LMI-BASED CONTROL DESIGN FOR DISCRETE POLYTOPIC LPV SYSTEMS

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Abstract

This paper presents two LMI-based designs for the stabilization and regulator problems of polytopic discrete linear parameter-varying (LPV) systems. The main advantage of the new approaches is that they are based on parameter-varying Lyapunov functions. Different Lyapunov functions are computed for each vertex of the polytope and the global Lyapunov function is a convex combination of local ones. The algorithms proposed are thus less conservative compared to the more usual approach of employing a common P matrix.

1 Introduction

The concept of linear parameter-varying (LPV) systems was first introduced in the context of gain scheduled control of nonlinear systems [9]. In gain scheduling, a group of linear systems are obtained by linearizing the nonlinear system to be controlled at several different operating points. A controller can be designed for each individual linear system and the global controller is then a combination of linear controllers with appropriate scheduling functions. Thus the design can be regarded as applicable for the underlying LPV system which is a weighted combination of linearized systems. This type of system was then widely studied by many researchers using different approaches [1, 7, 8].

In the control of LPV systems, as pointed in [10], the closed-loop stability can be guaranteed for slow-varying plants. However, in the actual controller design, it is difficult to quantify the *slow* property for the specified LPV or nonlinear plant. To avoid this problem, most of the design approaches involve finding a Lyapunov function for the closed-loop system, with which the global stability is assured. In the majority of approaches in the literature, linear matrix inequality (LMI) optimization is the main tool used.

LMIs now play an important role in designing stable controllers [2, 6]. In general, solving LMI problems gives the expected controller parameters and a Lyapunov function of the form,

$$V(x) = x^T P(.)\, x$$

where x is the system state and $P(.) > 0$ is a positive definite matrix which can be a constant or parameter-dependent matrix. If $P(.) = P$ is constant, the function $V(.)$ is called a quadratic Lyapunov function, which implies that all the linear sub-systems share a common Lyapunov function. Designs adopting a constant P matrix also are referred as common P approaches.

Clearly a parameter-dependent Lyapunov function gives more design freedom. However, in the literature, the parameter-dependent Lyapunov function is applied only in a few papers [5]. The difficulty of using a parameter-dependent Lyapunov function is that, if the parameter is time-varying, the rate of variation needs to be taken into account.

Recently, a new robust stability criterion was proposed for discrete linear systems with polytopic time-invariant uncertainty [4]. This work was later extended to linear system with time-varying uncertainties [3]. A parameter-dependent Lyapunov function can be obtained by solving a group of LMIs. The advantage of such approaches is that the difficulties introduced by the time-varying parameter need not be considered.

In this paper, motivated by the idea in [3], the control of a polytopic discrete LPV system is studied. LMI based solutions are given for the stabilization and regulator problems. The feedback gain for each vertex is computed by solving LMIs. The global gain is thus a convex combination of local ones. By applying LMIs, a parameter-varying Lyapunov function is obtained. It is shown that in the stabilization case, the feasible domain from the parameter-varying Lyapunov function is much larger than the one from quadratic stabilization. Further the performance is improved for the regulator problem.

A brief introduction of polytopic LPV system is first given in Section 2. In Sections 3 and 4, stabilization and regulator problems are discussed and the corresponding algorithms are given in the form of LMIs. The common and parameter-varying P approaches are compared in Section 5 by computing the feasibility regions. A simulation example for the regulator problem is given in Section 6. The conclusion is drawn in Section 7.

2 Preliminaries

Consider the LPV system described as

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) \\ y(k) &= C(k)x(k) \end{aligned} \quad (1)$$

where $x(k) \in R^{n_x}$ is the system state, $u(k) \in R^{n_u}$ is the control. The matrices $A(k), B(k)$ and $C(k)$ are time-varying and satisfy that

$$\begin{aligned} &[A(k) \mid B(k) \mid C(k)] \in \\ &\mathbf{Co}\{[A_1 \mid B_1 \mid C_1], [A_2 \mid B_2 \mid C_2], ..., [A_l \mid B_l \mid C_l]\} \end{aligned} \quad (2)$$

i.e., at any time instant k, they can be described by

$$[A(k) \mid B(k) \mid C(k)] = \sum_{i=1}^{l} \mu_i(k) [A_i \mid B_i \mid C_i] \quad (3)$$

and the parameter $\mu(k)^T = [\mu_1(k), \mu_2(k), ...\mu_l(k)]$ satisfies that

$$\begin{aligned} &\mu_i(k) > 0 \ \ \forall i = 1, 2, ..., l \\ &\sum_{i=1}^{l} \mu_i(k) = 1 \end{aligned} \quad (4)$$

System (1) with the condition (4) is referred as a polytopic LPV plant. In some papers, the triple $[A_i \mid B_i \mid C_i]$ is regarded as the ith local model. At any time instant k, the resulting system $[A(k) \mid B(k) \mid C(k)]$ is a convex combination of local models.

Equ. (1) also describes one type of Tagaki-Sugeno (T-S) fuzzy model [11]. In this case the parameter $\mu(k)$ is called normalized membership function, while in gain scheduled control it is referred to as the scheduling function. In the multiple modelling literature, $\mu(k)$ is called the weighting function.

This paper seeks the local state feedback gain F_i for each local model $[A_i \mid B_i \mid C_i]$ such that the controller $u(k) = F(k)x(k)$ is a solution for both the stabilization and regulator problems, where

$$F(k) = \sum_{i=1}^{l} \mu_i(k) F_i \quad (5)$$

Note that the weight $\mu_i(k)$ in equ. (5) is the same as in equ. (3).

3 Stabilization problem

To stabilize the LPV system (1), a feedback control $u(k) = F(k)x(k)$ must be found with the gain given in (5), such that the closed-loop system is stable. Applying this control to the LPV system produces a closed-loop system given by

$$\begin{aligned} x(k+1) &= (A(k) + B(k)F(k))x(k) \\ &= \sum_{i=1}^{l}\sum_{j=1}^{l} \mu_i(k)\mu_j(k) M_{i,j} x(k) \end{aligned} \quad (6)$$

where the matrix $M_{i,j}$ is defined as

$$M_{i,j} = \frac{1}{2}(A_i + B_i F_j + A_j + B_j F_i) \quad (7)$$

Thus, it is worth studying the stability of the system in the following form,

$$x(k+1) = \sum_{i=1}^{l}\sum_{j=1}^{l} \mu_i(k)\mu_j(k) M_{i,j} x(k) \quad (8)$$

Lemma 1 *The system (8) is stable if there exist $Q_{i,j} > 0$ with $i, j = 1, 2, ..., l$ and matrices $G_{i,j}$ with appropriate dimension such that*

$$\begin{bmatrix} G_{i,j} + G_{i,j}^T - Q_{i,j} & * \\ M_{i,j} G_{i,j} & Q_{m,n} \end{bmatrix} > 0 \quad (9)$$
$$\forall i, j, m, n = 1, 2, ..., l$$

Proof. See Appendix A. ■

Taking $G_{i,j} = G$ to be a constant matrix, a simplified criterion is obtained.

Corollary 2 *The system (8) is stable if there exist $Q_{i,j} > 0$ with $i, j = 1, 2, ..., l$ and matrix G with appropriate dimension such that*

$$\begin{bmatrix} G + G^T - Q_{i,j} & * \\ M_{i,j} & Q_{m,n} \end{bmatrix} > 0 \quad (10)$$
$$\forall i, j, m, n = 1, 2, ..., l$$

Obviously, this corollary is stricter than Lemma 1. From (17), it can be seen that in the proof of Lemma 1, $G_{i,j} + G_{i,j}^T - Q_{i,j}$ is used to approximate $G_{i,j}^T Q_{i,j}^{-1} G_{i,j}$, while for Corollary 2, it is required that $G + G^T - Q_{i,j} \le G^T Q_{i,j}^{-1} G$. Thus for a system in the form of (8), it may satisfy the condition in Lemma 1 and may not satisfy Corollary 2.

Now returning to the stabilization problem for system (1) and applying Corollary 2 with $M_{i,j}$ defined in (7) gives

Theorem 3 *System (1) is stabilized by state feedback controller $u(k) = F(k)x(k)$ if there exist matrices $G \in R^{n_x \times n_x}$, $Q_{i,j} > 0, i = 1, 2, ..., l, j \ge i$ and matrix $H_j \in R^{n_u \times n_x}$ such that*

$$\begin{bmatrix} G + G^T - Q_{i,j} & * \\ N_{i,j} & Q_{m,n} \end{bmatrix} > 0 \quad (11)$$
$$\forall i, m = 1, 2, ..., l$$
$$j \ge i, n \ge m$$

where $N_{i,j} = \frac{1}{2}[(A_i + A_j)G + B_i H_j + B_j H_i]$. The local feedback gains F_j are given by $F_j = H_j G^{-1}$.

Note that Corollary 2 is used to construct the stable controller instead of the more relaxed Lemma 1. Since,

$$N_{i,j} = \frac{1}{2}[(A_i + A_j)G_{i,j} + (B_i F_j + B_j F_i)G_{i,j}]$$

the terms $F_j G_{i,j}$ and $F_i G_{i,j}$ cannot be converted into LMI form and applying Lemma 1 will then result in a non-LMI problem.

The common P approach [13] can be seen to be a special case of Theorem 3. When $Q_{i,j} = Q_{m,n} = Q$ and the matrix G in (11) is chosen as $G = Q$, the LMIs (11) will be the same as in the common P approach. The quadratic Lyapunov function is then $V(x) = x^T Q^{-1} x$.

It can be concluded that if a common P matrix can be found for a polytopic LPV system, the LMI problem in (11) is feasible as well, which implies that the feasibility region of problem (11) is larger than that from the common P approach.

4 Regulator problem

The regulator problem is to find a state feedback controller such that the following objective function is minimized,

$$J_\infty = \sum_{i=0}^{\infty} \left(\|y(k)\|_{Q_1}^2 + \|u(k)\|_R^2 \right) \tag{12}$$

where $Q_1 \geq 0$ and $R > 0$ are suitable weight matrices.

Suppose that the Lyapunov function for the closed-loop system $V(x,k) = x^T P(k) x$ with $P(k) > 0$ and the control u satisfies the following inequality

$$\begin{aligned} V(x(k+1), k+1) - V(x(k), k) \quad &< \\ -\left(\|y(k)\|_{Q_1}^2 + \|u(k)\|_R^2 \right)& \end{aligned} \tag{13}$$

Summing (13) from $k = 0$ to ∞, because $y(\infty) = 0$, it follows that

$$V(x(0), 0) > J_\infty \tag{14}$$

Thus, the regulator problem now is casted to finding an upper bound $V(x(0), 0)$ for the objective function in equ. (12).

Theorem 4 *The upper bound for the objective function can be obtained by solving the following LMI optimization problem*

$$\min_{Q_{i,j}, G, Y_i} \gamma$$

subject to

$$\begin{bmatrix} 1 & * \\ x(0) & Q_{i,j} \end{bmatrix} > 0 \tag{15}$$

and

$$\begin{bmatrix} G + G^T - Q_{i,j} & * & * & * \\ N_{i,j} & Q_{m,n} & * & * \\ 0.5(C_i + C_j) Q_1^{1/2} G & 0 & \gamma I & * \\ 0.5 R^{1/2}(Y_i + Y_j) & 0 & 0 & \gamma I \end{bmatrix} > 0 \tag{16}$$
$$\forall i, m = 1, 2, ..., l,\ j \geq i, m \geq n$$

The local feedback gains are $F_i = Y_i G^{-1}$ and global gain is as given in equ. (5).

Proof. See Appendix B. ■

In Theorem 4, a group of local feedback gains is computed so that $V(x(0), 0)$ is minimized. Note that $V(x(0), 0)$ is an upper bound of the objective function. Actually the performance objective is not strictly minimized. Thus, Theorem 4 only gives a sub-optimal solution for the regulator problem.

In [14] and [12], two LMI-based algorithms for continuous time LPV systems are given. Similarly, both papers try to minimize the upper bound of $V(x(0), 0)$ by relaxing the stability condition. However, they employ quadratic Lyapunov functions and thus are more conservative than the algorithm in Theorem 4 above.

5 Example 1

Consider the LPV system containing two local models. The matrix coefficients are given as

$$\begin{aligned} A(1) &= \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -0.5 & 0.5 & 1 & 0 \\ 0.05 & -0.05 & 0 & 1 \end{bmatrix} \quad B(1) = \begin{bmatrix} 0 \\ 0 \\ 0.01 \\ 0 \end{bmatrix} \\ A(2) &= \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -0.1\frac{a}{b} & 0.1\frac{a}{b} & 1 & 0 \\ 0.1a & -0.1a & 0 & 1 \end{bmatrix} \quad B(2) = \begin{bmatrix} 0 \\ 0 \\ 0.1b \\ 0 \end{bmatrix} \\ C(1) &= C(2) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \end{aligned}$$

where $a \in [0, 100]$, $b \in (0, 5]$ are two time-varying parameters. The common P approach and the algorithm proposed in Theorem 3 are applied to the stabilization problems, respectively, with the feasibility regions shown in Fig. 1.

The feasibility region from the new algorithm is clearly much larger than the one produced by the common P approach.

6 Example 2

Consider the regulator problem of the system in example 1 with the weight matrices in the objective function given by

$$Q_1 = 1, R = 1$$

Suppose that $a = 30, b = 1$ in the second local model and that the initial condition is $[1, 1, 0, 0]$. The weight $\mu(k)$ varies according to the system output,

$$\begin{aligned} \mu_1(y(k)) &= \exp\left[-3(y-1)^2\right] \\ \mu_2(k) &= 1 - \mu_1(k) \end{aligned}$$

The control objective is to regulate the system from the initial condition to the origin. By applying the new design approach of this paper and the common P approach, a set of simulation results was obtained.

Fig. 3 shows that the control provided by the parameter-dependent design is initially more vigorous than the one from

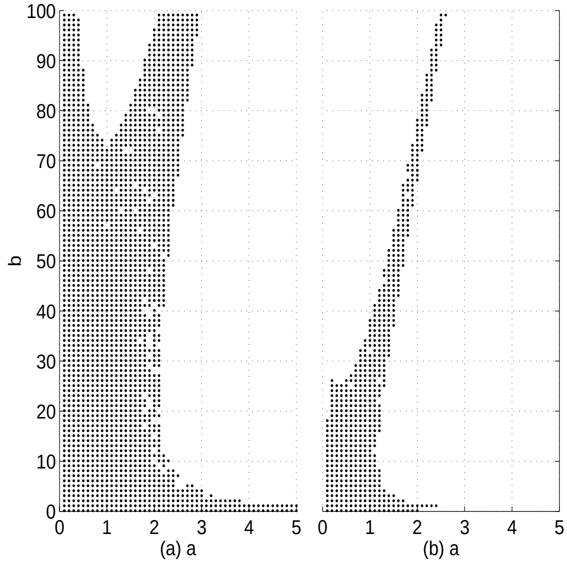

Figure 1: Feasible regions under different designs. (a) parameter-varying Lyapunov function. (b) quadratic Lyapunov function.

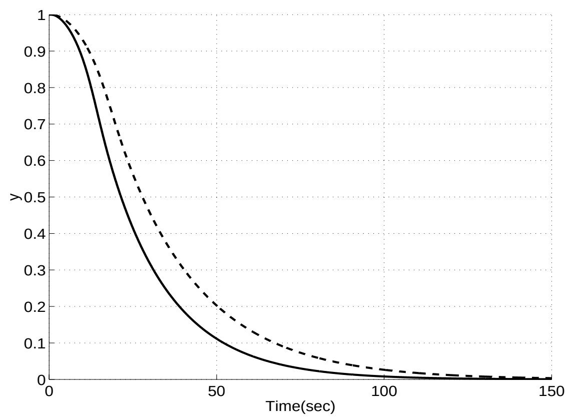

Figure 2: Trajectories of system output. The solid line is from parameter-varying Lyapunov function. The dashed line is from the common P approach.

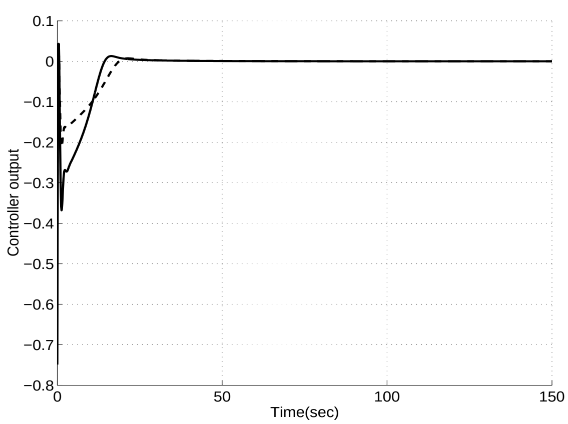

Figure 3: Trajectories of controller output. The solid line is from parameter-dependent Lyapunov function based design. The dashed line is based on common P approach.

the common P approach. Therefore, from Fig. 2, the output resulting from the parameter-varying based design settles faster than that from the common P approach.

7 Conclusion

A criterion for the stability of closed-loop LPV systems is given, with which a parameter-varying Lyapunov function can be found. Based on this result, stabilization and regulator problems for polytopic LPV systems are studied, and two LMI-based algorithms are derived. It is shown that the common P approach used in the literature is a special case and thus more conservative than the approaches proposed here. Simulation examples confirm that the feasibility region is enlarged for the stabilization problem and that better performance is achieved for the regulator problem.

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A Proof of Lemma 1

Proof. It can be easily seen that

$$G_{i,j} + G_{i,j}^T - Q_{i,j} > 0$$

which implies that $G_{i,j}$ is non-singular. Because $Q_{i,j} > 0$, it follows that

$$\left(Q_{i,j} - G_{i,j}\right)^T Q_{i,j}^{-1} \left(Q_{i,j} - G_{i,j}\right) \geq 0$$

which is equivalent to

$$G_{i,j}^T Q_{i,j}^{-1} G_{i,j} \geq G_{i,j} + G_{i,j}^T - Q_{i,j} \tag{17}$$

Thus the inequality (9) implies that

$$\begin{bmatrix} G_{i,j}^T Q_{i,j}^{-1} G_{i,j} & * \\ M_{i,j} G_{i,j} & Q_{m,n} \end{bmatrix} > 0 \tag{18}$$

Multiplying the inequality (18) from the left by $diag\left(G_{i,j}^{-T}, Q_{m,n}^{-1}\right)$ and from the right by $diag\left(G_{i,j}^{-1}, Q_{m,n}^{-1}\right)$, gives

$$\begin{bmatrix} Q_{i,j}^{-1} & * \\ Q_{m,n}^{-1} M_{i,j} & Q_{m,n}^{-1} \end{bmatrix} > 0 \tag{19}$$

Defining $P_{i,j} = Q_{i,j}^{-1}$, the inequality (19) can be written as

$$\begin{bmatrix} P_{i,j} & * \\ P_{m,n} M_{i,j} & P_{m,n} \end{bmatrix} > 0 \tag{20}$$

and consequently it follows that

$$\sum_{m=1}^{l} \sum_{n=1}^{l} \sum_{i=1}^{l} \sum_{j=1}^{l} \mu_m (k+1) \mu_n (k+1) \times \tag{21}$$

$$\mu_i (k) \mu_j (k) \begin{bmatrix} P_{i,j} & * \\ P_{m,n} M_{i,j} & P_{m,n} \end{bmatrix} > 0$$

Letting

$$M(k) = \sum_{i=1}^{l}\sum_{j=1}^{l}\mu_i(k)\mu_j(k)M_{i,j} \quad (22)$$
$$P(k) = \sum_{i=1}^{l}\sum_{j=1}^{l}\mu_i(k)\mu_j(k)P_{i,j}$$
$$P(k+1) = \sum_{m=1}^{l}\sum_{n=1}^{l}\mu_m(k+1)\mu_n(k+1)P_{m,n}$$

inequality (21) is now written as

$$\begin{bmatrix} P(k) & * \\ P(k+1)M(k) & P(k+1) \end{bmatrix} > 0$$

From the Schur complement, this is then equivalent to

$$M(k)^T P(k+1) M(k) - P(k) < 0 \quad (23)$$

Finally, taking

$$V(x(k), \mu(k)) = x^T(k) P(k) x(k)$$

as the Lyapunov function, inequality (23) confirms that the system (8) is stable as required. ■

B Proof of Theorem 4

Proof. First, letting $\gamma = V(x(0), 0)$, the optimization problem is reformulated as

$$\min_{P(0),\gamma} \gamma$$

subject to

$$\gamma = x(0)^T P(0) x(0) \quad (24)$$

Letting that $P(k) = \gamma Q(k)^{-1}$ and using the Schur complement, the optimization problem is equivalent to

$$\min_{Q(0),\gamma} \gamma$$

subject to

$$\begin{bmatrix} 1 & * \\ x(0) & Q(0) \end{bmatrix} > 0 \quad (25)$$

Defining $Q(k) = \sum_{i=1}^{l}\sum_{j=1}^{l}\mu_i(k)\mu_j(k)Q_{i,j}$, it can be seen that the left side of inequality (25) is the convex combination of the left side of inequalities (15). Inequalities (15) are thus established.

It is then proved that condition (13) is guaranteed by inequalities (16). Condition (13) can be written as

$$x(k+1)^T P(k+1) x(k+1) - x(k)^T P(k) x(k) + x(k)^T C(k)^T \times Q_1 C(k) x(k) + u(k)^T R u(k) < 0$$

Recalling the system equation (1) and that the control $u(k) = F(k)x(k)$, it follows that

$$x(k)^T \{[A(k) + B(k)F(k)]^T P(k+1) \times [A(k) + B(k)F(k)] - P(k) + C(k)^T Q_1 C(k) + F(k)^T R F(k)\} x(k) < 0$$

This is satisfied for all $x(k)$ if and only if

$$[A(k) + B(k)F(k)]^T P(k+1) \times [A(k) + B(k)F(k)] - P(k) + C(k)^T Q_1 C(k) + F(k)^T R F(k) < 0 \quad (26)$$

Since $P(k) = \gamma Q(k)^{-1}$, it is follows from the Schur complement that equ. (26) is equivalent to

$$\begin{bmatrix} Q(k) & * & * & * \\ [A(k)+B(k)F(k)]Q(k) & Q(k+1) & * & * \\ C(k)Q_1^{1/2}Q(k) & 0 & \gamma I & * \\ R^{1/2}F(k)Q(k) & 0 & 0 & \gamma I \end{bmatrix} > 0 \quad (27)$$

Since inequalities (16) hold, in a similar manner to the proof of Theorem 3, matrix G is non-singular. By multiplying (27) from the left by $diag\left(G^T Q(k)^{-1}, I, I, I\right)$ and from the right by $diag\left(Q(k)^{-1} G, I, I, I\right)$, inequality (27) is converted to,

$$\begin{bmatrix} G^T Q(k)^{-1} G & * & * & * \\ [A(k)+B(k)F(k)]G & Q(k+1) & * & * \\ C(k)Q_1^{1/2}G & 0 & \gamma I & * \\ R^{1/2}F(k)G & 0 & 0 & \gamma I \end{bmatrix} > 0 \quad (28)$$

Recalling that

$$Q(k) = \sum_{i=1}^{l}\sum_{j=1}^{l}\mu_i(k)\mu_j(k)Q_{i,j} > 0$$

since $(Q(k) - G)^T Q(k)^{-1} (Q(k) - G) \geq 0$, it follows that

$$G^T Q(k)^{-1} G \geq G + G^T - Q(k)$$

Thus, inequality (28) holds if

$$\begin{bmatrix} G + G^T - Q(k) & * & * & * \\ [A(k)+B(k)F(k)]G & Q(k+1) & * & * \\ C(k)Q_1^{1/2}G & 0 & \gamma I & * \\ R^{1/2}F(k)G & 0 & 0 & \gamma I \end{bmatrix} > 0 \quad (29)$$

Since $Y_i = F_i G$, the left side of inequality (29) is observed to be a convex combination of the left side of (16), i.e.,

$$T(k) = \sum_{m=1}^{l}\sum_{n=1}^{l}\sum_{i=1}^{l}\sum_{j=1}^{l}\mu_m(k+1) \times \mu_n(k+1)\mu_i(k)\mu_j(k)T_{i,j,m,n}$$

where $T_{i,j,m,n}$ and $T(k)$ are the left sides of inequality (16) and (29), respectively. The Theorem is therefore proved as required. ■

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