ROBUST RECEDING-HORIZON ESTIMATION FOR UNCERTAIN DISCRETE-TIME LINEAR SYSTEMS

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Abstract

The problem of estimating the state of discrete-time linear systems when uncertainties affect the system matrices is addressed. A quadratic cost function is considered, involving a finite number of recent measurements and a prediction vector. This leads to state the estimation problem in the form of a regularized least-squares one with uncertain data. The optimal solution (involving on-line scalar minimization) together with a suitable closed-form approximation are given. For both the resulting receding-horizon estimators convergence results are derived and an operating procedure to select the design parameters is proposed.

1 Introduction

Receding-horizon estimation has been the objective of numerous investigations since the appearance of the pioneering work [1] (see, also, [2, 3]). The interest for such a method stems from the capability of dealing with a limited amount of data, instead of using all the information available from the beginning.

Various methods have been proposed to perform receding-horizon estimation. A possible approach consists in constructing sliding-window estimators that provide maximum-likelihood or minimum-variance state estimates by assuming that the system and measurement noises are white and Gaussian distributed (see, among others, [4]). Alternative methods are based on the idea of estimating the state of the system by minimizing a least-squares cost function according to a sliding-window strategy, where the noises are regarded as unknown disturbances (see, among others, [5, 6, 7]).

Recent investigations mainly focused on least-squares methods that allow one to account for boundedness of both state and noises by applying on-line optimization [8]. The development of viable design procedures has been considered in [9]. However, despite the vast literature on the subject, no result on robustness in receding-horizon estimation is known to these authors. This has motivated our efforts of addressing robustness with respect to system uncertainty for the receding-horizon estimator proposed in [9]. Such a goal has been obtained by using recent results (see [10]) that are well-suited to treating the problem in our estimation framework.

In this paper, we address a receding-horizon estimation problem for discrete-time linear systems, affected by a bounded system uncertainty. We shall follow the approach originally stated in [11], where a guaranteed performance receding-horizon estimator was proposed for a quite general setting, making use of on-line optimization or resorting to approximate neural strategies. The linear case was treated in [9], where a solution in closed form was given when no uncertainties affect the system matrices. The proposed technique consists in minimizing a sliding-window quadratic cost function that is made up of two contributions. The first contribution is a weighted term penalizing the distance of the current estimated state from its prediction (both computed at the beginning of the sliding window); the second is the usual prediction error computed on the basis of the last measures. In the presence of uncertainty, estimation can be accomplished by minimizing a worst-case cost on line, according to [10].

We conclude this section with some notations we use throughout this paper. Given a generic, symmetric, positive definite matrix P, let us denote by $\underline{\sigma}(P)$ and $\bar{\sigma}(P)$ the minimum and maximum eigenvalues of P, respectively. Given a generic matrix M, M' and $M^\dagger$ indicate the matrix transpose and the pseudoinverse of M, respectively. Furthermore, $\|M\|_{\max} \triangleq \|M\| = [\bar{\sigma}(M'M)]^{1/2}$ and $\|M\|_{\min} \triangleq [\underline{\sigma}(M'M)]^{1/2}$. Given a generic vector v, $\|v\|$ denotes the Euclidean norm of v, and, given a positive definite matrix P, $\|v\|_P$ denotes the weighted norm of v, $\|v\|_P \triangleq (v'Pv)^{1/2}$. For a generic time-variant vector v_t, $v_{t-N}^t \triangleq \mathrm{col}\,(v_{t-N}, v_{t-N+1}, \ldots, v_t)$.

For the sake of brevity, the proofs of some of the results presented in the paper will be omitted.

2 Receding-horizon estimation for uncertain discrete-time linear systems

Let us consider an uncertain linear dynamic system described by the following discrete-time equations

$$x_{t+1} = (A + \delta A)\; x_t + \xi_t \qquad (1a)$$
$$y_t = (C + \delta C)\; x_t + \eta_t \qquad (1b)$$

where $t = 0, 1, \ldots$ is the time instant, $x_t \in \mathbb{R}^n$ is the state vector (the initial state x_0 is unknown), $\xi_t \in \Xi \subset \mathbb{R}^n$ is the system noise vector, $y_t \in \mathbb{R}^p$ is the vector of the measures, and $\eta_t \in H \subset \mathbb{R}^p$ is the measurement noise vector. The matrices δA and δC represent uncertainties in the knowledge of the system, and are supposed to belong to the known compact sets $\mathcal{A}$ and $\mathcal{C}$, respectively.

We assume the statistics of the random variables $x_0, \xi_0, \xi_1, \dots \eta_0, \eta_1, \dots$ to be unknown, and consider them as deterministic variables of an unknown kind. Moreover, we assume our estimates to be based on data obtained in the recent past, or, equivalently, we assume the estimator to be a finite-memory one. Then we define the information vector as

$$I_t^N \triangleq \mathrm{col}\,(y_{t-N}, \dots, y_t, u_{t-N}, \dots, u_{t-1}), \quad t = N, N+1, \dots .$$

We shall follow the receding-horizon strategy described in [11] for quite a general setting and specialized in [9] for linear systems with no uncertainties. More specifically, at any stage $t = N, N+1, \dots$, the objective is to find estimates of the state vectors $x_{t-N}, \dots, x_t$ on the basis of the information vector I_t^N and of a prediction $\bar{x}_{t-N}$ of the state x_{t-N}. Let us denote by $\hat{x}_{t-N,t}, \dots, \hat{x}_{t,t}$ the estimates of $x_{t-N}, \dots, x_t$, respectively, to be made at stage t. As we have assumed the statistics of $x_0, \xi_0, \xi_1, \dots, \eta_0, \eta_1, \dots$ to be unknown, a natural criterion to derive the estimator consists in resorting to a least-squares approach. Towards this end, we introduce the following loss function

$$J_t = \|\, \hat{x}_{t-N,t} - \bar{x}_{t-N} \,\|_M^2 + \sum_{i=t-N}^{t} \|\, y_i - (C + \delta C)\ \hat{x}_{i,t} \,\|^2 \qquad (2)$$

where the first term, weighted by the positive definite matrix M, expresses our belief in the prediction $\bar{x}_{t-N}$ as compared with the observation model. The matrix M is assumed to be positive definite and can be viewed as an extension of the scalar positive weight μ in [11] and [9], which was considered as a design parameter. Of course, resorting to a matrix M gives us many more degrees of freedom in the estimator design. A constructive procedure to select the matrix M will be given in the following (see Proposition 3). We assume that at stages $t = N+1, N+2, \dots$ the prediction $\bar{x}_{t-N}$ is determined via the state equation of the nominal system by the estimate $\hat{x}_{t-N-1,t-1}$, that is, $\bar{x}_{t-N} = A\,\hat{x}_{t-N-1,t-1}$. The vector $\bar{x}_0$ denotes an a priori prediction of x_0.

A notable simplification of the estimation scheme can be obtained by defining $\hat{x}_{t-N+1,t}, \dots, \hat{x}_{t,t}$ as estimates generated by the first estimate $\hat{x}_{t-N,t}$ through the state equation (1a), that is,

$$\hat{x}_{i+1,t} = (A + \delta A)\, \hat{x}_{i,t} \quad , \quad i = t-N, \dots, t-1 . \qquad (3)$$

By applying (3) repeatedly, we obtain that, at stage t, the cost J_t is a function of $\hat{x}_{t-N,t}$, δA, and δC, that is, $J_t = J_t\,(\hat{x}_{t-N,t}, \delta A, \delta C)$.

As to the uncertainty in the system matrices, we shall follow a minimax approach, then, at any stage $t = N, N+1, \dots$, the following problem has to be solved:

Problem E_t *For a given pair $(\bar{x}^\circ_{t-N}, I_t^N)$, find the optimal estimate*

$$\hat{x}^\circ_{t-N,t} = \arg \min_{\hat{x}_{t-N,t}} \max_{\delta A \in \mathcal{A}\,;\, \delta C \in \mathcal{C}} J_t\,(\hat{x}_{t-N,t}, \delta A, \delta C) . \qquad (4)$$

□

The optimal predictions are determined as

$$\bar{x}^\circ_0 = \bar{x}_0$$
$$\bar{x}^\circ_{t-N} = A\,\hat{x}^\circ_{t-N-1,t-1} , \quad t = N+1, N+2, \dots \qquad (5)$$

Now, in order to find an explicit solution for Problem E_t, we shall reformulate it as a regularized least-squares problem with uncertain data. Towards this goal, let us define the following matrices:

$$F_N \triangleq \begin{bmatrix} C \\ C\,A \\ \vdots \\ C\,A^N \end{bmatrix} , \quad \mathcal{F}_N \triangleq \begin{bmatrix} (C+\delta C) \\ (C+\delta C)\ (A+\delta A) \\ \vdots \\ (C+\delta C)\ (A+\delta A)^N \end{bmatrix} .$$

Using the definition of $\mathcal{F}_N$, it is possible to rewrite cost (2) as

$$J_t = \|\, \hat{x}_{t-N,t} - \bar{x}_{t-N} \,\|_M^2 + \left\| y_{t-N}^t - \mathcal{F}_N \hat{x}_{t-N,t} \right\|^2 . \qquad (6)$$

Owing to the compactness of the sets $\mathcal{A}$ and $\mathcal{C}$, a positive definite matrix Γ exists such that

$$(\mathcal{F}_N - F_N)'\,(\mathcal{F}_N - F_N) \le \Gamma , \quad \forall \delta A \in \mathcal{A} , \ \forall \delta C \in \mathcal{C} . \qquad (7)$$

A trivial choice for the matrix Γ is given by $\Gamma = \gamma^2 I$ where

$$\gamma = \max_{\delta A \in \mathcal{A}\,;\, \delta C \in \mathcal{C}} \|\mathcal{F}_N - F_N\| .$$

We can now state the following proposition.

Proposition 1 *Given a positive definite matrix Γ satisfying (7), $\forall \delta A \in \mathcal{A}$ and $\forall \delta C \in \mathcal{C}$ there exists a suitable matrix S such that:*

$$\|S\| \le 1 \quad , \quad \mathcal{F}_N - F_N = S\,\Gamma^{1/2}$$

where $\Gamma^{1/2}$ is the unique positive definite square root of the matrix Γ.

If we rewrite cost (6) as

$$J_t = \|\, \hat{x}_{t-N,t} - \bar{x}_{t-N} \,\|_M^2 + \left\| F_N \hat{x}_{t-N,t} + (F_N - \mathcal{F}_N)\, \hat{x}_{t-N,t} - y_{t-N}^t \right\|^2 ,$$

and, by exploiting the results of Proposition 1, we replace $\mathcal{F}_N - F_N$ with $S\,\Gamma^{1/2}$, we can formulate the following alternative version of Problem E_t.

Problem E_t' *For a given pair* $(\bar{x}^\circ_{t-N}, I_t^N)$ *, find the optimal estimate*

$$\hat{x}^\circ_{t-N,t} = \arg\min_{\hat{x}_{t-N,t}} \max_{\|S\|\leq 1} J_t'(\hat{x}_{t-N,t}, S)$$

where

$$J_t'(\hat{x}_{t-N,t}, S) \triangleq \|\hat{x}_{t-N,t} - \bar{x}^\circ_{t-N}\|_M^2 + \left\|F_N \hat{x}_{t-N,t} + S\,\Gamma^{1/2}\hat{x}_{t-N,t} - y^t_{t-N}\right\|^2 .$$

□

With a little abuse of notation, we denote by $\hat{x}^\circ_{t-N,t}$ the solutions of both Problem E_t and Problem E_t'. A similar consideration holds for the optimal predictions $\bar{x}^\circ_{t-N}$, $t = N, N+1, \dots$. The latter are determined as

$$\begin{aligned} &\bar{x}^\circ_0 = \bar{x}_0 \\ &\bar{x}^\circ_{t-N} = A\,\hat{x}^\circ_{t-N-1,t-1} \,, \quad t = N+1, N+2, \dots \,. \end{aligned}$$

Whereas on the one hand Proposition 1 ensures that every matrix $\mathcal{F}_N - F_N$ can be represented as $S\Gamma^{1/2}$ with a suitable choice of the contraction matrix S , on the other hand, in general, not every matrix of the form $S\Gamma^{1/2}$ corresponds to an admissible matrix $\mathcal{F}_N - F_N$. Hence Problem E_t' turns out to be a conservative reformulation of Problem E_t. However, choosing a suitable matrix Γ (e.g., by means of numerical simulations), it is possible to greatly reduce this element of conservativity. Furthermore, using the results shown in [10], it is possible to give a semi-explicit solution to Problem E_t'. More specifically, we can state the following theorem.

Theorem 1 *Problem E_t' has a unique solution given by*

$$\hat{x}^\circ_{t-N,t} = \left(\hat{M}_t + F_N'\hat{L}_t F_N\right)^{-1}\left(M\,\bar{x}^\circ_{t-N} + F_N'\,\hat{L}_t\,y^t_{t-N}\right) \tag{8}$$

where

$$\hat{M}_t \triangleq M + \lambda_t^\circ\Gamma \,, \quad \hat{L}_t \triangleq I + [(\lambda_t^\circ - 1)I]^\dagger \,,$$

and the scalar parameter λ_t° *is the unique solution of the one-dimensional optimization problem*

$$\lambda_t^\circ = \arg\min_{\lambda\geq 1}\left\{ \|x_t(\lambda)\|_M^2 + \left\|x_t(\lambda) - \bar{x}^\circ_{t-N}\right\|_\Gamma^2 + \left\|F_N' x(\lambda) - \left(y^t_{t-N} - F_N\bar{x}^\circ_{t-N}\right)\right\|^2_{\hat{L}(\lambda)} \right\} \tag{9}$$

where

$$\begin{aligned} x_t(\lambda) &\triangleq \left(\hat{M}(\lambda) + F_N'\hat{L}(\lambda)F_N\right)^{-1} \\ &\quad\times\left[F_N'\hat{L}(\lambda)\left(y^t_{t-N} - F_N\bar{x}^\circ_{t-N}\right) - \lambda^\circ\Gamma\,\bar{x}^\circ_{t-N}\right] , \\ \hat{M}(\lambda) &\triangleq M + \lambda\Gamma \,, \quad \hat{L}(\lambda) \triangleq I + [(\lambda - 1)I]^\dagger \,. \end{aligned}$$

□

As to the minimization in (9), if we exclude the boundary point $\lambda = 1$, we can explicitly solve the pseudo-inverse operation in the definition of $\hat{L}_t$, that is,

$$\hat{L}_t = \frac{\lambda_t^\circ}{\lambda_t^\circ - 1} I \,,$$

and hence we can rewrite the solution of Problem E_t' in a more compact expression:

$$\begin{aligned} \hat{x}^\circ_{t-N,t} &= \left(M + \lambda_t^\circ\Gamma + \frac{\lambda_t^\circ}{\lambda_t^\circ - 1}F_N'F_N\right)^{-1} \\ &\quad\times\left(M\bar{x}^\circ_{t-N} + \frac{\lambda_t^\circ}{\lambda_t^\circ - 1}F_N' y^t_{t-N}\right) . \end{aligned} \tag{10}$$

In general the proposed filter is nonlinear and time-variant, because of the dependence on the scalar parameter λ_t°, that has to be determined on line by means of a constrained line search. If, for some reasons (e.g., lack of computational time in the sampling period), this is not feasible, following [12], one can obtain a reasonable approximation of the optimal solution by assigning to the scalar parameter λ_t° a fixed value $1+\alpha$. The scalar parameter α can be suitably tuned off line by means of numerical simulations. This leads to an approximated solution of Problem E_t' given by:

$$\begin{aligned} \hat{x}^\circ_{t-N,t} &= \left(M + (1+\alpha)\Gamma + \frac{1+\alpha}{\alpha}F_N'F_N\right)^{-1} \\ &\quad\times\left(M\bar{x}^\circ_{t-N} + \frac{1+\alpha}{\alpha}F_N' y^t_{t-N}\right) . \end{aligned} \tag{11}$$

3 Stability of the estimator

In order to analyze the stability of the proposed estimation scheme, we shall first prove the exponential convergence both for the sub-optimal filter (11) and for the optimal one (10) when no noise acts on the system and measurement equations. Then we shall prove the (simple) stability in the noisy case. Let us consider the noiseless uncertain model of the form

$$x_{t+1} = (A + \delta A)\,x_t \tag{12a}$$
$$y_t = (C + \delta C)\,x_t \,. \tag{12b}$$

In the following, for the sake of brevity, we shall use the following definition:

$$\Phi(\lambda) \triangleq \left(M + \lambda\Gamma + \frac{\lambda}{\lambda - 1}F_N'F_N\right) .$$

First, we assume that the scalar weight λ_t° is set equal to a fixed value $1+\alpha$, hence we consider the approximate estimator (11).

Theorem 2 *Suppose that*

(i) *system (12a) is quadratically stable, that is, there exists a positive definite matrix* P *such that*

$$(A + \delta A)'\,P\,(A + \delta A) - P < 0 \,, \quad \forall \delta A \in \mathcal{A} \,,$$

(ii) *the matrix* $\Phi\,(1+\alpha)^{-1}\,MA$ *is asymptotically stable,*

then estimator (11) is an exponential observer for system (12).

Proof: If we consider the estimation error e_{t-N}, defined as $e_{t-N} \triangleq x_{t-N} - \hat{x}^{\circ}_{t-N,t}$, we have

$$\begin{aligned} e_{t-N} &= x_{t-N} - \Phi(1+\alpha)^{-1}\left(M\bar{x}^{\circ}_{t-N} + \frac{1+\alpha}{\alpha}F'_N y^t_{t-N}\right) \\ &= \Phi(1+\alpha)^{-1}\Big[\Phi(1+\alpha)x_{t-N} \\ &\qquad - \left(M\bar{x}^{\circ}_{t-N} + \frac{1+\alpha}{\alpha}F'_N y^t_{t-N}\right)\Big] \,. \end{aligned}$$

Owing to the fact that $y^t_{t-N} = \mathcal{F}_N x_{t-N}$, we obtain

$$\begin{aligned} e_{t-N} &= \Phi(1+\alpha)^{-1}\Big[M\left(x_{t-N} - \bar{x}^{\circ}_{t-N}\right) + (1+\alpha)\Gamma x_{t-N} \\ &\qquad + \frac{1+\alpha}{\alpha}F'_N F_N x_{t-N} - \frac{1+\alpha}{\alpha}F'_N \mathcal{F}_N x_{t-N}\Big] \\ &= \Phi(1+\alpha)^{-1}\Big[MAe_{t-N-1} + M\delta A x_{t-N-1} \\ &\qquad + (1+\alpha)\Gamma x_{t-N} + \frac{1+\alpha}{\alpha}F'_N\left(F_N - \mathcal{F}_N\right)x_{t-N}\Big] \,. \end{aligned}$$

Condition (ii) implies that a Lyapunov matrix P_e exists such that

$$\left[\Phi\,(1+\alpha)^{-1}\,MA\right]' P_e \left[\Phi\,(1+\alpha)^{-1}\,MA\right] - P_e < 0 \,. \quad (13)$$

By applying the operator $\|\cdot\|_{P_e}$ to the error dynamics we obtain

$$\begin{aligned} \|e_{t-N}\|_{P_e} &\le \left\|\Phi\,(1+\alpha)^{-1}\,MA\right\|_{P_e}\|e_{t-N-1}\|_{P_e} \\ &+ \left\|\Phi\,(1+\alpha)^{-1}\,M\delta A\right\|_{P_e}\|x_{t-N-1}\|_{P_e} \\ &+ \Bigg(\left\|(1+\alpha)\Phi\,(1+\alpha)^{-1}\,\Gamma\right\|_{P_e} \\ &+ \left\|\frac{1+\alpha}{\alpha}\Phi\,(1+\alpha)^{-1}\,F'_N\left(F_N - \mathcal{F}_N\right)\right\|_{P_e}\Bigg)\|x_{t-N}\|_{P_e} \,. \end{aligned}$$

For the sake of brevity, let us define the following quantities:

$$\begin{aligned} \varphi_P &\triangleq \|\Phi\,(1+\alpha)^{-1}\,MA\|_{P_e} \,, \\ c_1 &\triangleq \max_{\delta A \in \mathcal{A}} \|\Phi\,(1+\alpha)^{-1}\,M\delta A\|_{P_e} \,, \\ c_2 &\triangleq (1+\alpha)\|\Phi(1+\alpha)^{-1}\Gamma\|_{P_e} \,, \\ c_3 &\triangleq \frac{1+\alpha}{\alpha}\max_{\delta A \in \mathcal{A}\,;\,\delta C \in \mathcal{C}}\left\|\Phi\,(1+\alpha)^{-1}\,F'_N\left(F_N - \mathcal{F}_N\right)\right\|_{P_e} . \end{aligned}$$

Clearly, we have $\varphi_P < 1$ and, owing to the compactness of the sets $\mathcal{A}$ and $\mathcal{C}$, $c_1 < +\infty$, $c_2 < +\infty$, and $c_3 < +\infty$. If we define

$$a_P \triangleq \max_{\delta A \in \mathcal{A}} \|A + \delta A\|_P \,, \quad (14)$$

then we have

$$\|x_t\|_P \le a_P^t \|x_0\|_P \,.$$

It is worth noting that condition (i) ensures that $a_P < 1$, hence $\|x_t\|_P$ converges exponentially to zero. As

$$\|x_{t-N}\|_{P_e} \le \left[\underline{\sigma}(P_e)/\underline{\sigma}(P)\right]^{1/2}\|x_{t-N}\|_P \,,$$

the P_e-norm of the estimation error turns out to be bounded as

$$\|e_{t-N}\|_{P_e} \le \zeta_{t-N}$$

where the sequence ζ_{t-N}, defined as

$$\begin{aligned} \zeta_{t-N} &= \varphi_P\zeta_{t-N-1} + \left[\underline{\sigma}(P_e)/\underline{\sigma}(P)\right]^{1/2} \\ &\qquad \times (c_1 + c_2 a_P + c_3 a_P)\|x_0\|_P a_P^{t-N-1} \,, \end{aligned}$$

converges exponentially to zero, since $\varphi_P < 1$ and the second term converges exponentially to zero.

□

The following proposition provides an operating procedure to verify condition (i).

Proposition 2 *Suppose that the set* $\mathcal{A}$*, of all the admissible uncertainties on the system matrix* A*, is given by*

$$\mathcal{A} \triangleq \{\delta A \ : \ \delta A'\delta A \le \Gamma_A \,,\ \Gamma_A > 0\} \,. \quad (15)$$

Then system (12a) is quadratically stable if and only if there exist a positive definite matrix P *and a scalar weight* $\alpha \ge 0$ *such that*

$$\begin{bmatrix} A'PA - P + \alpha\Gamma_A & A'P \\ PA & P - \alpha I \end{bmatrix} < 0 \,. \quad (16)$$

□

Note that equation (15) gives a reasonable characterization of a compact set. More specifically, it is always possible to find a matrix Γ_A such that $\delta A'\delta A \le \Gamma_A$, $\forall \delta A \in \mathcal{A}$. In this case, a trivial choice is $\Gamma_A = \gamma_A^2$ where

$$\gamma_A \triangleq \max_{\delta A \in \mathcal{A}} \|\delta A\| \,.$$

Condition (16) is a Linear Matrix Inequality (LMI) in α and P and, hence, it is possible to verify its feasibility by means of efficient numerical routine (see [13] for details).

As to condition (ii) of Theorem 2, the following proposition can be formulated.

Proposition 3 *Suppose that the weight matrix* M *is given by*

$$M = \left\{\left(Y^{-1} - X^{-1}\right)X\left[(1+\alpha)\Gamma + \frac{1+\alpha}{\alpha}F'_N F_N\right]^{-1}\right\} \quad (17)$$

where X *and* Y *are two positive definite matrices such that*

$$\begin{bmatrix} X & A'Y \\ YA & X \end{bmatrix} > 0 \,, \quad X - Y > 0 \,. \quad (18)$$

Then the matrix $\Phi\,(1+\alpha)^{-1}\,MA$ *is asymptotically stable.*

□

Since conditions (18) are LMI in X and Y, Proposition 3 gives an operating procedure to choose the weight matrix M.

Now, in order to address the stability issue of the time-variant estimator given by (10), we need the following assumption:

A1.The pair (A, C) is completely observable in N steps.

We can state the following theorem.

Theorem 3 *Suppose that Assumption A1 is satisfied and that*

(i) *system (12a) is quadratically stable,*

(ii) *the weight matrix M is such that*

$$\bar{\sigma}(M) \leq \frac{\underline{\sigma}(M) + f^*}{a} \tag{19}$$

where

$$a \triangleq \|A\| , \quad f^* \triangleq \frac{f_{\min}}{\sqrt{\gamma_{\min}}} \left(\sqrt{\gamma_{\min}} + f_{\min}\right)^2 ,$$

$$\gamma_{\min} \triangleq \underline{\sigma}(\Gamma) , \quad f^2_{\min} \triangleq \underline{\sigma}\left(F'_N F_N\right) ,$$

then estimator (10) is an exponential observer for system (12).

Proof: The error dynamics associated with the estimator (10) is given by

$$e_{t-N} = \Phi(\lambda^\circ_t)^{-1} \Big[M A e_{t-N-1} + M \delta A x_{t-N-1} + \lambda^\circ_t \Gamma x_{t-N} + \frac{\lambda^\circ_t}{\lambda^\circ_t - 1} F'_N \left(F_N - \mathcal{F}_N\right) x_{t-N} \Big] . \tag{20}$$

If we apply the norm operator to equation (20), we obtain

$$\begin{aligned} \|e_{t-N}\| \leq & \left\| \Phi\left(\lambda^\circ_t\right)^{-1} M A \right\| \|e_{t-N-1}\| \\ & + \left\| \Phi\left(\lambda^\circ_t\right)^{-1} M \delta A \right\| \|x_{t-N-1}\| \\ & + \Big(\left\| \lambda^\circ_t \Phi\left(\lambda^\circ_t\right)^{-1} \Gamma \right\| \\ & + \left\| \frac{\lambda^\circ_t}{\lambda^\circ_t - 1} \Phi\left(\lambda^\circ_t\right)^{-1} F'_N \left(F_N - \mathcal{F}_N\right) \right\| \Big) \|x_{t-N}\| . \end{aligned}$$

Clearly, we have

$$\begin{aligned} \left\|\Phi\left(\lambda^\circ_t\right)^{-1} M A\right\| & \leq \left\|\Phi\left(\lambda^\circ_t\right)^{-1}\right\| \|M\| \|A\| \\ & = \frac{1}{\underline{\sigma}\left[\Phi\left(\lambda^\circ_t\right)\right]} \bar{\sigma}(M)\, a . \end{aligned}$$

Moreover, the quantity $\underline{\sigma}\left[\Phi\left(\lambda^\circ_t\right)\right]$ is bounded below as

$$\begin{aligned} \underline{\sigma}\left[\Phi\left(\lambda^\circ_t\right)\right] & \geq \underline{\sigma}(M) + \lambda^\circ_t \underline{\sigma}(\Gamma) + \frac{\lambda^\circ_t}{\lambda^\circ_t - 1} \underline{\sigma}(F'_N F_N) \\ & = \underline{\sigma}(M) + \lambda^\circ_t \gamma_{\min} + \frac{\lambda^\circ_t}{\lambda^\circ_t - 1} f^2_{\min} , \end{aligned}$$

and, since the function $f(\lambda^\circ_t)$, defined as

$$f(\lambda^\circ_t) \triangleq \lambda^\circ_t \gamma_{\min} + \frac{\lambda^\circ_t}{\lambda^\circ_t - 1} f^2_{\min} ,$$

has a global minimum $f^* = f(\lambda^*)$ in $\lambda^* \triangleq 1 + f_{\min}/\sqrt{\gamma_{\min}}$ in the interval $(1, +\infty)$, we have

$$\underline{\sigma}\left[\Phi\left(\lambda^\circ_t\right)\right] \geq \underline{\sigma}(M) + f^* ,$$

and, hence,

$$\left\|\Phi\left(\lambda^\circ_t\right)^{-1} M A\right\| \leq \frac{\bar{\sigma}(M)\, a}{\underline{\sigma}(M) + f^*} . \tag{21}$$

Let us now define

$$\begin{aligned} d_1 & \triangleq \sup_{\lambda^\circ_t \geq 1 \,;\, \delta A \in \mathcal{A}} \left\|\Phi\left(\lambda^\circ_t\right)^{-1} M \delta A\right\| , \\ d_2 & \triangleq \sup_{\lambda^\circ_t \geq 1} \left\|\lambda^\circ_t \Phi\left(\lambda^\circ_t\right)^{-1} \Gamma\right\| , \\ d_3 & \triangleq \sup_{\lambda^\circ_t \geq 1 \,;\, \delta A \in \mathcal{A} \,;\, \delta C \in \mathcal{C}} \left\| \frac{\lambda^\circ_t}{\lambda^\circ_t - 1} \Phi\left(\lambda^\circ_t\right)^{-1} F'_N \left(F_N - \mathcal{F}_N\right) \right\| . \end{aligned}$$

Clearly, equation (21) implies $d_1 < +\infty$. As to the constant d_2, we have

$$\begin{aligned} d_2 & \leq \|\Gamma\| \sup_{\lambda^\circ_t \geq 1} \lambda^\circ_t \left\|\Phi\left(\lambda^\circ_t\right)^{-1}\right\| \\ & \leq \|\Gamma\| \sup_{\lambda^\circ_t \geq 1} \frac{\lambda^\circ_t}{\underline{\sigma}(M) + \lambda^\circ_t \gamma_{\min} + \frac{\lambda^\circ_t}{\lambda^\circ_t - 1} f^2_{\min}} \\ & \leq \|\Gamma\| / \gamma_{\min} < +\infty . \end{aligned}$$

In a similar way, it is possible to see that

$$\begin{aligned} d_3 & \leq \max_{\delta A \in \mathcal{A} \,;\, \delta C \in \mathcal{C}} \left\|F'_N (F_N - \mathcal{F}_N)\right\| \\ & \qquad \times \sup_{\lambda^\circ_t \geq 1} \left\{ \frac{\lambda^\circ_t}{\lambda^\circ_t - 1} \left\|\Phi\left(\lambda^\circ_t\right)^{-1}\right\| \right\} \\ & \leq 1/f^2_{\min} \max_{\delta A \in \mathcal{A} \,;\, \delta C \in \mathcal{C}} \left\|F'_N (F_N - \mathcal{F}_N)\right\| < +\infty . \end{aligned}$$

Therefore, the norm of the estimation error is bounded above by the sequence ζ_{t-N}, defined as

$$\begin{aligned} \zeta_{t-N} = & \frac{\bar{\sigma}(M)\, a}{\underline{\sigma}(M) + f^*} \zeta_{t-N-1} \\ & + 1/\underline{\sigma}(P)^{1/2} \left(d_1 + d_2 a_P + d_3 a_P\right) \|x_0\|_P \, a_P^{t-N-1} , \end{aligned}$$

which converges exponentially to zero provided that condition (19) is satisfied.

□

Note that condition (19) can be easily satisfied for any value of a. More specifically, if $a \leq 1$, for every choice of the parameter $\underline{\sigma}(M)$ it is always possible to choose $\bar{\sigma}(M)$ such that $\bar{\sigma}(M) \geq \underline{\sigma}(M)$ and condition (19) is verified. Instead, when

$a > 1$, the region of the plane $(\underline{\sigma}(M), \bar{\sigma}(M))$ in which condition (19) is satisfied is the triangle of vertices $(0, 0)$, $(0, f^*/a)$, and $(f^*/(1-a), f^*/(1-a))$.

Let us now consider the noisy system (1) and let us suppose that the disturbances acting on the state and measurement equations are norm-bounded. More specifically, we make the following assumption:

A2. Ξ and H are compact sets.

Then, owing to the exponential convergence of the estimator (11), the following theorem can be stated.

Theorem 4 *Suppose that Assumption A2 is satisfied and that*

(i) *system (12a) is quadratically stable,*

(ii) *the matrix* $\Phi\,(1+\alpha)^{-1}\,MA$ *is asymptotically stable,*

then estimator (11) applied to system (1) provides a bounded estimation error.

□

A similar behavior can be shown for the estimator (10). More specifically the following theorem can be stated.

Theorem 5 *Suppose that Assumptions A1 and A2 are satisfied and that*

(i) *system (12a) is quadratically stable,*

(ii) *the weight matrix* M *satisfies condition (19),*

then estimator (10) applied to system (1) provides a bounded estimation error.

□

4 Conclusions

A receding-horizon method for estimating the state of uncertain discrete-time linear systems have been presented. The proposed technique relies on the minimization of a worst-case quadratic cost function. The latter involves a weighted term penalizing the distance of the current estimated state from its prediction (both computed at the beginning of the sliding window) and the usual prediction error computed on the basis of the last measures.

The estimation problem has been reduced to a regularized least-squares minimization with uncertain data, and the optimal solution has been provided, involving on-line scalar minimization. Moreover, a suitable closed-form approximate solution have been given. For both the optimal and approximate receding-horizon estimators, conditions for the exponential convergence have been stated when no noise acts on the system and measurement equations, and operating procedures to choose the design parameters have been proposed, based on a linear matrix inequality. Finally, the boundedness of the estimation error has been proved in the noisy case.

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