$. Using (5), (6), we see that along the corresponding trajectory

$$\left| \int_0^t w\left(\phi(s, x_*, u), u(s)\right) ds \right| \le D^* \beta^{-1}\left(|x - x_*|\right).$$

This completes the proof.•

Proposition 1 shows that a number of existing tests for local small-time controllability can also be applied to determine the w-uniform reachability property.

The following nonlocal version of w-uniform reachability will be also used below.

Definition 3. Let $\mathcal{M}, \mathcal{N}$ are subsets of X. We will say that $\mathcal{M}$ is w-uniformly reachable from $\mathcal{N}$ (equivalently, $\mathcal{N}$ is w-uniformly controllable to $\mathcal{M}$), if $\mathcal{M} \subset \mathcal{R}(\mathcal{N})$, and there exists $K < +\infty$ such that for each $\mu \in \mathcal{M}$ there exists $\eta \in \mathcal{N}$ and $u \in \mathcal{U}_\eta^{[0,t)}$, $t \ge 0$ such that $\phi(t, \eta, u) = \mu$ and

$$\left| \int_0^t w\left(\phi(s, \eta, u), u(s)\right) ds \right| \le K.$$

2.3 Weak Accessibility

Definition 4. [2] The state $\xi \in X$ is said to be weakly accessible from the state $\eta \in X$ if there exists a finite number of states $x_0, x_1, \ldots, x_n \in X$ such that $x_0 = \xi$, $x_n = \eta$, and $x_{i+1} \in \mathcal{R}(x_i) \cup \mathcal{C}(x_i)$ for any integer $0 \le i \le n-1$.

It is worth noting that thus defined weak accessibility is an equivalence relation, while the reachability and the controllability are not. More precisely, both the controllability and the reachability are reflexive and transitive relations, but in general they are not symmetric, while the weak accessibility is clearly symmetric (to prove this note that if $x \in \mathcal{R}(y)$ then $y \in \mathcal{C}(x)$, and vice versa). Since the weak accessibility is an equivalence relation, one can consider the following partition of the state space

$$X = \bigcup_{i \in \Xi} W_i,$$

where W_i are classes of equivalence with respect to weak accessibility relation. Note that each set W_i is invariant with respect to trajectories of the controlled system.

We now provide a construction that will be used in the sequel. For a given $\xi \in X$ define the sequence of sets Ω_i, $i = 0, 1, \ldots$ as follows:

$$\Omega_0(\xi) = \mathcal{R}(\xi),$$

and

$$\Omega_i(\xi) = \mathcal{R}\left(\mathcal{C}\left(\Omega_{i-1}(\xi)\right)\right).$$

for $i = 1, 2, \ldots$. Thus, each set $\Omega_i(\xi)$, $i = 0, 1, 2, \ldots$, consists of points accessible from ξ by concatenation of k possible forward and backward motions of the system (1), where $k \in \{0, 1, \ldots, 2i - 1\}$. It is easy to see that each set $\Omega_i(\xi)$, $i = 0, 1, 2, \ldots$, is forward invariant with respect to trajectories of the controlled system (1), and

$$\mathcal{W}(\xi) = \bigcup_{i = 0, 1, 2, \ldots} \Omega_i(\xi), \tag{7}$$

where $\mathcal{W}(\xi)$ is the class of states equivalent to ξ with respect to the weak accessibility relation.

2.4 Local w-uniform accessibility

Let $w(x,u)$ be a scalar function continuous on $X \times U$.

Definition 5. The system is called locally w-uniformly accessible at the state $\xi \in X$, if for any $\epsilon > 0$ and for any neighbourhood U of ξ the set

$$\mathcal{A}_{U,\left|\int w\right|<\epsilon}(x_*) := \Big\{ x \in U : \exists t \geq 0, u \in \mathcal{U}_\xi^{[0,t)}, \\ \text{s.t. } \phi(t,\xi,u) = x, \left| \int_0^t w\left(\phi(s,\xi,u),u(s)\right) ds \right| < \epsilon \Big\}$$

has a nonempty interior.

The introduced local w-uniform accessibility is also related to the more traditional accessibility-type property. Indeed, using the notation (3), the following statement can be formulated.

Proposition 2. The system (1) is locally w-uniform accessible in the state $x_* \in X$ if there exists $L < \infty$ such that for any $T > 0$

$$\operatorname{int} \mathcal{R}_{T,L}(x_*) \neq \emptyset. \tag{8}$$

Proof. Take an arbitrary $\delta > 0$ such that $\bar{B}_\delta(x_*) := \{x \in X : |x - x_*| \leq \delta\} \subset U$. We have (see the proof of Proposition 1)

$$\sup_{\substack{x \in \bar{B}_\delta(x_*) \\ |u| \leq L}} |F(x,u)| = F^* < \infty,$$
$$\sup_{\substack{x \in \bar{B}_\delta(x_*) \\ |u| \leq L}} |w(x,u)| = D^* < \infty.$$

Put

$$T^* = \min\left\{ \frac{\delta}{F^*}, \frac{\epsilon}{D^*} \right\}.$$

For any $t \in [0, T^*)$ we have

$$\mathcal{R}_{t,L}(x_*) \subset \mathcal{A}_{U,\left|\int w\right|<\epsilon}(x_*),$$

therefore

$$\operatorname{int} \mathcal{R}_{t,L}(x_*) \subset \operatorname{int} \mathcal{A}_{U,\left|\int w\right|<\epsilon}(x_*).$$

The proof is complete. •

Proposition 2 shows that the local w-uniform accessibility property follows from a version of the local accessibility property widely studied in the literature [9, 11, 2]. In particular, the local w-uniform accessibility can be checked by calculating rank of the corresponding Lie algebra. Let $\mathcal{L}$ be the Lie algebra generated by the set of vector fields $\mathcal{F}_u := \{F(\cdot,u),\ u \in U\}$. Using standard line of reasoning [9, 11, 2], one can easily prove the following consequence of Propositon 2.

Corollary 1. The system (1) is locally w-uniform accessible in the state $x_* \in X$ if

$$\operatorname{rank} \mathcal{L}(x_*) = n.$$

3 Main results

Our main result is presented by the following theorem.

Theorem 2. Suppose the system (1) is dissipative at some state $x_* \in X$ with respect to supply rate $w(x,u)$ and the following properties are satisfied:

i) the system (1) is locally w-uniformly reachable at x_*;

ii) each set $\mathcal{C}\left(\Omega_i(x_*)\right)$, $i = 1, 2, \ldots$ is w-uniformly controllable to $\Omega_i(x_*)$, and each set $\Omega_i(x_*)$, $i = 1, 2, \ldots$, is w-uniformly reachable from $\mathcal{C}\left(\Omega_{i-1}(x_*)\right)$.

iii) the system (1) is locally w-uniformly accessible on the set

$$\Omega^*(x_*) := \bigcup_{i=0,1,2,\ldots} \left(\left(\partial\Omega_i(x_*) \cap \mathcal{C}\left(\Omega_i(x_*)\right) \right) \cup \left(\left(\partial\mathcal{C}\left(\Omega_i(x_*)\right) \cap \Omega_{i+1}(x_*) \right) \right) \right).$$

Then there exists a continuous virtual storage function defined on the set $\mathcal{W}(x_*)$.

Proof. For each state $x \in \mathcal{R}(x_*) = \Omega_1(x_*)$ define a function V as follows:

$$V(x) = \inf_{\substack{t \geq 0,\ u \in \mathcal{U}_{x_*}^{[0,t)}, \\ \phi(t,x_*,u) = x}} \int_0^t w\left(\phi(s,x_*,u),u(s)\right) ds \tag{9}$$

Thus defined function V is called the required supply [14, 5]. Taking into account the dissipativity property and assumption ii), we see that

$$-\beta(x_*) \leq V(x) \leq K_1 - \beta(x_*)$$

on the set $\Omega_1(x_*)$ for some $K_1 < \infty$. By construction, the set $\Omega_1(x_*)$ is connected, and by i) it is open. Also, the function V is continuous on the set $\Omega_1(x_*)$ (see for example [10]).

Consider now the set $\mathcal{C}\left(\Omega_1(x_*)\right)$. This set is clearly open and connected. For each $x \in \mathcal{C}\left(\Omega_1(x_*)\right) \setminus \Omega_1(x_*)$ define $V(x)$ by the formula

$$V(x) = \sup_{\substack{t \geq 0,\ u \in \mathcal{U}_x^{[0,t)}, \\ \tilde{x} \in \Omega_1(x_*), \\ \phi(t,x,u) = \tilde{x},}} \left(V(\tilde{x}) - \int_0^t w\left(\phi(s,x,u),u(s)\right) ds \right).$$

Due to ii), we see that

$$-\beta(x_*) - K_2 \leq V(x) \leq K_1 + K_2 - \beta(x_*)$$

on the set $\mathcal{C}\left(\Omega_1(x_*)\right)$ for some $K_2 < +\infty$. To show that defined in this way the function V satisfies the dissipation inequality (2), take any point $x_0 \in \mathcal{C}\left(\Omega_1(x_*)\right)$. Take an arbitrary $\hat{u} \in \mathcal{U}_{x_0}^{[0,t_1)}$, $t_1 > 0$ and suppose $\phi(t_1, x_0, \hat{u}) =$

$x_1 \in \mathcal{C}(\Omega_1(x_*)) \setminus \Omega_1(x_*)$. Then

$$\begin{aligned} V(x_0) = & \sup_{\substack{t \geq 0,\ u \in \mathcal{U}_{x_0}^{[0,t)}, \\ \phi(t, x_0, u) \in \Omega_1(x_*),}} \left(V(\phi(t, x_0, u)) - \int_0^t w(\phi(s, x_0, u), u(s))\, ds \right) \\ & \geq - \int_0^{t_1} w(\phi(s, x_0, \hat{u}), \hat{u}(s))\, ds \\ & + \sup_{\substack{t \geq 0,\ u \in \mathcal{U}_{x_1}^{[0,t)}, \\ \phi(t, x_1, u) \in \Omega_1(x_*),}} \left(V(\phi(t, x_1, u)) - \int_0^t w(\phi(s, x_1, u), u(s))\, ds \right) \\ & = - \int_0^{t_1} w(\phi(s, x_0, \hat{u}), \hat{u}(s))\, ds + V(x_1). \end{aligned}$$

Otherwise, suppose $\phi(t_1, x_0, \hat{u}) = x_1 \in \Omega_1(x_*)$, then

$$\begin{aligned} V(x_0) = & \sup_{\substack{t \geq 0,\ u \in \mathcal{U}_{x_0}^{[0,t)}, \\ \phi(t, x_0, u) \in \Omega_1(x_*),}} \left(V(\phi(t, x_0, u)) - \int_0^t w(\phi(s, x_0, u), u(s))\, ds \right) \\ & \geq V(x_1) - \int_0^{t_1} w(\phi(s, x_0, \hat{u}), \hat{u}(s))\, ds. \end{aligned}$$

Now we claim that under the conditions of the theorem the function V is continuous on the set $\mathcal{C}(\Omega_1(x_*))$. Indeed, continuity of V on the set $\operatorname{int}(\mathcal{C}(\Omega_1(x_*)) \setminus \Omega_1(x_*))$ clearly follows from the continuity of V on the set $\Omega_1(x_*)$ and continuity of $w(x,u)$. To prove continuity of V on the set $\partial\Omega_1(x_*) \cap \mathcal{C}(\Omega_1(x_*))$, take any point $x_0 \in \partial\Omega_1(x_*) \cap \mathcal{C}(\Omega_1(x_*))$ and fix $\epsilon > 0$. It is easy to see using standard continuous dependence arguments that there exists $\delta > 0$ such that for any $x \in \mathcal{C}(\Omega_1(x_*)) \setminus \Omega_1(x_*)$, $|x - x_0| < \delta$ we have $|V(x) - V(x_0)| < \epsilon$. On the other hand, take an arbitrary $x \in \Omega_1(x_*)$ sufficiently close to x_0. Suppose $\hat{u} \in \mathcal{U}_{x_0}^{[0,\hat{t})}$ is an arbitrary control such that $\phi(\hat{t}, x_0, \hat{u}) \in \Omega_1(x_*)$ and

$$V(\phi(\hat{t}, x_0, \hat{u})) - \int_0^{\hat{t}} w(\phi(s, x_0, \hat{u}), \hat{u}(s))\, ds \geq V(x_0) - \frac{\epsilon}{3}.$$

If x is sufficiently close to x_0, then we have $\phi(\hat{t}, x, \hat{u}) \in \Omega_1(x_*)$,

$$\left| V(\phi(\hat{t}, x, \hat{u})) - V(\phi(\hat{t}, x_0, \hat{u})) \right| \leq \frac{\epsilon}{3},$$

and

$$\left| \int_0^{\hat{t}} w(\phi(s, x_0, \hat{u}), \hat{u}(s))\, ds - \int_0^{\hat{t}} w(\phi(s, x, \hat{u}), \hat{u}(s)) \right| \leq \frac{\epsilon}{3}.$$

Combining the above formulas with the dissipation inequality

$$V(\phi(\hat{t}, x, \hat{u})) \leq V(x) + \int_0^{\hat{t}} w(\phi(s, x, \hat{u}), \hat{u}(s)),$$

we see that

$$V(x) \geq V(x_0) - \epsilon. \quad (10)$$

To prove the inequality opposite to (10), take a sufficiently small open neighborhood $\Upsilon(x_0)$ of the point x_0 such that for any $x_1, x_2 \in \Upsilon(x_0) \bigcap \Omega_1(x_*)$ we have $|V(x_1) - V(x_2)| < \epsilon/2$. By the assumption iii), there exists a nonempty open subset $\Upsilon_0 \subset \Upsilon(x_0)$ with the following property: for any state $\xi \in \Upsilon_0$ there exists a control $u \in \mathcal{U}_{x_0}^{[0,t)}$, $t \geq 0$ such that $\phi(t, x_0, u) = \xi$, and

$$\left| \int_0^t w(\phi(s, x_0, u), u(s))\, ds \right| < \frac{\epsilon}{2}.$$

First, we claim that

$$\Upsilon_0 \bigcap \Omega_1(x_*) \neq \emptyset. \quad (11)$$

Indeed, if $\Upsilon_0 \bigcap \Omega_1(x_*) = \emptyset$, then $\Upsilon_0 \subset \operatorname{int}(X \setminus \Omega_i(x_*))$. Then for any state $\tilde{x}_0 \in \Omega_1(x_*)$ sufficiently close to x_0, we have $\phi(t, \tilde{x}_0, u) \in \Upsilon_0 \subset \operatorname{int}(X \setminus \Omega_i(x_*))$, which contradicts the fact that $\Omega_i(x_*)$ is invariant with respect to trajectories of the controlled system. Now, take any point $x_1 \in \Upsilon_0 \bigcap \Omega_1(x_*)$. By definition of Υ_0 we have $V(x_1) \leq V(x_0) + \epsilon/2$. Therefore, by definition of $\Upsilon(x_0)$ for any $x \in \Upsilon(x_0) \bigcap \Omega_1(x_*)$ we have

$$V(x_0) \geq V(x) - \epsilon. \quad (12)$$

Combining (10) and (12), we get that the function V is continuous on the set $\partial\Omega_1(x_*) \cap \mathcal{C}(\Omega_1(x_*))$. Therefore, it is continuous on $\mathcal{C}(\Omega_1(x_*))$.

Consider now the set $\Omega_2(x_*) := \mathcal{R}(\mathcal{C}(\Omega_1(x_*)))$. Define an extension of the function V on the set $\Omega_2(x_*) \setminus \mathcal{C}(\Omega_1(x_*))$ as follows

$$V(x) = \inf_{\substack{\tilde{x} \in \Omega_1(x_*), \\ t \geq 0, \\ u \in \mathcal{U}_{\tilde{x}}^{[0,t)}, \\ \phi(t, \tilde{x}, u) = x,}} \left(V(\tilde{x}) + \int_0^t w(\phi(s, x, u), u(s))\, ds \right).$$

By ii) we see that thus defined function V is uniformly bounded on the set $\Omega_2(x_*)$. Using the same line of reasoning as above, one can prove that V is continuous on the set $\Omega_2(x_*)$ and satisfies the dissipation inequality (2) along the trajectories of the system.

Thus we have proven that the function V is a continuous virtual storage function defined on the set $\Omega_2(x_*)$. Continuing this line of reasoning, one can extend V to the sets

Ω_3, $\Omega_4, \ldots$ etc. Taking into account formula (7), we get the result of Theorem. •

Corollary 2. Under the assumptions of theorem 2, for any set $\Omega_i(x_*)$, $i = 1, 2, \ldots$, there exists a continuous storage function defined on $\Omega_i(x_*)$.

Proof. By construction of Theorem 1, the continuous virtual storage function V is uniformly bounded on each set $\Omega_i(x_*)$, $i = 1, 2, \ldots$. Therefore, for any $i \in \{1, 2, \ldots\}$ the function V can be made nonnegative on $\Omega_i(x_*)$ simply by adding an appropriate constant.•

4 Concluding Remarks

In this paper we have shown that if a nonlinear system of the general form (1) is dissipative in the state x_* and locally w-uniformly reachable from the same state x_*, then under additional assumptions ii), iii) of Theorem 1 there exists a continuous virtual storage function defined on the set of points weakly accessible from x_*. By construction, this function is bounded from below on each set $\Omega_i(x_*)$, $i = 1, 2, \ldots$, therefore it can be made nonnegative on each such a set simply by adding an appropriate positive constant. We also show that most of our assumptions follow from the well-studied controllability type properties and can be checked by the similar tests. The development of analogous conditions for existence of more regular (for example, smooth) storage functions should be a topic for future research.

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