ON ROBUST OPTIMIZATION AND THE OPTIMAL CONTROL OF CONSTRAINED LINEAR SYSTEMS WITH BOUNDED STATE DISTURBANCES

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Abstract

The first part of this paper studies a specific class of uncertain quadratic and linear programs, where the uncertainty enters the constraints in an affine manner and the uncertainty set is a polytope. It is shown that one can convert the resulting semi-infinite optimization problem into a standard QP or LP with a finite number of decision variables and a finite number of constraints. This transformation is achieved in a computationally tractable way by solving as many LPs as there are constraints in the optimization problem without uncertainty. It is also shown that if the uncertainty set is given by upper and lower bounds only, then one need not solve any LPs in order to do this transformation; computing the 1-norms of the rows of the matrix by which the uncertainty enters the constraints is sufficient. The second part of the paper reviews and extends some definitions and results on input-to-state stability for nonlinear discrete-time systems. The third part of the paper shows how one can translate a class of robust finite-horizon optimal control problems (RFHOCPs) into the class of robust convex optimization problems that was studied in the first part of the paper. It is assumed that the system under consideration is linear, that there is a persistent, but bounded state disturbance that assumes values in a polytope and that there are mixed affine constraints on the input and state. By using the results from the previous sections, it is shown that one can set up a receding horizon controller (RHC) that is input-to-state stable (ISS) and guarantees robust constraint satisfaction for all time. The number of decision variables and constraints in the RFHOCP that define the RHC law increases linearly with the horizon length.

1 Introduction

The field of optimal control of constrained systems without uncertainty is fairly well developed and understood, but the optimal control of constrained systems with uncertainty is still a very open area with very few useful results [17, 18, 19]. One of the reasons for the difficulty in implementing robust controllers for constrained systems is that the incorporation of the effect of the uncertainty in the model often translates into computationally intractable problems. Various researchers, especially those working in the field of model predictive control, have suggested compromise solutions to addressing this problem [1, 7, 13, 14, 15, 16, 22]. However, some of these solutions are, arguably, still computationally intensive or conservative.

Recent research in the optimization literature on *robust convex optimization* problems [3, 4, 5, 22] might shed some light on the tractability of a large class of robust optimal control problems. This paper studies a very specific class of robust quadratic programming problems, where the uncertainty enters the problem only via the constraints and the uncertainty set is a polytope. The results from this study will be applied to a class of robust finite-horizon optimal control problems (RFHOCPs), where the uncertainty is a bounded, persistent state disturbance that can take on values from a polytopic uncertainty set. The solution of the RFHOCP will then be used to define a robustly stabilizing receding horizon control (RHC) law.

Section 2 defines the class of robust optimization problems that will be studied. Section 3 reviews and extends some basic results in input-to-state stability of nonlinear discrete-time systems. Section 4 introduces a class of RFHOCPs. It will be shown that if the cost is assumed to be quadratic and the disturbance is zero in cost, but included in the constraints, then the RHC law can be computed efficiently using the results from Section 2. It is also shown that the resulting closed-loop is input-to-state stable (ISS) and that mixed constraints on the state and input are robustly satisfied for all time.

Many of the results in this paper are available in the literature in one form or another. This paper makes a contribution by bringing these results together in a single paper, presenting them with a slightly different emphasis and extending the results where necessary.

Notation: If $x \in \mathbb{R}^n$ is a vector, then x' denotes its transpose, $\|x\|_p$ and $\|x\|$ denotes the p-norm and Euclidean norm of x, respectively. $\mathbf{1} := [1 \ \cdots \ 1]'$ is a column vector of ones of suitable dimension. $e_i := [0 \ \cdots \ 1 \ \cdots \ 0]'$ is the i'th standard basis vector in Euclidean space, i.e. a column vector with 1 as the i'th component and all other components being zero. If x and y are vectors, then $x \leq y$ will be used to denote component-wise inequality. If $S \in \mathbb{R}^{q\times t}$ is a matrix and $\mathcal{W} \subset \mathbb{R}^t$ is a non-empty compact set, then the column vector $\operatorname{vec}\min_{\mathbf{w}\in\mathcal{W}} S\mathbf{w} := [\min_{\mathbf{w}\in\mathcal{W}} e_1' S\mathbf{w} \ \cdots \ \min_{\mathbf{w}\in\mathcal{W}} e_q' S\mathbf{w}]'$, where $e_i' S$ is clearly the i'th row of S. The matrix $|S|$ is formed by taking the absolute values of the components of S, i.e. if $S := [s_{ij}] \in \mathbb{R}^{q\times t}$, then

$|S| := [|s_{ij}|] \in \mathbb{R}^{q\times t}$. Where it will not lead to confusion, $\omega(k)$ will denote the actual value of the sequence of variables $\omega(\cdot)$ at time k, while ω_k will be used to denote the prediction of $\omega(\tau+k)$ at a time instant k steps into the future if $\omega(\tau) = \omega_0 = \omega$ is the current value of the variable.

2 A Class of Uncertain Parametric QPs

This paper will consider *uncertain parametric quadratic programs* (UPQPs) [3, 5] in the form

$$V_N^*(x) := \min_{\mathbf{v}\in\mathcal{C}_N(x)} \mathbf{v}'H\mathbf{v} + \mathbf{v}'Gx + x'Yx + c'\mathbf{v}, \tag{1a}$$

and the *robustly feasible set* of decision variables is defined as

$$\mathcal{C}_N(x) := \left\{\mathbf{v} \in \mathbb{R}^d \mid L\mathbf{v} \le b + Mx + S\mathbf{w}, \forall \mathbf{w} \in \mathcal{W}\right\} \tag{1b}$$

where the matrices $H \in \mathbb{R}^{d\times d}$, $G \in \mathbb{R}^{d\times n}$, $Y \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{q\times d}$, $M \in \mathbb{R}^{q\times n}$, $S \in \mathbb{R}^{q\times t}$ and the vectors $b \in \mathbb{R}^q$, $c \in \mathbb{R}^d$. The minimizer is

$$\mathbf{v}^*(x) := \arg \min_{\mathbf{v}\in\mathcal{C}_N(x)} \mathbf{v}'H\mathbf{v} + \mathbf{v}'Gx + x'Yx + c'\mathbf{v}. \tag{1c}$$

In the above, $\mathbf{v} \in \mathbb{R}^d$ is the *decision variable*, $x \in \mathbb{R}^n$ is the *parameter* that affects the solution of the optimization problem, $\mathbf{w} \in \mathbb{R}^t$ is the *uncertain data* and $\mathcal{W} \subset \mathbb{R}^t$ is the *uncertainty set*. The reason for including N is to be consistent with the notation in subsequent sections; the role of N will become clearer. Note that if H, G and Y are zero and c is non-zero, then (1) is an *uncertain parametric linear program* (UPLP) [4, 22].

Assumption 1. $\mathcal{W}$ is a polytope (i.e. a bounded, closed and convex polyhedron) containing the origin and is given by a finite number of affine inequalities. The set of parameters for which there exists a robustly feasible decision variable

$$X_N^{\mathbf{v}} := \left\{x \in \mathbb{R}^n \mid \mathcal{C}_N(x) \neq \emptyset\right\} \tag{2}$$

is non-empty and compact. The set $\mathcal{C}_N(x)$ is compact for all $x \in X_N^{\mathbf{v}}$. If $c = 0$, then H and Y are positive definite. If $c \neq 0$, then H, G and Y are zero.

Remark 1. Note that [3, 4, 5, 22] do not consider *parametric* programs explicitly in their formulation. However, it is trivial to extend the relevant ideas for standard uncertain programs to uncertain parametric programs. Note also that even though the uncertainty $\mathbf{w}$ does not enter the cost in the class of UPQPs considered here, it is still possible to formulate certain classes of min-max problems in the form of (1) [3, 4, 15, 22]. However, in the sequel we will be restricting our attention to optimal control problems in which the uncertainty only enters via the constraints.

Remark 2. The notation used above is slightly inconsistent with conventional notation used in the parametric programming literature, where x is conventionally used to denote the decision variable and another variable is used to denote the parameter. However, it is consistent with conventional notation used in control theory, where x often denotes the current state of the system. Since the main aim of this paper is to show how results in robust optimization can be used to efficiently solve a class of control problems, the current state x will be treated as the parameter of the UPQP that results from formulating the associated control problem; in the sequel $\mathbf{w}$ will denote a sequence of disturbances that acts on the system over a finite horizon of length N and $\mathbf{v}^*(\cdot)$ will be used to define a control policy.

For a given x, the optimization problem defined in (1) can be seen to be a semi-infinite optimization problem, where there is a finite number of decision variables, but an infinite number of constraints. The rest of this section will show how one can efficiently convert the optimization problem in (1) into an optimization problem with a finite number of decision variables and affine inequality constraints.

Since $\mathcal{W}$ is assumed to be given by a finite number of affine inequalities, the following well-known result implies that, by solving q LPs (recall that q is the number of rows in L, M, S and components of b), one can obtain an expression for $\mathcal{C}_N(x)$ in terms of q affine inequalities:

Proposition 1 (Robustly feasible set). *If $\mathcal{C}_N(x)$ is given by* (1b), *then*

$$\mathcal{C}_N(x) = \left\{\mathbf{v} \in \mathbb{R}^d \;\middle|\; L\mathbf{v} \le b + Mx + \operatorname{vec}\min_{\mathbf{w}\in\mathcal{W}} S\mathbf{w}\right\}. \tag{3}$$

Proof. Note that one can interchange the order of the universal quantifiers to get that

$$\begin{aligned}\mathcal{C}_N(x) &= \left\{\mathbf{v} \in \mathbb{R}^d \;\middle|\; \begin{array}{r} e_i'(L\mathbf{v} - b - Mx - S\mathbf{w}) \le 0, \\ \forall i \in \{1,\dots,q\}, \forall \mathbf{w} \in \mathcal{W} \end{array}\right\} \\ &= \left\{\mathbf{v} \in \mathbb{R}^d \;\middle|\; \begin{array}{r} e_i'(L\mathbf{v} - b - Mx - S\mathbf{w}) \le 0, \\ \forall \mathbf{w} \in \mathcal{W}, \forall i \in \{1,\dots,q\} \end{array}\right\}.\end{aligned}$$

For any continuous scalar function $\psi_i : \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^t \to \mathbb{R}$, $\psi_i(\mathbf{v}, x, \mathbf{w}) \le 0, \forall \mathbf{w} \in \mathcal{W} \Leftrightarrow \max_{\mathbf{w}\in\mathcal{W}} \psi_i(\mathbf{v}, x, \mathbf{w}) \le 0$. The result follows from letting $\psi_i(\mathbf{v}, x, \mathbf{w}) := e_i'(L\mathbf{v}-b-Mx-S\mathbf{w})$ for all $i \in \{1,\dots,q\}$ and noting that $\max_{\mathbf{w}\in\mathcal{W}}(e_i'L\mathbf{v} - e_i'b - e_i'Mx - e_i'S\mathbf{w}) = e_i'L\mathbf{v} - e_i'b - e_i'Mx + \max_{\mathbf{w}\in\mathcal{W}}(-e_i'S\mathbf{w}) = e_i'L\mathbf{v} - e_i'b - e_i'Mx - \min_{\mathbf{w}\in\mathcal{W}}(e_i'S\mathbf{w})$. □

Remark 3. Proposition 1 implies that solving the uncertain problem (1) is only slightly more involved than solving the equivalent *certain* problem (in other words, when $\mathcal{W} := \{0\}$ and there is no uncertainty in the data); the number of decision variables and the number of constraints are the same. Note also that the q LPs that define $\operatorname{vec}\min_{\mathbf{w}\in\mathcal{W}} S\mathbf{w}$ are independent of the parameter x and therefore do not need to be solved for different values of x.

If $\mathcal{W}$ is given by simple upper and lower bounds only (which is often the case in practical applications), then the use of an LP solver in obtaining a computationally tractable expression for $\mathcal{C}_N(x)$ is not necessary. This is because $\operatorname{vec}\min_{\mathbf{w}\in\mathcal{W}} S\mathbf{w}$ can be computed far more efficiently by computing the 1-norms of each of the rows of S. To see why this is the case, recall the following result:

Proposition 2 (LP with box constraints). *If the vector* $a \in \mathbb{R}^t$ *and* $\mathcal{W}$ *is a hypercube given by*

$$\mathcal{W} := \left\{ \mathbf{w} \in \mathbb{R}^t \mid \|\mathbf{w}\|_\infty \leq \eta \right\}, \tag{4}$$

where η *is a positive scalar, then* $\min_{\mathbf{w}\in\mathcal{W}} a'\mathbf{w} = -\eta\|a\|_1$ *and* $\max_{\mathbf{w}\in\mathcal{W}} a'\mathbf{w} = \eta\|a\|_1$.

Proof. Note that the constraints on component $\mathbf{w}_i$ of $\mathbf{w}$ are independent of constraints on the other components $\mathbf{w}_j$, $i \neq j$. In other words $\mathcal{W} = \left\{ \mathbf{w} \in \mathbb{R}^t \mid |\mathbf{w}_i| \leq \eta, i = 1, \ldots, t \right\}$. It then follows that $\max_{\mathbf{w}\in\mathcal{W}} a'\mathbf{w} = \max_{\mathbf{w}\in\mathcal{W}} \sum_{i=1}^t a_i\mathbf{w}_i = \sum_{i=1}^t \max_{|\mathbf{w}_i|\leq\eta} a_i\mathbf{w}_i = \sum_{i=1}^t \max\{\eta a_i, -\eta a_i\} = \sum_{i=1}^t \eta|a_i| = \eta\|a\|_1$. The proof is completed by recalling that $\min_{\mathbf{w}\in\mathcal{W}} a'\mathbf{w} = -\max_{\mathbf{w}\in\mathcal{W}}(-a'\mathbf{w})$. □

From Propositions 1 and 2 it follows that:

Corollary 1 (Uncertainty set is a hypercube). *If* $\mathcal{W}$ *is given by* (4) *and* $\mathcal{C}_N(x)$ *is defined in* (1b)*, then one need only compute the* 1*-norms of each of the rows of* S *in order to generate the* q *affine inequalities in* (3)*, i.e.*

$$\mathcal{C}_N(x) = \left\{ \mathbf{v} \in \mathbb{R}^d \mid L\mathbf{v} \leq b + Mx - \eta|S|\mathbf{1} \right\}. \tag{5}$$

Note that $|S|\mathbf{1}$ *is a column vector formed by stacking the* 1*-norms of the rows of* S *on top of each other.*

Remark 4. Proposition 2 and Corollary 1 can be stated without loss of generality. If $\mathcal{W}$ is not a hypercube, but a hyper-rectangle given by asymmetric upper and lower bounds on the components of $\mathbf{w}$, then it is possible to redefine S, b and $\mathcal{W}$ in (1b) via a suitable mapping. Similarly, it is trivial extending the result to the more general case when $\mathcal{W}$ is the affine mapping of a hypercube and the mapping is known, i.e. if $\mathcal{W}$ is the translation of a scaled, rotated and/or projected hypercube. For example, if $\mathcal{W} := \left\{ T\tilde{\mathbf{w}} + y \mid \|\tilde{\mathbf{w}}\|_\infty \leq \eta \right\}$, then $\min_{\mathbf{w}} \left\{ a'\mathbf{w} \mid \mathbf{w} \in \mathcal{W} \right\} = \min_{\tilde{\mathbf{w}}} \left\{ a'(T\tilde{\mathbf{w}} + y) \mid \|\tilde{\mathbf{w}}\|_\infty \leq \eta \right\} = -\eta\|T'a\|_1 + a'y$.

Once (1) has been converted into an equivalent finite-dimensional optimization problem by replacing (1b) with (3) or (5), one can compute a solution in a number of standard ways. One way is: given x, compute $V_N^*(x)$ and $\mathbf{v}^*(x)$ using off-the-shelf LP or QP solvers. Alternatively, one can obtain explicit expressions for the functions $V_N^*(\cdot)$ and $\mathbf{v}^*(\cdot)$ using parametric programming techniques [2, 6].

Lemma 1. *If Assumption 1 holds and* (1b) *is replaced with* (3) *or* (5)*, then the value function* $V_N^* : X_N^{\mathbf{v}} \to \mathbb{R}$ *is Lipschitz continuous on* $X_N^{\mathbf{v}}$*. Furthermore, if* H *and* Y *are positive definite, then the minimizing function* $\mathbf{v}^* : X_N^{\mathbf{v}} \to \mathbb{R}^d$ *is also Lipschitz continuous on* $X_N^{\mathbf{v}}$.

Proof. Immediate from the results in [2, 6], where it is shown that $V_N^*(\cdot)$ is a convex, piecewise quadratic (piecewise affine if c is non-zero and H, G and Y are zero) function and $\mathbf{v}^*(\cdot)$ is a continuous, piecewise affine function if $c = 0$ and H and Y are positive definite. Note that, contrary to [8], [2, 6] do not require that the solution of the dual of (1), with (1b) replaced by (3) or (5), be unique. □

3 Input-to-State Stability

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{K}$-function if it is strictly increasing and $\alpha(0) = 0$; it is a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and $\alpha(z) \to \infty$ as $z \to \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$-function if, for each $k \geq 0$, the function $\beta(\cdot, k)$ is a $\mathcal{K}$-function and, for each $z \geq 0$, the function $\beta(z, \cdot)$ is decreasing and $\beta(z, k) \to 0$ as $k \to \infty$.

Consider now the following nonlinear, discrete-time system

$$x(k+1) = f(x(k), w(k)), \tag{6}$$

where $x \in \mathbb{R}^n$ is the state and $w \in \mathbb{R}^r$ is a disturbance that takes on values in a compact set $W \subset \mathbb{R}^r$ containing the origin. Given the state x at time 0 (note that since the system is time-invariant, we can always regard current time as zero) and a disturbance sequence $w(\cdot)$, let $\phi(k, x, w(\cdot))$ denote the solution to (6) at time instant k and let $\mathcal{M}_W$ denote the set of infinite disturbance sequences taking values in W.

It is assumed that the state is measured at each time instant, that $f(0, 0) = 0$ but that $f(0, W) \neq \{0\}$; in other words, the origin is an equilibrium for the undisturbed system, but it is not a common fixed point for all $w \in W$. For systems of this type, a useful notion of stability is input-to-state stability [9, 10, 16, 21]. We therefore introduce the following definition, which is a slight modification of the one in [10], in order to have a *regional* definition of input-to-state stability:

Definition 1 (ISS). System (6) is *(regionally) input-to-state stable* (ISS) in a set $\mathcal{X} \subset \mathbb{R}^n$ containing the origin in its interior if a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ exist such that, for all $x \in \mathcal{X}$ and all $w(\cdot) \in \mathcal{M}_W$, the solution of (6) satisfies $\phi(k, x, w(\cdot)) \in \mathcal{X}$ and

$$\|\phi(k, x, w(\cdot))\| \leq \beta(\|x\|, k) + \gamma\left(\sup_{\tau\in\{0,\ldots,k-1\}} \|w(\tau)\| \right) \tag{7}$$

for all $k \in \mathbb{N}$. If $\mathcal{X} = \mathbb{R}^n$, then we say that system (6) is *globally* ISS.

Remark 5. It can be seen from (7) that the ISS property implies that the origin is an asymptotically stable fixed point of the undisturbed system $x(k+1) = f(x(k), 0)$ with region of attraction $\mathcal{X}$. Note also that ISS implies that all trajectories of (6) are bounded for all bounded disturbance sequences and that (6) is "converging-disturbance converging-state", i.e. every trajectory $\phi(k, x, w(\cdot)) \to 0$ if $w(k) \to 0$ as $k \to \infty$. Furthermore, the bound on the state trajectory is proportional to the bounds on the disturbance and initial condition. See also [10] and the extensive literature on ISS [11, 21] for more results and interpretations of input-to-state stability.

In order to be self-contained, we present the following result:

Lemma 2. *Let* $\mathcal{X}$ *contain the origin in its interior and be a robustly positively invariant set for system* (6)*, i.e.* $f(x, w) \in \mathcal{X}$ *for all* $x \in \mathcal{X}$ *and all* $w \in W$*. If there exist* $\mathcal{K}_\infty$*-functions* α_1*,* α_2*,* α_3*, a* $\mathcal{K}$*-function* σ *and a continuous function* $V : \mathcal{X} \to$

$\mathbb{R}_{\geq 0}$, *such that, for all* $x \in \mathcal{X}$ *and all* $w \in W$ *the following holds:*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{8a}$$
$$V(f(x,w)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|w\|), \tag{8b}$$

then system (6) *is ISS in* $\mathcal{X}$.

Proof. See [10, Lem. 3.5]. □

Remark 6. A $V(\cdot)$ that satisfies (8) is often referred to as an *ISS-Lyapunov function*.

We conclude this section with the following result, which, to the best of our knowledge, is new:

Lemma 3. *Let* $\mathcal{X}$ *contain the origin in its interior and be a robustly positively invariant set for system* (6)*, i.e.* $f(x,w) \in \mathcal{X}$ *for all* $x \in \mathcal{X}$ *and all* $w \in W$*. If* $f : \mathcal{X} \times W \to \mathbb{R}^n$ *is Lipschitz continuous on* $\mathcal{X} \times W$ *and there exist* $\mathcal{K}_\infty$*-functions* α_1, α_2 *and* α_3 *and a function* $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ *that is Lipschitz continuous on* $\mathcal{X}$ *such that, for all* $x \in \mathcal{X}$ *the following holds:*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{9a}$$
$$V(f(x,0)) - V(x) \leq -\alpha_3(\|x\|), \tag{9b}$$

then $V(\cdot)$ *is an ISS-Lyapunov function and system* (6) *is ISS in* $\mathcal{X}$.

Proof. Since $\|V(f(x,w)) - V(f(x,0))\| \leq L_V \|f(x,w) - f(x,0)\| \leq L_V L_f \|w\|$, where L_V and L_f are the Lipschitz constants of $V(\cdot)$ and $f(\cdot)$, respectively, it follows that $V(f(x,w)) - V(x) = V(f(x,0)) - V(x) + V(f(x,w)) - V(f(x,0)) \leq -\alpha_3(\|x\|) + L_V L_f \|w\|$. The proof is completed by letting $\sigma(z) := L_V L_f z$ in Lemma 2. □

Remark 7. Note that, if $\mathcal{X}$ in Lemmas 2 and 3 is compact, then the condition that α_1, α_2 and α_3 be of class $\mathcal{K}_\infty$ can be relaxed to the condition that they only be of class $\mathcal{K}$.

Remark 8. Lemma 3 can be interpreted as a discrete-time analogue of [11, Lem. 4.6]. If the conditions of Lemma 3 are satisfied, then the origin is also an exponentially stable fixed point of the undisturbed system $x(k+1) = f(x(k), 0)$ with a region of attraction $\mathcal{X}$. Many robust stability results, similar to those in [10], can be derived based on this fact [11, 20].

4 ISS Receding Horizon Control

Consider the LTI discrete-time system

$$x(k+1) = Ax(k) + Bu(k) + Ew(k), \tag{10}$$

where the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $E \in \mathbb{R}^{n\times r}$, the state $x \in \mathbb{R}^n$, the control input $u \in \mathbb{R}^m$ and the disturbance $w \in \mathbb{R}^r$.

Assumption 2. The pair (A, B) is stabilizable and measurements of the state are available. The disturbance is unknown, persistent, but only takes on values in a polytope $W \subset \mathbb{R}^r$ containing the origin.

The goal is to design a time-invariant, state feedback, receding horizon control (RHC) policy $u = \kappa_N(x)$ such that the closed-loop system (6) with $f(x,w) := Ax + B\kappa_N(x) + Ew$ is ISS and the following mixed constraints on the state and input are satisfied for all disturbance sequences $w(\cdot) \in \mathcal{M}_W$:

$$Cx(k) + Du(k) \leq g, \ \forall k \in \mathbb{N}, \tag{11}$$

where $C \in \mathbb{R}^{s\times n}$, $D \in \mathbb{R}^{s\times m}$, $g \in \mathbb{R}^s$ and s is the number of constraints. Note that if the constraints are not mixed, but are only simple upper and lower bounds on the state and input, then $s = 2(m+n)$.

Assumption 3. The set $\mathcal{Z} := \{(x,u) \mid Cx + Du \leq g\}$ is nonempty, compact and contains the origin in its interior.

We follow the same useful idea as in [1, 7, 14, 15] of pre-stabilizing (10) with a linear state feedback gain and optimizing over a sequence of perturbations to this control law. A gain $K \in \mathbb{R}^{m\times n}$ is chosen such that the eigenvalues of $A + BK$ are strictly inside the unit disk and

$$u(k) = Kx(k) + v(k), \tag{12}$$

where $v \in \mathbb{R}^m$ is the input perturbation, is substituted into (10) to get the system

$$x(k+1) = A_K x(k) + Bv(k) + Ew(k), \tag{13}$$

where $A_K := A + BK$. With a slight abuse of earlier notation, let $\phi(k, x, \mathbf{v}, \mathbf{w})$ denote the solution of (13) at time k if the initial state is x at time 0 and $\mathbf{v} := [v_0' \ \cdots \ v_{N-1}']' \in \mathbb{R}^{Nm}$ and $\mathbf{w} := [w_0' \ \cdots \ w_{N-1}']' \in \mathbb{R}^{Nr}$ are, respectively, sequences of input perturbations and disturbances over the horizon $k = 0, \ldots, N-1$, i.e.

$$\phi(k, x, \mathbf{v}, \mathbf{w}) := A_K^k x + \sum_{i=0}^{k-1} A_K^i (Bv_{k-1-i} + Ew_{k-1-i}) \tag{14}$$

for all $k \in \{1, \ldots, N\}$ with $\phi(0, x, \mathbf{v}, \mathbf{w}) := x$.

Let $\mathcal{W} := W^N := \overbrace{W \times \cdots \times W}^{N \text{ times}}$, $t := Nr$, $d := Nm$ and the robust finite horizon optimal control problem (RFHOCP) be defined as:

$$V_N^*(x) := \min_{\mathbf{v} \in \mathcal{C}_N(x)} \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k + x_N' P x_N, \tag{15a}$$

where $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ and $P \in \mathbb{R}^{n\times n}$ are positive definite, $x_k := \phi(k, x, \mathbf{v}, 0)$, $u_k := K\phi(k, x, \mathbf{v}, 0) + v_k$ for all $k \in \{0, \ldots, N-1\}$ and $x_N := \phi(N, x, \mathbf{v}, 0)$. The set of robustly feasible input perturbation sequences of length N is

$$\mathcal{C}_N(x) := \left\{ \mathbf{v} \in \mathbb{R}^d \;\middle|\; \begin{array}{l} C\phi(k, x, \mathbf{v}, \mathbf{w}) + Du_k \leq g, \\ u_k = K\phi(k, x, \mathbf{v}, \mathbf{w}) + v_k, \\ k = 0, \ldots, N-1, \\ \phi(N, x, \mathbf{v}, \mathbf{w}) \in X_f, \ \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}. \tag{15b}$$

The terminal constraint set $X_f \subset \mathbb{R}^n$ is given by p affine inequality constraints, and will be defined below. The minimizer in (15a) is

$$\begin{aligned}\mathbf{v}^*(x) &:= [v_0^*(x)' \ \cdots \ v_{N-1}^*(x)']' \\ &:= \arg\min_{\mathbf{v}\in\mathcal{C}_N(x)} \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k + x_N' P x_N.\end{aligned} \tag{15c}$$

Remark 9. Note that the *cost* in (15a) and (15c) assumes that the disturbance sequence is zero. However, it is still required that all the *constraints* in (15b) be satisfied for all finite disturbance sequences $\mathbf{w} \in \mathcal{W}$. The motivation often given for not including the disturbance in the cost is that, in practice, the model is usually a good estimate of the true plant; minimizing the worst-case cost over all disturbance sequences may result in an unnecessarily conservative control action. As will be shown below, one can still define a receding horizon controller that makes the closed-loop system ISS, despite not having included the disturbance in the cost function of the RFHOCP.

It is straightforward, by substituting (14) and the p affine inequalities defining X_f into (15), to derive an H, G, Y, L, M, S, b and c such that the RFHOCP (15) is equivalent to the UPQP (1) with H and Y being positive definite and $c = 0$. One can then efficiently compute equivalent expressions for $\mathcal{C}_N(x)$ as in (3) and (5), where the number of affine inequality constraints is $q = sN + p$.

Before proceeding, note that the set of states for which a solution to the RFHOCP (15) exists is the compact set $X_N^{\mathbf{v}}$ defined in (2) (assuming $X_N^{\mathbf{v}}$ is non-empty). The receding horizon control law $\kappa_N : X_N^{\mathbf{v}} \to \mathbb{R}^m$ can now be defined in the usual manner [19] by the first component in the minimizer (15c) of the RFHOCP:

$$\kappa_N(x) := Kx + v_0^*(x), \ \forall x \in X_N^{\mathbf{v}}. \tag{16}$$

Since the control $u = \kappa_N(x)$ adopts a receding horizon policy, additional assumptions on the RFHOCP are needed in order to guarantee that the closed-loop system

$$x(k+1) = Ax(k) + B\kappa_N(x(k)) + Ew(k) \tag{17}$$

is ISS in $X_N^{\mathbf{v}}$ and that the constraints (11) are satisfied for all time and for all disturbance sequences.

Assumption 4. The terminal constraint set X_f in (15b) is a (non-empty) polytope containing the origin in its interior, is *robustly positively invariant* for the *disturbed* closed-loop system $x(k+1) = (A + BK)x(k) + Ew(k)$, i.e.

$$(A + BK)x + Ew \in X_f, \ \forall x \in X_f, \forall w \in W, \tag{18}$$

and $X_N^{\mathbf{v}}$ is contained inside the set

$$X_K := \left\{ x \in \mathbb{R}^n \mid (C + DK)x \leq g \right\}. \tag{19}$$

Remark 10. For methods of computing an X_f that satisfies (18), see [12]. In particular, [12] shows how one can compute X_f such that it is the *maximal* positively invariant set contained in X_K.

The above assumption allows us to state our first main result:

Theorem 1. *If Assumptions 2–4 hold, then the set of states* $X_N^{\mathbf{v}}$ *for which a robustly feasible input perturbation sequence exists is non-empty, contains the origin in its interior and is robustly positively invariant set for the closed-loop system* (17)*, i.e.*

$$Ax + B\kappa_N(x) + Ew \in X_N^{\mathbf{v}}, \ \forall x \in X_N^{\mathbf{v}}, \forall w \in W. \tag{20}$$

Furthermore, an increase in the horizon length N does not result in a decrease in the size of $X_N^{\mathbf{v}}$*, i.e.*

$$X_f \subseteq X_1^{\mathbf{v}} \subseteq \cdots \subseteq X_{N-1}^{\mathbf{v}} \subseteq X_N^{\mathbf{v}}. \tag{21}$$

Proof. Proving (20) follows the standard procedure [7, 15, 19] of showing that the *shifted* perturbation sequence $\tilde{\mathbf{v}}(x) := [v_1^*(x)' \ \cdots \ v_{N-1}^*(x)' \ 0']'$ is a feasible perturbation sequence contained in $\mathcal{C}_N(Ax + B\kappa_N(x) + Ew)$ if $x \in X_N^{\mathbf{v}}$ and $w \in W$. Proving (21) is done by induction and follows similar arguments as in proving (20), but showing instead that the *appended* perturbation sequence $\hat{\mathbf{v}}(x) := [v_0^*(x)' \ v_1^*(x)' \ \cdots \ v_{N-1}^*(x)' \ 0']'$ is a feasible perturbation sequence contained in $\mathcal{C}_{N+1}(Ax + B\kappa_N(x) + Ew)$ if $x \in X_N^{\mathbf{v}}$ and $w \in W$. □

The next assumption allows us to use the value function $V_N^*(\cdot)$ as an ISS-Lyapunov function for (17):

Assumption 5. The feedback gain K and positive definite matrix P is chosen such that $F(x) := x'Px$ is a control Lyapunov function in X_f for the *undisturbed* closed-loop system $x(k+1) = (A + BK)x(k)$, i.e.

$$F((A+BK)x) - F(x) \leq -x'(Q+K'RK)x, \ \forall x \in X_f. \tag{22}$$

Remark 11. Clearly, a K and P that satisfy (22) are easily obtained by solving the unconstrained infinite horizon LQR problem with weights Q and R, i.e. $K = -(R + B'PB)^{-1}B'PA$ and $P = (A + BK)'P(A + BK) + K'RK + Q$. Alternatively, if A is strictly stable, then one can set $K = 0$ and let P be the solution of the Lyapunov equation $P = A'PA + Q$.

We can now state our final, main result:

Theorem 2. *If Assumptions 2–5 hold and the receding horizon control policy* $u = \kappa_N(x)$ *is given by* (16)*, then the closed-loop system* (17) *is ISS in* $X_N^{\mathbf{v}}$ *and the constraints* (11) *are satisfied for all time and for all disturbance sequences* $w(\cdot) \in \mathcal{M}_W$.

Proof. Let $f(x, w) := Ax + B\kappa_N(x) + Ew$. Recall from Theorem 1 that $X_N^{\mathbf{v}}$ is robustly positively invariant, hence the constraints are satisfied for all time and all $w(\cdot) \in \mathcal{M}_W$. It follows from Lemma 1 that V_N^* and v_0^* are Lipschitz continuous on $X_N^{\mathbf{v}}$, hence f is Lipschitz continuous on $X_N^{\mathbf{v}} \times W$. Since V_N^* is a continuous, positive definite function and $0 \in \text{int}\left(X_N^{\mathbf{v}}\right)$, there exist $\mathcal{K}$-functions α_1 and α_2 such that (9a) holds for V_N^* [11, Lem. 4.3]. Using standard arguments [18, 19] one can show that V_N^* is a Lyapunov function for the *undisturbed* closed-loop system $x(k+1) = Ax(k) + B\kappa_N(x(k))$, hence V_N^* satisfies (9b) with $\alpha_3(z) := \lambda_{\min}(Q)z^2$. It follows from Lemma 3 that (17) is ISS. Note that since $X_N^{\mathbf{v}}$ is compact, it is sufficient to require α_1 and α_2 to be of class $\mathcal{K}$, rather than $\mathcal{K}_\infty$. □

Remark 12. Similar results to Theorem 2 have been obtained for the receding horizon control of linear discrete-time systems [9] and nonlinear discrete-time systems [16]. The results presented in this paper and the techniques used for proving them are stronger than those in [9, 16]. For example, we do not require V_N^* to be continuously differentiable, as in [9]; continuous differentiability of requires additional assumptions on (1) [6]. In contrast to [16], since we are dealing with linear, rather than nonlinear systems, we can guarantee the inclusion property in (21) and that an exact expression for $\mathcal{C}_N(x)$ can be computed.

5 Conclusions

Section 2 presented a tractable method for computing the solution to a very specific class of uncertain parametric QPs and LPs. It was shown that, in order to turn the problem from a semi-infinite optimization problem to a finite dimensional optimization problem, one need only solve a small number of LPs or, if the uncertainty set is given by upper and lower bounds, only compute the 1-norms of the rows of the matrix by which the uncertainty enters the constraints. Once this has been done, one can find the solution to the robust optimization problem using standard QP or LP solvers. It was also shown that, under some mild assumptions, the solution to the problem is Lipschitz in the parameter.

Section 3 showed that if a nonlinear, discrete-time system is Lipschitz in the state and the disturbance and the origin is exponentially stable for the undisturbed system, then the disturbed system is input-to-state stable.

Section 4 showed that one can use the results of Section 2 to efficiently translate a class of robust finite-horizon optimal control problems (RFHOCPs) into a finite dimensional parametric QP. The resulting optimization problem was shown to have the same number of constraints and decision variables as the optimal control problem with no disturbance and that the number of decision variables and constraints increases only linearly with the horizon length N. Under some mild assumptions, it was shown that one can use the solution of the RFHOCP to define a robust receding horizon controller that makes the closed-loop system ISS and guarantees robust constraint satisfaction.

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