On Reduced-Order H^∞ Filtering for Nonlinear Systems with Sampled Measurements

YEN-FANG LI*, **CHEE-FAI YUNG**† and **HSIN-TENG SHEU**‡
*Department of Electrical Engineering
Ming Hsin University of Science and Technology
Hsin-Chu, Taiwan
E-mail : yfli@must.edu.tw
Tel/Fax : +886-3-4642794.
†Department of Electrical Engineering
National Taiwan Ocean University
Keelung, Taiwan
E-mail : yung@mail.ntou.edu.tw
Tel/Fax : +886-3-24626993.
‡Department of Electrical Engineering
National Taiwan University of Science and Technology
Taipei, Taiwan

Keywords: H^∞ filtering, Reduced-order filters, Sampled-data systems, Nonlinear systems, Differential game.

Abstract

This paper deals with the reduced-order H^∞ filtering problem for nonlinear continuous-time systems with sampled measurements. Using the concepts of dissipativity and differential game, sufficient conditions are derived for the existence of filters that satisfy a specified H^∞ performance bound. These conditions are expressed in terms of the solution of a differential Hamilton-Jacobi inequality with jumps. This differential Hamilton-Jacobi inequality is exactly the one used in the construction of full-order H^∞ filters. When these conditions hold, state-space formulae are also given for such filters.

I. Introduction

The celebrated Kalman filter is the optimal state estimator that minimizes either the average root-mean-square power of the estimation error or the variance of the terminal state estimation error, where the signal generating system is assumed to be driven by a white noise process and the measured output is also assumed to be corrupted by a white noise process, both with known statistical properties, while the purpose of an H^∞ filter is to ensure that the L^2-energy gain from the disturbance, which is assumed to be unknown deterministic but of finite energy, to the estimation error is less than a prespecified level. In contrast to the traditional Kalman filter, an H^∞ filter has some practical advantages. First, it does not require knowledge of the statistical properties of the noise; instead, the only requirement for the noise is that it has bounded energy. Consequently, H^∞ filters are less sensitive to the noise uncertainty. Second, it is more robust than Kalman filter to the unmodeled uncertainties of signal systems.

In this paper, we address the problem of filtering for nonlinear systems with sampled measurements in an H^∞ setting. This problem is to estimate the states of a continuous-time system using only sampled measurements at discrete instants of time. Motivation of studying this problem comes from the fact that in many practical situations the underlying plant is continuous-time while the measurements are usually taken only at discrete-time instants, and virtually all physical systems are nonlinear in nature. Moreover, such systems are important in practice because of the widespread use of digital computers in implementation. Typically, sampled-data filtering is designed either by discretizing an analog design by, e.g., the bilinear transformation, or directly through their discrete-time behavior by the use of the modified Z-transform. However, many performances, such as disturbance and noise attenuation, overshoot, require a deeper insight into the intersampling behavior. The H^∞ performance criterion is defined directly in terms of the continuous-time signals and thus intersampling behavior is taken into account. Our goal is to design a filter that achieves a given bound on the ratio between the L^2-energy of a given function of the estimation error and the L^2-energy of the disturbances that consist of the continuous-time process noise and the discrete-time measurement noise.

The H^∞ filtering problem was first addressed by Elsayed and Grimble [4] and Grimble [7] for the scalar case and later generalized by Grimble et al.[8] to the multivariable case. A game theoretic approach has been given by Yaesh and Shaked [15] and a state-space approach has been offered by Nagpal and Khargoneker

[12] for the linear continuous-time system, and Sun et al.[14] for the linear sampled-data case.

On the other hand, using a game theoretic approach, the nonlinear H^∞ filtering problems have been extensively studied by Berman and Shaked [1], Fridman and Shaked [5], and Yung et al.[19] for continuous-time systems, and Li et al.[11] for sampled-data systems.

The filters obtained from the aforementioned papers have a state dimension greater than or equal to that of the system model which is built from the physical plant and some of its weighting functions. This limits the use of full-order filters in practical applications, since a high order filter usually results in high implementation cost and tends to be numerically ill conditioned. Reduced-order filters, i.e., filters of order lower than the order of system, are often desirable to reduce the complexity and computational burden of the real-time filtering process. For this reason, the reduced-order filter design is very important and necessary, especially when fast data processing is desired or when the estimation of small number of state variables are actually required.

Recently, a number of papers have dealt with reduced-order (or fixed-order) H^∞ filtering problem. Utilizing Bounded Real Lemma approach, Bernstein et al.[2], Bettayeb et al.[3], Hsu et al.[9], Kim et al.[10], and Yu et al.[17] tackled the reduced-order H^∞ filtering problem for the continuous-time time-invariant systems. Yu and Hsu [16], using Bounded Real Lemma method, settled the reduced-order H^∞ filtering problem for the continuous-time time-varying systems. Grigoriadis et al.[6] have also addressed this filtering problem for the continuous- and discrete-time time-invariant systems by linear matrix inequality(LMI) method. In addition, Rawson et al.[13] and Yu and Hsu [18] have solved this filtering problem for the discrete-time time-varying systems. In these papers, the problem was studied only for linear systems. To the best of our knowledge, no discussion has yet been given on how to design reduced-order H^∞ filters for nonlinear systems.

The purpose of this study is to continue this line of research to address the problem of H^∞ filtering design for nonlinear sampled-data systems [11]. By extending the technique developed by Yung and Wang [20] for nonlinear H^∞ controller reduction, we present sufficient conditions for the existence of an H^∞ filter with a state dimension less than that of the plant. The conditions obtained are expressed in terms of the solution to a Hamilton-Jacobi inequality in only $n+1$ independent variables. The Hamilton-Jacobi inequality is exactly the one used in the construction of the full-order H^∞ filters obtained in [11].

II. Problem Formulation and Preliminaries

Consider a time-invariant nonlinear sampled-data system described by the dynamic equations

$$\dot{x}(t) = f(x) + k_1(x)w(t), \qquad (1)$$
$$y(kT) = h_1(x(kT)) + v(kT), \quad k = 0, 1, 2, ..., \qquad (2)$$

where x represents the state defined on a neighborhood of the origin in $I\!R^n$, and $w \in I\!R^m$ represents a continuous-time process noise which is assumed to be a member of $L^2[0, \Gamma, I\!R^m] := \{w : \|w\|^2_{L^2} := \int_0^\Gamma \|w(t)\|^2 dt < \infty$ for a fixed $\Gamma > 0\}$. Here $\|\cdot\|$ denotes the Euclidean norm. Eq.(2) defines the discrete-time measured variable $y \in I\!R^p$ which is available at sampling instants kT with the sampling period T, and $v \in I\!R^p$ represents measurement noise which is assumed to be a member of $l^2[0, \Gamma, I\!R^p] := \{v : \|v\|^2_{l^2} := \sum_{k=0}^{\lfloor \Gamma/T \rfloor} \|v(kT)\|^2 < \infty$ for a fixed $\Gamma > 0\}$. Here $\lfloor e \rfloor$ denotes the integer part of $e \in I\!R$. We assume that f, k_1, and h are all smooth functions. We also assume that $f(0) = 0$, and $h_1(0) = 0$.

Our task is to design a causal filter using the output measurement $y(kT)$ to estimate a function of state

$$z(t) = h_2(x(t)), \qquad (3)$$

where $h_2(x)$ is a smooth function in x with $h_2(0) = 0$. Suppose that $\hat{z}(t)$ is the estimate of $z(t)$, and that the estimation error is

$$\varepsilon(t) := z(t) - \hat{z}(t). \qquad (4)$$

The objective of H^∞ filtering design is to ensure that the L^2 gain from the disturbances consisting of the exogenous input w and the measurement noise $v(kT)$ to the estimation error $\varepsilon(t)$ is bounded by a prespecified valve γ, namely

$$\int_0^\Gamma \|\varepsilon(t)\|^2 dt \le \gamma^2 (N + \int_0^\Gamma \|w(t)\|^2 dt + \sum_{k=0}^{\lfloor \Gamma/T \rfloor} \|v(kT)\|^2) \qquad (5)$$

for some positive function N, which is a function of initial states, see below for details. For clarity, we denote $N = N_f$ for full-order filter case and $N = N_r$ for reduced-order filter case.

The following result quoted from [11] provides a full-order H^∞ filter of the form

$$\begin{aligned} \dot{\hat{x}}(t) &= f(\hat{x}(t)), \qquad t \neq kT, \\ \hat{x}(kT) &= \hat{x}(kT^-) + g(\hat{x}(kT^-))(y(kT) - h_1(\hat{x}(kT^-))), \end{aligned} \qquad (6)$$

with $\hat{x} \in I\!R^n$, that solves the problem in question.

Proposition 1 *: Suppose that there exists a positive definite function $Q(x,t)$ with $Q(x(0) - \hat{x}(0), 0) = \gamma^2 N_f(x(0), \hat{x}(0))$, locally defined on $\Psi_1 \times [0, \Gamma]$ with Ψ_1 a neighborhood of the origin in $I\!R^n$, which is T-periodic, piecewise differentiable with respect to t, and C^3 with respect to x, and is such that the Hessian matrix $Q_{xx}(0,T)$ is nonsingular,*

$$g^T(0) Q_{xx}(0,T) g(0) - 2\gamma^2 I < 0, \qquad (7)$$

and for all $t \in [0,T]$ the function $\bar{K}_1(x,t)$ is negative definite near $x = 0$ with nonsingular Hessian matrix at $x = 0$, where the function $\bar{K}_1(x,t)$ is piecewise continuous defined as

$$\begin{aligned} \bar{K}_1(x,t) &= Q_t(x,t) + Q_x(x,t) f(x) + h_2^T(x) h_2(x) \\ &\quad + \frac{1}{4\gamma^2} Q_x(x,t) k_1(x) k_1(x)^T Q_x^T(x,t), \end{aligned} \qquad (8)$$

for $0 \le t < T$, *and*

$$\bar{K}_1(x,T) = Q(x,T) - Q(x,T^-) - \gamma^2 h_1^T(x)h_1(x). \quad (9)$$

Then the H^∞ *nonlinear sampled-data filtering problem is solved by the full-order filter (6) with*

$$\hat{z}(t) = h_2(\hat{x}(t)),$$

and filter gain $g(x)$ *satisfying*

$$Q_x(x,T)g(x) = 2\gamma^2 h_1^T(x).$$

III. Solution of Reduced-Order Filtering Problem

The reduced-order H^∞ filtering problem considered in this paper involves finding a state estimator of the form

$$\begin{array}{rcll} \dot{\xi}(t) & = & \hat{f}(\xi(t)), & t \neq kT \\ \xi(kT) & = & \xi(kT^-) + \hat{g}(y(kT) - \hat{h}_1(\xi(kT^-))), & \\ \hat{z}(t) & = & \hat{h}_2(\xi(t)), & \end{array} \quad (10)$$

where $\xi \in I\!\!R^r$ $(r \le n)$ is defined on a neighborhood of the origin, with $\hat{f}(0) = 0$, $\hat{h}_1(0) = 0$, and $\hat{h}_2(0) = 0$, such that the estimation error (4) against the discrepancy between x and ξ satisfies the dissipativity inequality (5).

More precisely, we seek a state estimator (10) such that there exists a neighborhood Π of $(x,\xi) = (0,0)$, and for all $(x(0),\xi(0)) \in \Pi$ and for each input $w(\cdot) \in L^2[0,\Gamma,I\!\!R^m]$ and $v(\cdot) \in l^2[0,\Gamma,I\!\!R^p]$, the trajectory $X^e(t) := col(x(t),\xi(t))$ of the augmented system

$$\begin{array}{rcll} \dot{X^e} & = & F^e(X^e) + G(X^e)w(t), & t \neq kT \\ X^e(kT) & = & F_d^e(X^e(kT^-)) + G_d(X^e(kT^-))v(kT), & \end{array} \quad (11)$$

with

$$F^e(X^e) = \begin{bmatrix} f(x) \\ \hat{f}(\xi) \end{bmatrix},$$

$$F_d^e(X^e) = \begin{bmatrix} x \\ \xi + \hat{g}(\xi)(h_1(x) - \hat{h}_1(\xi)) \end{bmatrix},$$

$$G(X^e) = \begin{bmatrix} k_1(x) \\ 0 \end{bmatrix},$$

and

$$G_d(X^e) = \begin{bmatrix} 0 \\ \hat{g}(\xi) \end{bmatrix},$$

remains in Π for all $t \in [0,\Gamma]$, and the dissipativity inequality (5) is satisfied

A preliminary lemma will be needed in the sequel.

Lemma 2 *: Suppose that there exists a smooth function* $\phi : I\!\!R^n \times I\!\!R \to I\!\!R^r$*, locally defined on a neighborhood of the origin* $(x,t) = (0,0)$ *in* $I\!\!R^n \times I\!\!R$*, with* $\phi(0,t) = 0$ *and* $\frac{\partial\phi}{\partial x}(0,t)(\frac{\partial\phi}{\partial x})^T(0,t) = I$*. For a given* $\gamma > 0$ *and the augmented system (11), a sufficient condition for* $\hat{g}(\xi)$ *to solve the* H^∞ *filtering problem is that there exists a function* $V(X^e,t)$*, locally defined on* $\Psi \times [0,\Gamma]$ *with* Ψ *a neighborhood of the origin in* $I\!\!R^{n+r}$*,* C^2 *with respect to* X^e*, T-periodic (i.e.,* $V(X^e,t) = V(X^e,t+T)$ *for all t), and piecewise differentiable with respect to t, which vanishes at* $X^e = col(x,\phi(x,t))$ *for all* $t \in [0,T]$*, is positive elsewhere, satisfies*

$$V(X^e(0),0) = \gamma^2 N_r(x(0),\xi(0)) \quad (12)$$

and is such that the following conditions are satisfied.

(a) *The quantity*

$$G_d^T(0)V_{X^eX^e}(0,T)G_d(0) - 2\gamma^2 I < 0. \quad (13)$$

(b) *The function*

$$\begin{array}{rcl} J^e(X^e,t) & = & (h_2(x) - h_2(\xi))^T(h_2(x) - h_2(\xi)) \\ & & + \frac{1}{4\gamma^2}V_{X^e}G(X^e)G^T(X^e)V_{X^e}^T \\ & & + V_t + V_{X^e}F^e(X^e) \end{array} \quad (14)$$

vanishes at $X^e = col(x,\phi(x,t))$ *and is negative elsewhere for all* $t \in [0,\Gamma]$*, with* $t \neq kT$*.*

(c) *The function*

$$\begin{array}{rcl} J_d^e(X^e) & = & V(F_d^e(X^e,T) + G_d(X^e)v^*,T) \\ & & - V(X^e,T^-) - \gamma^2 {v^*}^T v^* \end{array} \quad (15)$$

is less than or equal to zero, where v^* *is the unique solution with* $v^*(0) = 0$ *of the implicit equation*

$$\frac{\partial V}{\partial \alpha}\,|_{\alpha = F_d^e(X^e) + G_d(X^e)v}\; G_d(X^e) - 2\gamma^2 v^T = 0. \quad (16)$$

Proof. We first observe that the problem in question can be cast as a two players, zero sum, differential game with a value functional

$$\begin{array}{rcl} \Theta(\hat{g},(w,v)) & \triangleq & \|\varepsilon\|_{L^2}^2 - \gamma^2[N_r(x(0),\xi(0)) + \|w\|_{L^2}^2 \\ & & + \|v\|_{l^2}^2], \end{array}$$

where $\|\cdot\|_{L^2}^2$ denotes the L^2-energy on $[0,\Gamma]$, $\|\cdot\|_{l^2}^2$ denotes the l^2-energy on $[0,\Gamma]$, w and v are the maximizers, and $\hat{g}$ is the minimizer. Associated with this differential game setup, we define two Hamiltonian function $\bar{M}_1 : I\!\!R^{n+r} \times I\!\!R^m \times I\!\!R \to I\!\!R$,

$$\begin{array}{rcl} \bar{M}_1(X^e,w,t) & \triangleq & V_t + V_{X^e}(F^e(X^e) + G(X^e)w(t)) \\ & & -\gamma^2 w^T(t)w(t) + \varepsilon^T(t)\varepsilon(t), \end{array} \quad (17)$$

where $t \neq kT$, and $\bar{M}_{1d} : I\!\!R^{n+r} \times I\!\!R^p \to I\!\!R$,

$$\begin{array}{rcl} \bar{M}_{1d}(X^e,v) & \triangleq & V(F_d^e(X^e) + G_d(X^e)v,T) \\ & & -V(X^e,T^-) - \gamma^2 v^T v, \end{array} \quad (18)$$

for $t = kT$. Then it can be shown that $\bar{M}_1(X^e,w,t) \le 0$ and $\bar{M}_{1d}(X^e,v) \le 0$. To see this, observe that $\bar{M}_1$ can be rewritten as

$$\bar{M}_1(X^e,w,t) = J^e(X^e,t) - \gamma^2\|w - w^*\|^2,$$

where $w^* := \frac{1}{2\gamma^2} G^T(X^e) V_{X_e}^T$ is the worst disturbance. Since $J^e(X^e, t) \leq 0$ by hypothesis, we have $\bar{M}_1(X^e, w, t) \leq 0$. Also, a simple calculation shows that

$$\frac{\partial \bar{M}_{1d}(X^e, v)}{\partial v} = \frac{\partial V}{\partial \alpha} \mid_{\alpha = F_d^e(X^e) + G_d(X^e)v} G_d(X^e) - 2\gamma^2 v^T, \tag{19}$$

and

$$\frac{\partial^2 \bar{M}_{1d}(X^e, v)}{\partial v^2} = G_d^T \frac{\partial^2 V}{\partial \alpha^2} \mid_{\alpha = F_d^e(X^e) + G_d(X^e)v} G_d - 2\gamma^2 I. \tag{20}$$

Since the Hessian matrix is nonsingular at $(X^e, v) = (0, 0)$ by hypothesis, there exists a unique solution $v^*(X^e)$, defined on a neighborhood of $X^e = 0$, satisfying

$$\frac{\partial \bar{M}_{1d}(X^e, v)}{\partial v} \mid_{v = v^*} = 0,$$

and

$$v^*(X^e) \mid_{X^e = 0} = 0,$$

provided by the implicit function theorem. Using the Taylor expansion, $\bar{M}_{1d}$ can be expressed as

$$\bar{M}_{1d}(X^e, v) = J_d^e(X^e) + \frac{1}{2} \|v - v^*\|_{r_{11}}^2 + O(\|v - v^*\|^3),$$

where $r_{11} := \frac{\partial^2 \bar{M}_{1d}(X^e, v)}{\partial v^2} \mid_{(X^e, v) = (0,0)}$ is negative definite. Since $J_d^e \leq 0$ by hypothesis, we have $\bar{M}_{1d} \leq 0$.

Combination of (17) and (18) and integration on $[0, \Gamma]$ with the initial condition (12) give

$$V(X^e(\Gamma), \Gamma) - \gamma^2 N_r(x(0), \xi(0)) + \int_0^\Gamma \|\varepsilon(t)\|^2 dt - \gamma^2 \left[\int_0^\Gamma \|w(t)\|^2 dt + \sum_{k=0}^{\lfloor \Gamma/T \rfloor} \|v(kT)\|^2 \right] \leq 0.$$

Since $V(X^e(t), t) \geq 0$, we obtain (5). This completes the proof. □

The filter given above is a nonlinear system with finite jumps at discrete instants of time. At the sampling instants, the measurement $y(kT)$ is used to update the estimate with filter gain $\hat{g}$, and the function J_d^e is only available at sampling instant kT. The combination (14) and (15) can be regarded as a differential Hamiltonian function with jumps.

The conditions in **Lemma 2** can be further simplified by providing an alternative set of sufficient conditions for the solution of this H^∞ filtering problem, which involves a new Hamilton Jacobi inequality having fewer independent variables, without involving the estimation gain $\hat{g}(\xi)$. Also, the "jump condition" is explicitly given. This is summarized in the following statement.

Theorem 3 *: Suppose that the hypotheses of* **Proposition 1** *hold and suppose that there exists a smooth function $\phi : I\!R^n \times I\!R \to I\!R^r$, locally defined on a neighborhood of the origin $x = 0$ in $I\!R^n$, with $\phi(0, t) = 0$ and $\frac{\partial \phi}{\partial x}(0, t)(\frac{\partial \phi}{\partial x})^T(0, t) = I$. Suppose also that there exists a smooth positive definite function $\hat{Q}(\xi, t)$, locally defined on $\Psi_2 \times [0, \Gamma]$ with Ψ_2 a neighborhood of the origin in $I\!R^r$, which is T-periodic, piecewise differentiable with respect to t, C^3 with respect to ξ, and satisfies $\frac{\partial^2 \hat{Q}}{\partial \hat{\xi}^2}(0, t)\frac{\partial \phi}{\partial x}(0, t) = \frac{\partial \phi}{\partial x}(0, t)\frac{\partial^2 Q}{\partial x^2}(0, t)$. Then, if*

$$N_r(x(0), \xi(0)) = N_f(x(0), \hat{x}(0)),$$

and the following conditions are satisfied:

$$\hat{h}_1(\phi(x, t), t) = h_1(x, t), \tag{21}$$

$$\hat{h}_2(\phi(x, t), t) = h_2(x, t), \tag{22}$$

$$\hat{f}(\phi(x, t), t) = \phi_x(x, t) f(x, t) + \phi_t(x, t), \tag{23}$$

$$\hat{g}(\phi(x, t), t) = \phi_x(x, t) g(x, t), \tag{24}$$

then the nonlinear sampled-data H^∞ reduced-order filtering problem is solved by the filter (10).

Proof. Clearly, $V(X^e, t) = \hat{Q}(\xi - \phi(x, t), t)$ is positive definite with respect to $(\xi - \phi(x, t))$, T-periodic, piecewise differentiable with respect to t, and C^3 with respect to $(\xi - \phi(x, t))$. Then we make a change of variable

$$\hat{\xi} = \xi - \phi(x, t), \tag{25}$$

where $\hat{\xi} \in I\!R^r$ and $\xi \in I\!R^r$ are defined on a neighborhood of the origin. Thus, equation (13) can be rewritten as

$$\hat{g}^T(0) \hat{Q}_{\hat{\xi}\hat{\xi}}(0, T) \hat{g}(0) - 2\gamma^2 I < 0. \tag{26}$$

Applying (24) to (26), it can be easily checked that (26) is the same as (7) of **Proposition 1**, thus the condition *(a)* of **Lemma 2** holds.

Furthermore, let

$$J_1(x, \hat{\xi}, t) := J^e(X^e, t) \mid_{\xi = \hat{\xi} + \phi(x,t), V(X^e, t) = \hat{Q}(\hat{\xi}, t)}$$

for all $t \neq 0$. It is straightforward to verify that

$$J_1 \mid_{\hat{\xi} = 0} = 0, \qquad \frac{\partial J_1}{\partial \hat{\xi}} \mid_{\hat{\xi} = 0} = 0,$$

and

$$\begin{aligned} \frac{\partial^2 J_1}{\partial \hat{\xi}^2} = & \frac{1}{2\gamma^2} \hat{Q}_{\hat{\xi}\hat{\xi}}(0, t) \phi_x(0, t) k_1(0) k_1^T(0) \phi_x^T(0, t) \hat{Q}_{\hat{\xi}\hat{\xi}}(0, t) \\ & + \hat{f}_{\hat{\xi}}^T(0, t) \hat{Q}_{\hat{\xi}\hat{\xi}}(0, t) + \hat{Q}_{\hat{\xi}\hat{\xi}}(0, t) \hat{f}_{\hat{\xi}}(0, t) \\ & + 2\hat{h}_{1_{\hat{\xi}}}^T(0, t) \hat{h}_{1_{\hat{\xi}}}(0, t) \end{aligned} \tag{27}$$

at $(x, \hat{\xi}) = (0, 0)$. Moreover, substituting (21)-(24) into (27) gets

$$\begin{aligned} \frac{\partial^2 J_1}{\partial \hat{\xi}^2} = & \phi_x(0, t)(f_x^T(0, t) Q_{xx}(0, t) + Q_{xx}(0, t) f_x(0, t) \\ & + 2h_{1_x}^T(0) h_{1_x}(0)) \phi_x^T(0, t) \\ & + \frac{1}{2\gamma^2} Q_{xx}(0, t) k_1(0) k_1^T(0) Q_{xx}(0, t). \end{aligned} \tag{28}$$

A routine calculation shows that

$$\frac{\partial^2 J_1(x, \hat{\xi}, t)}{\partial \hat{\xi}^2} \mid_{\hat{\xi} = 0, x = 0} = \phi_x(0, t) \frac{\partial^2 \bar{K}_1(x, t)}{\partial x^2} \mid_{x = 0} \phi_x^T(0, t).$$

By Taylor expansion theorem around $(x, \hat{\xi}) = (0, 0)$, we have

$$
\begin{aligned}
J_1(x, \hat{\xi}, t) &= \frac{1}{2}\hat{\xi}^T \frac{\partial^2 J_1(0,0,t)}{\partial \hat{\xi}^2}\hat{\xi} + h.o.t. \\
&= \frac{1}{2}\hat{\xi}^T(\phi_x(0,t) \ \frac{\partial^2 \bar{K}_1}{\partial x^2}|_{x=0} \ \phi_x^T(0,t))\hat{\xi} \\
&\quad + h.o.t.,
\end{aligned}
$$

where "$h.o.t$" means higher order terms and $t \neq 0$. Since $\frac{\partial^2 \bar{K}_1(x,t)}{\partial x^2}|_{x=0}$ is negative definite for all $t \in [0, \Gamma]$ by hypothesis, it is concluded that for all $t \in [0, \Gamma]$ the function $J^e(X^e, t)$ vanishes at $\phi(x,t) = \xi$ and is negative elsewhere. Thus, the condition (b) of **Lemma 2** holds.

At the jump points $t = kT$, a routine manipulation shows that the worst disturbance is equal to

$$v^*_{\hat{\xi}}(0) = \hat{h}_{1_{\hat{\xi}}}(0)$$

by setting the filter gain as

$$\hat{g}^T(0)\hat{Q}_{\hat{\xi}\hat{\xi}}(0, T) = 2\gamma^2 \hat{h}_{1_{\hat{\xi}}}(0).$$

A similar argument leads to

$$J^e_d|_{\hat{\xi}=0} = 0, \qquad \frac{\partial J^e_d}{\partial \hat{\xi}}|_{\hat{\xi}=0} = 0,$$

and

$$\frac{\partial^2 J^e_d}{\partial \hat{\xi}^2} = \hat{Q}_{\hat{\xi}\hat{\xi}}(0, T) - \hat{Q}_{\hat{\xi}\hat{\xi}}(0, T^-) - 2\gamma^2 v^{*T}_{\hat{\xi}}(0) v^*_{\hat{\xi}}(0)$$

at $(x, \hat{\xi}) = (0, 0)$. Using Taylor expansion again, it can be shown that

$$
\begin{aligned}
J^e_d(X^e) &= \frac{1}{2}\hat{\xi}^T(\phi_x(0,T) \ \frac{\partial^2 \bar{K}_1(x,T)}{\partial x^2}|_{x=0} \ \phi_x^T(0,T))\hat{\xi} \\
&\quad + h.o.t..
\end{aligned}
$$

Thus, the condition (c) of **Lemma 2** is satisfied. This completes the proof. □

IV. Conclusions

The reduced-order H^∞ filtering problem for nonlinear sampled-data systems has been addressed. It has been shown that reduced-order H^∞ filters can be built from the solution of a standard differential Hamilton-Jacobi inequality with jumps, together with some auxiliary equations.

Acknowledgements

The work was supported by the National Science Council of the Republic of China under Grant NSC-89-2213-E-019-012.

References

[1] N. Berman, and U. Shaked, "H^∞ nonlinear filtering", *Int. J. Robust and Nonlinear Control*, **volume 6**, pp. 281-295, (1996).

[2] D.S. Bernstein, and W.M. Haddad, "Steady-state Kalman filtering with an H^∞ error bound", *Systems and Control Letters*, **volume 12**, pp.9-16, (1989).

[3] M. Bettayeb, and Kavranoglu, "Reduced-order H^∞ filtering",. *Proceedings of the American Control Conference*, Baltimore,MD, pp.1884-1888, June, (1994).

[4] A. Elsayed, and M. J. Grimble, "A new approach to H^∞ design of optimal digital linear filter", *IMA J. Math. Contr. Informat.*, **volume 6**, pp. 233-251, (1989).

[5] E. Fridman, and U. Shaked, "On regional H^∞-nonlinear filtering", *Systems and Control Letters*, **volume 29**, pp. 233-240, (1997).

[6] K.M. Grigoriadis, and J.T. Watson, "Reduced-Order H^∞ and $L_2 - L_\infty$ filtering via Linear Matrix Inequalities", *IEEE Trans. Aerospace ans Electronic systems*, **volume 33**, No.4, pp. 1326-1338, (1997).

[7] M. J. Grimble, "Minimax design optimal linear filters", *Presented at IFAC Symp.*, Glasgow, U.K., April, (1989).

[8] M. J. Grimble, D. Ho, and A. Elsayed, "H^∞ robust linear estimator", *Presented at MTNS Conf.*, AZ, (1987).

[9] C.S. Hsu, X. Yu, H.H. Yeh, and S.S. Banda, "H^∞ compensator design with minimal order observers", *Proceedings of the American Control Conference*, San Francisco, California, pp.2185-2189, (1993).

[10] H. Kim, C.S. Sims, and K.M. Nagpal, "Reduced-order filtering in an H^∞ setting", *Proceedings of the American Control Conference*, Chicago, Illinois, pp.1876-1877, (1992).

[11] Y. F. Li, C. F. Yung, and H. T. Sheu, "H^∞ Filtering of Nonlinear Systems with Sampled Measurements", *Journal of the Franklin Institute*, **volume 338**, pp. 91-100, (2001).

[12] K. M. Nagpal, and P. P. Khargonekar, "Filtering and smoothing in an H^∞ setting", *IEEE Trans. Automat. Contr.*, **volume 36**, pp.152-166, (1991).

[13] J.L. Rawson, and C.S. Hsu, "Reduced-order H^∞ filter for Discrete Linear Systems", *Proceedings of the 36th IEEE CDC,* San Diego, California USA, pp.3311-3316, (1997).

[14] W. Sun, K. M. Nagpal, and K. M. Khargonekar, "H^∞ control and filtering for sampled-data systems", *IEEE Trans. Automat. Contr.*, **volume 38**, pp. 1162-1175, (1993).

[15] I. Yaesh, and U. Shaked, "Game theory approach to optimal linear estimation in the minimum H^∞ norm sense", *Proc. IEEE Conference on Decision and Control*, pp. 421-425, (1989).

[16] X. Yu, and C.S. Hsu, "Reduced-order H^∞ filter design for time varying systems", *Proceedings of the 33rd IEEE CDC,* Lake Buena Vista, FL, pp. 2231-2236, December, (1994).

[17] X. Yu, C.S. Hsu, H.H. Yeh, and S.S. Banda, "A searation theory of Luneburger observer-based H^∞ controllers", *Proceedings of the 32nd IEEE CDC,* San Antonio, Texas, pp.662-667, (1993).

[18] X. Yu, and C.S. Hsu, "Reduced-order H^∞ filter design for discrete time variant systems", *Int. Journal of Robust and Nonlinear Control*, **volume 7**, pp.797-809, (1997).

[19] C. F. Yung, Y. F. Li, and H. T. Sheu, "H^∞ filtering and solution bound for nonlinear systems", *Int. Journal of Control*, **volume 74**, pp.565-570, (2001).

[20] C.F. Yung and H.S. Wang, "H^∞ controller reduction for nonlinear systems," *Automatica*, **volume 37**, pp. 1797-1802, (2001).