STABILITY ANALYSIS AND SYNTHESIS FOR SLOWLY TIME VARYING SYSTEMS BASED ON NON-COMMON LYAPUNOV MATRICES

Gan CHEN†, Hiroshi SHIBATA∗ and Toru FUJINAKA†

† Graduate School of Engineering, University of Osaka Prefecture
Sakai, Osaka 599-8531 JAPAN
E-mail: kan@cs.osakafu-u.ac.jp, fuji@cs.osakafu-u.ac.jp
∗ Dept. of Mechanical Engineering, Doshisha University
Kyotanabe, Kyoto 610-0321 JAPAN
E-mail: hshibata@mail.doshisha.ac.jp

Keywords: non-common Lyapunov matrices, linear matrix inequalities, redundant descriptor form, slowly time-varying systems, real μ synthesis.

Abstract

Recently, new stability analysis and controller synthesis methods based on non-common Lyapunov matrices are proposed. In this paper, we give another simple and easy proof for them using redundant descriptor form. Furthermore, we propose a sufficient condition for stability of slowly time-varying systems that have real rational uncertainties using the technique in the proof. The proposed condition is not more conservative than former methods and the quadratic stability.

1 Introduction

Recently, with the development of effective computational method for linear matrix inequality (LMI) conditions, a lot of analysis and synthesis methods are proposed in LMI's. LMI's can describe difficult problems such as multi objective synthesis, synthesis for parameter dependent systems and so on. However, there is a problem called "common Lyapunov matrix" in these synthesis methods based on LMI's. For example, the Lyapunov matrix for H_∞ performance analysis of a system is not equivalent to that for H_2 analysis of the same system. It is clear that we should prepare two Lyapunov matrices for less conservative synthesis of the H_∞/H_2 mixed problem. However, it is difficult to choose Lyapunov matrices independently for the mixed synthesis based on LMI's, because the change of variable is required for multi objective synthesis in many cases. This means that there is a gap of conservativeness between analysis and synthesis.

For this problem, Oliveira et al.[1, 2] proposed new analysis and synthesis methods using non-common Lyapunov matrices for discrete time systems. For continuous time systems, Peaucelle et al.[3], Apkarian et al.[4], Shimomura et al.[6], Ebihara et al.[5] proposed new methods, independently. These results are applicable to multi objective synthesis and give less conservative controllers.

In this paper, we show another simple and easy proof for these results using descriptor form. Furthermore, we give analysis and synthesis method for slowly time varying systems that have real rational uncertainties, based on the technique used in our proof.

The notation is standard. The notation $\mathrm{He}\{M\}$ stands for $M+M^T$, $\mathrm{diag}\{M_1, M_2, \ldots, M_m\}$ is the block diagonal matrix of M_1, M_2, …, M_m. $\|G(s)\|_\infty$ is H_∞ norm of $G(s)$. $\mu_U(G)$ stands for the structured singular value of a matrix G for given class of uncertainty U.

2 Former results

The followings are the result of Ebihara et al. and Shimomura et al., respectively.

Lemma 1 *[5] For a continuous linear dynamical system;*

$$\dot{x}(t) = Ax(t) \tag{1}$$

the following statements are equivalent.

1. *The system (1) is stable.*
2. *(Lyapunov stability) There exists $X > 0$ such that $AX + XA^T < 0$ holds.*
3. *There exist $X > 0$, F_1 and F_2 such that the following matrix inequality holds.*

$$\begin{bmatrix} X & -X \\ -X & 0 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} A - \frac{1}{2}I \\ I \end{bmatrix} \begin{bmatrix} F_1 & F_2 \end{bmatrix} \right\} < 0 \tag{2}$$

4. *There exist $X > 0$ and G such that the following matrix inequality holds.*

$$\begin{bmatrix} X & -X \\ -X & 0 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} A - \frac{1}{2}I \\ I \end{bmatrix} G \begin{bmatrix} I & -I \end{bmatrix} \right\} < 0 \tag{3}$$

Lemma 2 *[6] The system (1) is stable if and only if there exist $\varepsilon > 0$, $X > 0$ and V such that the following matrix inequality holds.*

$$\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} + \mathrm{He}\{\begin{bmatrix} A \\ I \end{bmatrix} V \begin{bmatrix} I & \varepsilon I \end{bmatrix}\} < 0 \tag{4}$$

The goal of this paper is to give another simple and easy proof of these results using redundant descriptor form.

3 Another proof based on descriptor form

In this section, we give another simple and easy proof for the former results (Lemma 1, 2) using descriptor form.

3.1 Stability condition

Theorem 1 *The system (1) is stable if and only if there exist $X_{11} > 0$, X_{21}, X_{22}, α and $\beta \neq 0$ such that the following matrix inequality holds.*

$$\begin{bmatrix} 2\alpha X_{11} & \beta X_{11} \\ \beta X_{11} & 0 \end{bmatrix} + \mathrm{He}\{\begin{bmatrix} A-\alpha I \\ -\beta I \end{bmatrix} \begin{bmatrix} X_{21} & X_{22} \end{bmatrix}\} < 0 \tag{5}$$

Proof (Sufficiency) *Let $x_2(t)$ be $x_2(t) = x(t)$, we have the following descriptor system that is equivalent to the system (1).*

$$\hat{E}\begin{bmatrix} \dot{x}(t) \\ \dot{x}_2(t) \end{bmatrix} = \hat{A}\begin{bmatrix} x(t) \\ x_2(t) \end{bmatrix} \tag{6}$$

$$\hat{E} := \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \hat{A} := \begin{bmatrix} \alpha I & A-\alpha I \\ \beta I & -\beta I \end{bmatrix} \tag{7}$$

A descriptor system (6) is stable if there exists X such that the following matrix inequality holds [9].

$$\mathrm{He}\{\hat{A}X\} < 0 \tag{8}$$
$$\hat{E}X = (\hat{E}X) \geq 0 \tag{9}$$

Here, $(1,2)$-block of X must be 0 because of the structure of $\hat{E}$ and Eq.(9). Then we have the following X;

$$X := \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{bmatrix}, X_{11} > 0 \tag{10}$$

Substituting $\hat{A}$ and Eq. (10) for the matrix inequality (8), we obtain the matrix inequality (5).

[Necessity] There exist $X_{11} > 0$, α, β and minute matrix[1], $X_{22} = X_{22}^T$, (such that $\beta X_{22} > 0$ holds) such that the following matrix inequality holds, if the system (1) is stable.

$$\mathrm{He}\{AX_{11}\} + (A-\alpha I)\frac{1}{2\beta}X_{22}(A-\alpha I)^T < 0 \tag{11}$$

[1] In this paper, "minute matrix" implies that the matrix whose spectral radius is minute.

Using Schur complement, we have the following matrix inequality.

$$\begin{bmatrix} 2\alpha X_{11} + \mathrm{He}\{(A-\alpha I)X_{11}\} & (A-\alpha I)X_{22} \\ X_{22}(A-\alpha I)^T & -2\beta X_{22} \end{bmatrix} < 0 \tag{12}$$

Letting $X_{21} = X_{11}$, we have the following inequality;

$$\begin{bmatrix} 2\alpha X_{11} + \mathrm{He}\{(A-\alpha I)X_{11}\} & (A-\alpha I)X_{22} + \beta(X_{11}-X_{21})^T \\ X_{22}(A-\alpha I)^T + \beta(X_{11}-X_{21}) & -2\beta X_{22} \end{bmatrix} < 0 \tag{13}$$

It is equivalent to the matrix inequality (5). □

Theorem 1 is equivalent to the result of Ebihara et al.[5] when $\alpha = \frac{1}{2}$ and $\beta = -1$, and is equivalent to the result of Shimomura et al.[6], when $\alpha = 0$, $\beta = 1$, $X_{21} = -V$ and $X_{22} = \varepsilon X_{21} = -\varepsilon V$, ($\varepsilon$ is a minute positive scalar). Note that we can take α and β as specified values without loss of generality. Eq.(5) is also necessary and sufficient condition for stability when we fix α and β to be certain values. It implies that Eq.(5) is LMI condition and computable.

3.2 State feedback synthesis

Now, we consider the following system;

$$\dot{x}(t) = Ax(t) + B_u u(t) \tag{14}$$
$$u(t) = Kx(t) \tag{15}$$

Letting $X_{22} = \varepsilon X_{21}$, ($\varepsilon$ is a minute positive scalar) in Theorem 1, it enables us to synthesize state feedback gain in the same way as the former methods.

Theorem 2 *There exists stabilizing state feedback gain K, if there exist $X_{11} > 0$, X_{21}, Y, α, $\beta \neq 0$ and ε such that the following matrix inequality holds;*

$$\mathrm{He}\{\begin{bmatrix} A-\alpha I \\ -\beta I \end{bmatrix}\begin{bmatrix} X_{21} & \varepsilon X_{21} \end{bmatrix} + \begin{bmatrix} B_u \\ 0 \end{bmatrix}\begin{bmatrix} Y & \varepsilon Y \end{bmatrix}\} + \begin{bmatrix} 2\alpha X_{11} & \beta X_{11} \\ \beta X_{11} & 0 \end{bmatrix} < 0 \tag{16}$$

When there exist the above variables such that (16) holds, the stabilizing state feedback gain K is given by $K := YX_{21}^{-1}$.

As we have shown in the proof of the necessity of Theorem 1, X_{22} can be taken to be a minute matrix without loss of generality and this fact does not make the gap of conservativeness between analysis condition and synthesis condition. The parameters α and β can be taken to be certain values without loss of generality. However, the optimal value of ε is unknown in general. This implies that the synthesis method requires line search as in the former methods (When ε is fixed, Eq. (16) becomes LMI condition). Note that this problem demands a lot of computation.

3.3 H_∞ norm condition

We can also explain the former results for the performance problem using descriptor form. Now, we consider the H_∞ norm condition for the following system $G(s) := C(sI-A)^{-1}B+D$.

$$\dot{x} = Ax + Bw \quad (17)$$
$$z = Cx + Dw \quad (18)$$

We have the following descriptor system that is equivalent to the system (17), (18), whose coefficient matrices are defined in the same way as in the proof of Theorem 1.

$$\hat{E}\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}w \quad (19)$$
$$z = \hat{C}\hat{x} + Dw \quad (20)$$
$$\hat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix},\ \hat{C} = \begin{bmatrix} 0 & C \end{bmatrix} \quad (21)$$

For the descriptor system, we apply the following H_∞ norm condition.

Lemma 3 *[9] For a descriptor system $\hat{G}(s) := \hat{C}(s\hat{E}-\hat{A})\hat{B}+D$, $\|\hat{G}(s)\|_\infty < 1$ holds if there exists X such that*

$$\begin{bmatrix} \mathrm{He}\{\hat{A}X\} & \hat{B} & (\hat{C}X)^T \\ \hat{B}^T & -I & D^T \\ \hat{C}X & D & -I \end{bmatrix} < 0,\ \hat{E}X = (\hat{E}X)^T \geq 0 \quad (22)$$

holds.

Substituting the $\hat{A}$, $\hat{B}$, $\hat{C}$, D and $\hat{E}$ for Eq. (22), we obtain the following condition that is equivalent to the results of Ebihara et al. and Shimomura et al.

Theorem 3 *$\|G(s)\|_\infty < 1$ holds if there exist $X_{11} > 0$, X_{21} and X_{22} such that*

$$\mathrm{He}\{\begin{bmatrix} A-\alpha I \\ -\beta I \\ 0 \\ C \end{bmatrix} \begin{bmatrix} X_{21} & X_{22} & 0 & 0 \end{bmatrix}\} + \begin{bmatrix} 2\alpha X_{11} & \beta X_{11} & B & 0 \\ \beta X_{11} & 0 & 0 & 0 \\ B^T & 0 & -I & D^T \\ 0 & 0 & D & -I \end{bmatrix} < 0 \quad (23)$$

holds.

For other performance conditions (e.g., H_2 norm condition), we can also obtain non-common Lyapunov type conditions, by using the descriptor system (19)–(21) and substituting them for the performance conditions for descriptor form in LMI term.

4 Analysis and synthesis for slowly time varying systems

In this section, we consider the stability analysis for the following slowly time varying linear dynamical system.

$$\dot{x}(t) = Ax(t) + Bw(t) \quad (24)$$
$$z(t) = Cx(t) + Dw(t) \quad (25)$$
$$w(t) = \Delta(t)z(t),\ \Delta(t) \in \mathscr{U} \quad (26)$$
$$\mathscr{U} := \{\mathrm{diag}\{r_1(t)I, r_2(t)I, \cdots, r_N(t)I\} \mid |r_i(t)| \leq 1,\ \left|\frac{d}{dt}r_i(t)\right| \leq \bar{v}_i\} \quad (27)$$

Note that this system has real rational uncertainties. It is difficult to deal the uncertainties without conservativeness. When $\bar{v}_i = 0$, the problem is real μ analysis. It is well known that there is no effective method to calculate the exact value of μ for the system.

Theorem 4 *The system (24)–(27) is stable if there exist $X_{11}(r) > 0$, $X_{ij}(r)$, $(r := [r_1(t), r_2(t), \cdots, r_N(t)])$, α and $\beta \neq 0$ such that the following matrix inequality holds.*

$$\mathrm{He}\{\begin{bmatrix} A-\alpha I & B\Delta \\ -\beta I & 0 \\ C & -I+D\Delta \end{bmatrix} \begin{bmatrix} X_{21}(r)^T & X_{31}(r)^T \\ X_{22}(r)^T & X_{32}(r)^T \\ X_{23}(r)^T & X_{33}(r)^T \end{bmatrix}^T\} + \begin{bmatrix} 2\alpha X_{11}(r) - \frac{d}{dt}X_{11}(r) & \beta X_{11}(r) & 0 \\ \beta X_{11}(r) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} < 0 \quad (28)$$

Proof *We can obtain the following equivalent descriptor system using the same way as in the previous section.*

$$\tilde{E}\begin{bmatrix} \dot{x}(t) \\ \dot{x}_2(t) \\ \dot{z}(t) \end{bmatrix} = \tilde{A}\begin{bmatrix} x(t) \\ x_2(t) \\ z(t) \end{bmatrix} \quad (29)$$

$$\tilde{E} := \mathrm{diag}\{\hat{E}, 0\},\ \tilde{A}(t) := \begin{bmatrix} \hat{A} & \begin{bmatrix} B\Delta(t) \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & C \end{bmatrix} & -I+D\Delta(t) \end{bmatrix} \quad (30)$$

A time-varying descriptor system $\tilde{E}\dot{\tilde{x}} = \tilde{A}\tilde{x}$ is stable if there exists $\tilde{X}(t)$ such that the following matrix inequality holds.

$$\mathrm{He}\{\tilde{A}(t)\tilde{X}(t)\} - \frac{d}{dt}\{\tilde{E}\tilde{X}(t)\} < 0,\ \tilde{E}\tilde{X}(t) = (\tilde{E}\tilde{X}(t))^T \geq 0 \quad (31)$$

The same way as the previous section, we have the following $\tilde{X}(t)$.

$$\tilde{X}(t) := \begin{bmatrix} X_{11}(t) & 0 & 0 \\ X_{21}(t) & X_{22}(t) & X_{23}(t) \\ X_{31}(t) & X_{32}(t) & X_{33}(t) \end{bmatrix},\ X_{11}(t) > 0 \quad (32)$$

Substituting Eq. (30) and Eq. (32) for Eq. (31) and replacing $X_{ij}(t)$ with $X_{ij}(r)$, we obtain the matrix inequality (28). □

Using the descriptor system (29), (30) and Eq. (19)–(21), we can obtain the L_2 gain (sufficient) condition for the system with real rational uncertainty.

Note that the condition (28) contains not the rational terms of uncertainties $\Delta(t)$ but affine terms of $\Delta(t)$. It is easier to deal the affine uncertainties than the rational ones. Actually, with the restriction for the variables as in [7, 8], the condition becomes convex and we can obtain computable sufficient conditions (it is enough to check LMI conditions at all vertices of Δ and $\overline{v_i}$). Furthermore, if $D=0$, we can prove that the proposed method is not more conservative than the quadratic stability.

Theorem 5 *There exists a time invariant matrix $\tilde{X}$ such that (28) holds, if the system (24)–(27) with $D=0$ is quadratically stable (i.e. stable for $\overline{v_i}=+\infty$).*

Proof *There exist $X_{11}>0$, α, β, minute matrices $X_{22}=X_{22}^T$, ($\beta X_{22}>0$) and $X_{33}>0$ such that the following matrix inequality holds, when the system (24)–(27) with $D=0$ is quadratically stable.*

$$(A-\alpha I+B\Delta C)\frac{1}{2\beta}X_{22}(A-\alpha I+B\Delta C)^T \\ +B\Delta\frac{1}{2}X_{33}\Delta^T B^T+\mathrm{He}\{(A+B\Delta C)X_{11}\}<0 \quad (33)$$

Using Schur complement, we have the following matrix inequality.

$$\mathrm{He}\begin{bmatrix} (A-\alpha I+B\Delta C)X_{11} & 0 & B\Delta X_{33} \\ ((A-\alpha I+B\Delta C)X_{22})^T & -\beta X_{22} & 0 \\ 0 & 0 & -X_{33} \end{bmatrix} \\ +\begin{bmatrix} 2\alpha X_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}<0 \quad (34)$$

Letting X_{31} and X_{32} to be

$$X_{31}:=CX_{11},\ X_{32}:=CX_{22}, \quad (35)$$

we have the following matrix inequality.

$$\mathrm{He}\begin{bmatrix} (A-\alpha I)X_{11}+B\Delta X_{31} & -\beta X_{11} & B\Delta X_{33} \\ ((A-\alpha I)X_{22}+B\Delta X_{32})^T & -\beta X_{22} & 0 \\ CX_{11}-X_{31} & CX_{22}-X_{32} & -X_{33} \end{bmatrix} \\ +\begin{bmatrix} 2\alpha X_{11} & \beta X_{11} & 0 \\ \beta X_{11} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}<0 \quad (36)$$

It is equivalent to the following matrix inequality.

$$\mathrm{He}\{\begin{bmatrix} A-\alpha I & B\Delta \\ -\beta I & 0 \\ C & -I \end{bmatrix}\begin{bmatrix} X_{11} & X_{22} & 0 \\ X_{31} & X_{32} & X_{33} \end{bmatrix}\} \\ +\begin{bmatrix} 2\alpha X_{11} & \beta X_{11} & 0 \\ \beta X_{11} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}<0 \quad (37)$$

□

When $D=0$, it is also applicable (actually computable) for stability analysis of the slowly time varying systems using the former methods (Theorem 1, Lemma 1, 2) with matrix A substituted by $A+B\Delta C$.

Corollary 1 *The slowly time varying system (24)–(27) with $D=0$ is stable if there exists $X_{11}(r)>0$, $X_{21}(r)$ and $X_{22}(r)$ such that the following matrix inequality holds.*

$$\mathrm{He}\{\begin{bmatrix} A+B\Delta C-\alpha I \\ -\beta I \end{bmatrix}\begin{bmatrix} X_{21}(r) & X_{22}(r) \end{bmatrix}\} \\ +\begin{bmatrix} 2\alpha X_{11}(r)-\frac{d}{dt}X_{11}(r) & \beta X_{11}(r) \\ \beta X_{11}(r) & 0 \end{bmatrix}<0 \quad (38)$$

When $D=0$, the system has not rational uncertainties but only affine uncertainties. It is easy to deal the affine uncertainty and there are less conservative analysis methods. However, we can show that the proposed method is not more conservative than the methods based on Corollary 1.

Theorem 6 *For the slowly time varying system (24)–(25) with $D=0$, there is $\tilde{X}(r)$ such that (28) holds, if there exist $X_{11}(r)$, $X_{21}(r)$ and $X_{22}(r)$ such that the matrix inequality (38) holds.*

Proof *When the matrix inequality (38) holds, there exists $X_{11}(r)$, $X_{21}(r)$, $X_{22}(r)$ and $X_{33}(r)>0$ such that the following matrix inequality holds.*

$$\begin{bmatrix} 2\alpha X_{11}(r)-\frac{d}{dt}X_{11}(r)+B\Delta\frac{1}{2}X_{33}(r)\Delta^T B & \beta X_{11}(r) \\ \beta X_{11}(r) & 0 \end{bmatrix} \\ +\mathrm{He}\{\begin{bmatrix} A+B\Delta C-\alpha I \\ -\beta I \end{bmatrix}\begin{bmatrix} X_{21}(r) & X_{22}(r) \end{bmatrix}\}<0 \quad (39)$$

Using Schur complement, we have the following matrix inequality.

$$\begin{bmatrix} (*) & \begin{bmatrix} B\Delta X_{33}(r) \\ 0 \end{bmatrix} \\ \begin{bmatrix} X_{33}(r)\Delta^T B & 0 \end{bmatrix} & -2X_{33}(r) \end{bmatrix}<0 \quad (40)$$

Here, $(*)$ means the left hand side of the matrix inequality (38). Letting $X_{31}(r)$ and $X_{32}(r)$ to be

$$X_{31}(r):=CX_{21}(r),\ X_{32}(r):=CX_{22}(r), \quad (41)$$

we have the following matrix inequality.

$$\mathrm{He}\{\begin{bmatrix} A-\alpha I & B\Delta \\ -\beta I & 0 \\ C & -I \end{bmatrix}\begin{bmatrix} X_{21}(r)^T & X_{31}(r)^T \\ X_{22}(r)^T & X_{32}(r)^T \\ 0 & X_{33}(r)^T \end{bmatrix}^T\} \\ +\begin{bmatrix} 2\alpha X_{11}(r)-\frac{1}{2}X_{11}(r) & \beta X_{11}(r) & 0 \\ \beta X_{11}(r) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}<0 \quad (42)$$

□

Note that this result shows that the extension of the system dimension used in Theorem 4 does not increase the conservativeness.

Restricting $X_{21}(r)$, $X_{22}(r)$ and $X_{23}(r)$ as

$$\begin{bmatrix} X_{21}(r) & X_{22}(r) & X_{23}(r) \end{bmatrix}=\begin{bmatrix} X_{21} & \varepsilon_1 X_{21} & \varepsilon_2 X_{21} \end{bmatrix} \quad (43)$$

in Theorem 4 (Note that X_{21} is time invariant and nonsingular.), the proposed method can be used to synthesize state feedback gain (both constant matrix gain and gain scheduling). From the proof of Theorem 6, we may fix $\varepsilon_2 = 0$. This implies that the synthesis method becomes LMI and computable (however, it needs line search for ε_1).

5 State feedback μ-synthesis

In this section, we consider the static state feedback μ synthesis problem for the following closed loop system with $G_c(s) := (C+D_uK)(sI-(A+B_uK))^{-1}B+D$ and Δ.

$$\dot{x} = Ax+Bw+B_uu \tag{44}$$
$$z = Cx+Dw+D_uu \tag{45}$$
$$w = \Delta z \tag{46}$$
$$u = Kx \tag{47}$$
$$w = \Delta z,\ \Delta \in U \tag{48}$$
$$U := \{\mathrm{diag}\{r_1I, r_2I, \cdots, r_NI\} \mid |r_i| \le 1\} \tag{49}$$

For this problem, we proposed the following analysis and synthesis conditions.

Lemma 4 *(real μ analysis) [8] When K is given, $\mu_U(G_c(j\omega)) < 1$, $\forall\omega \in [0,\infty]$ holds if and only if there exist $X_{11}(r) > 0$, $X_{21}(r)$, $X_{22}(r)$ such that*

$$\mathrm{He}\{\begin{bmatrix} (A+B_uK)X_{11}(r) - B\Delta X_{21}(r) & -B\Delta X_{22}(r) \\ (C+D_uK)X_{11}(r) + (I-D\Delta)X_{21}(r) & (I-D\Delta)X_{22}(r) \end{bmatrix}\} < 0,\ \forall\Delta \in U \tag{50}$$

holds.

Lemma 5 *(real μ synthesis) [8] By using the state feedback $u = Kx$, $\mu_U(G_c(j\omega)) < 1, \forall\omega \in [0,\infty]$ holds, if there exist $X_{11} > 0$, $X_{21}(r)$, $X_{22}(r)$, W such that*

$$\mathrm{He}\{\begin{bmatrix} AX_{11} + B_uW - B\Delta X_{21}(r) & -B\Delta X_{22}(r) \\ CX_{11} + D_uW + (I-D\Delta)X_{21}(r) & (I-D\Delta)X_{22}(r) \end{bmatrix}\} < 0, \forall\Delta \in U \tag{51}$$

holds. If this condition is satisfied, the state feedback gain K which achieves $\mu_U(G_c(j\omega)) < 1$, $\forall\omega \in [0,\infty]$ is given by $K = WX_{11}^{-1}$.

Note that the analysis condition (Lemma 4) gives necessary and sufficient condition. However, synthesis condition (Lemma 5) is sufficient condition because we fixed the matrix X_{11} to reduce the condition to LMI using the change of variable, $W = KX_{11}$. This implies that the synthesis condition has some conservativeness.

It is known that the closed loop system with $G_c(s)$ and $\forall\Delta \in U$ is stable if and only if $\mu_U(G_c(j\omega)) < 1$, $\forall\omega \in [0,\infty]$ holds. Now, we propose that less conservative μ synthesis condition using redundant descriptor form.

Let descriptor variable $\hat{x} := [x^T, x^T, z^T]^T$, we have the following descriptor system that is equivalent to the closed loop system with $G_c(s)$ and Δ.

$$\mathrm{diag}\{1,0,0\} \begin{bmatrix} x \\ x \\ z \end{bmatrix} = \begin{bmatrix} 0 & A+B_uK & B\Delta \\ I & -I & 0 \\ 0 & C+D_uK & -I+D\Delta \end{bmatrix} \begin{bmatrix} x \\ x \\ z \end{bmatrix} \tag{52}$$

Using the stability condition for descriptor systems, we have the following stability condition.

$$\mathrm{He}\{\begin{bmatrix} 0 & A+B_uK & B\Delta \\ I & -I & 0 \\ 0 & C+D_uK & -I+\Delta \end{bmatrix} \begin{bmatrix} X_{11}(r) & 0 & 0 \\ X_{21}(r) & X_{22}(r) & X_{23}(r) \\ X_{31}(r) & X_{32}(r) & X_{33}(r) \end{bmatrix}\} < 0 \tag{53}$$

To reduce the condition to LMI, we restrict $X_{21}(r)$, $X_{22}(r)$ and $X_{23}(r)$ as follows,

$$\begin{bmatrix} X_{21}(r) & X_{22}(r) & X_{23}(r) \end{bmatrix} = \begin{bmatrix} V & \varepsilon_1 V & \varepsilon_2 V \end{bmatrix} \tag{54}$$

and use the change of variable; $Y := KV$. Then we have the following LMI condition (with plain search parameter ε_1 and ε_2).

Theorem 7 *By using the state feedback $u = Kx$, $\mu_U(G_c(j\omega)) < 1$, $\forall\omega \in [0,\infty]$ holds, if there exist $X_{11}(r) > 0$, $X_{3i}(r)$'s, V, ε_1 and ε_2 such that*

$$\mathrm{He}\{\begin{bmatrix} 0 & AV+B_uY & B\Delta \\ I & -V & 0 \\ 0 & CV+D_ur & -I+\Delta \end{bmatrix} \begin{bmatrix} X_{11}(r) & 0 & 0 \\ I & \varepsilon_1 I & \varepsilon_2 I \\ X_{31}(r) & X_{32}(r) & X_{33}(r) \end{bmatrix}\} < 0,\ \forall\Delta \in U \tag{55}$$

holds. If this condition is satisfied, the state feedback gain K which achieves $\mu_U(G_c(j\omega)) < 1$, $\forall\omega \in [0,\infty]$ is given by $K = YV^{-1}$.

By pre- and post-multiplying

$$\begin{bmatrix} I & A+B_uK & 0 \\ 0 & C+D_uK & I \end{bmatrix} \tag{56}$$

and its transpose to Eq. (55), respectively, we have

$$\mathrm{He}\begin{bmatrix} A+B_uK & B\Delta \\ C+D_uK & -I+D\Delta \end{bmatrix} \begin{bmatrix} X_{11}(r) & 0 \\ X_{31}(r) & X_{33}(r) \end{bmatrix} < 0 \tag{57}$$

By replacing $X_{31}(r)$ and $X_{33}(r)$ with $X_{21}(r)$ and $X_{22}(r)$, respectively, Eq. (57) implies Eq. (50). This shows that the LMI condition (55) of Theorem 7 allow us to use the parameter dependent Lyapnov matrix $X_{11}(r)$ for state feedback synthesis.

Note that Theorem 7 is a special case of Theorem 4. We can prove that Theorem 7 is not more conservative than Lemma 5 using the same way as in the previous section. However, it is omitted for sake of space.

6 Another proof for discrete time systems

The new analysis and synthesis approach for discrete time systems was proposed by Oliveira et al[1, 2]. In this section, we give another proof for the result for discrete time systems using descriptor form.

Lemma 6 *[1, 2] For the discrete time system;*

$$x(k+1) = Ax(k) \tag{58}$$

the following statements are equivalent.

1. *There exists $P > 0$ such that $-P + A^T PA < 0$ holds.*
2. *There exist $P > 0$ and G such that the following matrix inequality holds.*

$$\begin{bmatrix} -P & A^T G^T \\ GA & P - G - G^T \end{bmatrix} < 0 \tag{59}$$

Proof (sufficient condition based on descriptor form)
Letting $x_2(k)$ be $x_2(k) = Ax(k)$, we have the following descriptor system that is equivalent to the system (58).

$$\hat{E}\begin{bmatrix} x(k+1) \\ x_2(k+1) \end{bmatrix} = \hat{A}\begin{bmatrix} x(k) \\ x_2(k) \end{bmatrix} \tag{60}$$

$$\hat{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \hat{A} = \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix} \tag{61}$$

Letting Lyapunov matrix candidate $\hat{P}$ be

$$\hat{P} = \begin{bmatrix} P & G \\ G^T & 0 \end{bmatrix}, \; P > 0 \tag{62}$$

and substituting it for the following stability condition for discrete time systems $\hat{E}\hat{x}(k+1) = \hat{A}\hat{x}(k)$,

$$-\hat{E}^T \hat{P}\hat{E} + \hat{A}^T \hat{P}\hat{A} < 0 \tag{63}$$

we obtain (59). □

7 Conclusion

We have given another simple and easy proof for non-common Lyapunov type stability condition and its application (state feedback synthesis, H_∞ performance condition) based on descriptor form. Using the technique in the proof, we derived a new stability condition for slowly time varying systems that have real rational uncertainty. For the system with $D = 0$, the proposed method is not more conservative than former methods and the quadratic stability. Furthermore, a new static feedback μ synthesis method has been given.

References

[1] M. C. Oliveira, J. Bernussou, and J. C. Geromel: A New Discrete-time Robust Stability Condition; Systems & Control Letters, Vol.37, pp.261-265 (1999)

[2] M. C. Oliveira, J. C. Geromel and J. Bernussou: Extended H_2 and H_∞ norm characterizations and controller parameterizations for discrete-time systems; International Journal of Control, Vol.75, No.9, pp.666-679 (2002)

[3] D. Peaucelle and D. Arzelier: New LMI-based conditions for robust H_2 performance analysis; Proc. 19th American Control Conference, pp.317-321 (2000)

[4] P. Apkarian, H. D. Tuan and J. Bernussou: Continuous-Time Analysis, Eigenstructure Assignment, and H_2 Synthesis With Enhanced Linear Matrix Inequalities (LMI) Characterizations; IEEE Trans. AC, Vol.46, No.12, pp.1941-1946 (2001)

[5] Y. Ebihara and T. Hagiwara: New Dilated LMI Characterizations for Continuous-Time Control Design and Robust Multiobjective Control; Proc. American Control Conference, pp.47-52 (2002)

[6] T. Shimomura, M. Takahashi and T. Fujii: Extended-Space Control Design with Parameter-Dependent Lyapunov Functions; Proc. IEEE CDC, WeP02-1, pp.2157-2162 (2001)

[7] G. Chen and T. Sugie: An upper bound of mu based on the parameter dependent multipliers; Proc. 16th American Control Conference, pp.2604-2608 (1997)

[8] G. Chen and T. Sugie: Mu-Analysis and Synthesis of State Feedback System based on Multipliers and LMI's; Proc. 17th American Control Conference, pp.537-541 (1998)

[9] I. Masubuchi, Y. Kamitane, A. Ohara and N. Suda: H_∞ control for descriptor systems: matrix inequalities approach Automatica, Vol.33, No.4, pp.669-673 (1997)