H^{∞} Controller Reduction for Nonlinear Sampled-Data Systems

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Abstract

In this paper, the problem of designing reduced-order H^{∞} controllers is studied for nonlinear continuous-time systems with sampled measurements. Using the concepts of dissipativity and differential gain, sufficient conditions are derived for the existence of such reduced-order H^{∞} controllers. These conditions are expressed in terms of the solutions of two Hamilton-Jacobi inequalities, comprising a standard Hamilton-Jacobi inequality and a differential Hamilton-Jacobi inequality with jumps. These Hamilton-Jacobi inequalities are exactly the one used in the construction of full-order H^{∞} controllers. When these conditions hold, state-space formulae are also given for such reduced-order controllers.

I. Introduction

Over the last two decades, much attention has been given to the extensions of the results of linear H^{∞} control theory [3], [8] to nonlinear settings; see, e.g., [2], [12], [13], [14], [21], [26], [27], [28], [29], [33], and [35]. In particular, Van der Schaft [26] has shown that the solution of the H^{∞} state feedback control problem for affine nonlinear systems can be obtained by solving one Hamilton-Jacobi equation, which is the nonlinear version of the Riccati equation considered in the corresponding linear H^{∞} control theory (see [3]). On the other hand, Ball et al. [2], Isidori [12], and Isidori and Astolfi [13] have presented sufficient (or necessary) conditions based on two Hamilton-Jacobi equations (or inequalities) for the solution to the H^{∞} control problem for affine nonlinear systems in the case of output feedback. Furthermore, Isidori and Kang [14], Van der schaft [28], and Yung et al. [36] have studied the H^{∞} control problem for general nonaffine nonlinear systems. Moreover, Lin and Byrnes (see [19] and [20]) have obtained some corresponding results for discrete-time nonlinear systems. Recently, extending the results obtained by Sun et al. [25] for linear systems, Suzuki et al. [24] and Guillard [5] have considered the H^{∞} control problem with sampled measurements for nonlinear systems.

The controllers obtained from the aforementioned papers have a state dimension greater than or equal to that of the system model which is built from the physical plant and some of its weighting functions, thus have limited use in practical applications, since a high order controller usually incurs a high implementation cost and is prone to be numerically ill conditioned. Therefore, a lower order controller should be sought when the resulting performance degradation is kept within an acceptable magnitude. Recently, a number of papers have appeared that deal with reduced-order (or fixed-order) H^{∞} controller design for linear systems (see, e.g., [4], [6], [7], [9], [10], [11], [15], [16], [17], [18], [22], [23], [30] and [31]) and nonlinear systems (see, e.g., [32] and [34]).

The purpose of this paper is to continue this line of research to address the H^{∞} controller reduction problem for nonlinear systems with sampled measurements. By extending the technique developed by Yung and Wang [34] for nonlinear H^{∞} controller reduction, we present sufficient conditions for the existence of H^{∞} controllers with a state dimension less than that of the plant for nonlinear sampled-data systems. The conditions obtained are expressed in terms of the solution to two Hamilton-Jacobi inequalities, which comprise a standard Hamilton-Jacobi inequality and a differential Hamilton-Jacobi inequality with jumps. The Hamilton-Jacobi inequalities are exactly the one used in the construction of the full-order H^{∞} controller for nonlinear sampled-data systems obtained in [5]. When these conditions hold, state-space formulae are also given for such controllers.

II. Preliminaries and Problem Formulation

Consider a time-invariant nonlinear sampled-data system (Figure 1) described by the dynamic equations:

$$G: \left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x) + g_1(x)w(t) + g_2(x)u(t), \\ z(t) & = & h_1(x) + k_{12}(x)u(t), \\ y(ih) & = & h_2(x(iT)) + v(iT), \qquad i = 1, 2, 3, ..., \end{array} \right. \tag{1}$$

where x represents the state defined on a neighborhood of the origin in $I\!\!R^n$, $w \in I\!\!R^{m_1}$ a continuous-time noise process which is assumed to be a member of $L^2[0, \Gamma, I\!\!R^m] := \{w : \|w\|^2_{L^2} := \int_0^\Gamma \|w(t)\|^2 dt < \infty$ for a fixed $\Gamma > 0\}$, $u \in I\!\!R^{m_2}$ a continuous control input to achieve the prescribed performance specifications, $z \in I\!\!R^{p_1}$ a controlled output, $y \in I\!\!R^p$ the measured variable which is available at sampling instants iT with the sampling period T, and $v \in I\!\!R^{p_2}$ represents measurement noise which is assumed to be a member of $l^2[0, \Gamma, I\!\!R^p] := \{v : \|v\|^2_{l^2} := \sum_{i=0}^{\lfloor \Gamma/T \rfloor} \|v(iT)\|^2 < \infty$ for a fixed $\Gamma > 0\}$. Here $\lfloor e \rfloor$ denotes the integer part of $e \in I\!\!R$. Throughout, we assume that the origin is an equilibrium, i.e. $f(0) = 0$; without loss of generality, we assume also that $h_1(0) = 0$, $h_2(0) = 0$, and that f, g_1, g_2, k_{12}, h_1 and h_2 are all smooth functions. Moreover, in this paper, for ease of presentation, we restrict ourselves to the consideration of systems satisfying the following standing assumption considered in, e.g., [24] [5].

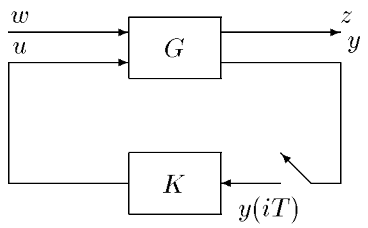

Figure 1 : H^∞ control with sampled measurements

Assumption (A1):

$$k_{12}^T(x)k_{12}(x) = I,$$

and

$$k_{12}^T(x)h_1(x) = 0$$

for all x near $x = 0$.

The standard H^∞ control problem is concerned with constructing a controller using the measurements y, such that the resulting closed-loop system has a locally asymptotically stable equilibrium at the origin and has L^2-gain $\leq \gamma$, i.e. the whole system satisfies the dissipativity inequality

$$\int_0^\Gamma \|z(t)\|^2 dt \leq \gamma^2 \Big(\int_0^\Gamma \|w(t)\|^2 dt + \sum_{i=0}^{\lfloor \frac{\Gamma}{T} \rfloor} \|v(iT)\|^2 \Big) \tag{2}$$

(see below for detailed definition of L^2-gain).

Let us denote for a function $V(x,t)$, under suitable condition of differentiability,

$$V_x := \frac{\partial V}{\partial x}, \qquad V_t := \frac{\partial V}{\partial t}, \qquad V_{tx} := \frac{\partial}{\partial t}\Big(\frac{\partial V}{\partial x}\Big).$$

We conclude this section by recalling from [5] the following results which will be useful in the sequel.

Proposition 1 *: Consider a nonlinear dynamic system with jumps described by the equations*

$$\begin{array}{rclr} \dot{x}(t) & = & f(x(t)), & t \neq iT, \\ x(iT) & = & f_T(x(iT)), & i \geq 1, \end{array} \tag{3}$$

with $f(0) = 0$, and $f_T(0) = 0$. Suppose that there exists a positive definite function $S(x,t)$, locally defined on $\Psi_1 \times [0, \Gamma]$ with Ψ_1 a neighborhood of the origin in $I\!\!R^n$, which is T-periodic, piecewise differentiable with respect to t, and C^1 with respect to x, and satisfies

$$\begin{array}{rcl} S_t(x,t) + S_x(x,t)f(x) & < & 0, \quad \forall\, x \neq 0,\ \forall\, t \in [0, T), \\ S(f_T(x), T) - S(x, T^-) & \leq & 0, \quad \forall\, t = T. \end{array} \tag{4}$$

Then system (3) has a locally asymptotically stable equilibrium at $x = 0$.

Proposition 2 *: Consider system (1) and suppose Assumption* **(A1)** *is satisfied. Suppose the following hypotheses hold.*

(H1) *There exists a C^3, positive definite function $V(x)$, locally defined on a neighborhood of the origin in $I\!\!R^n$, such that the function*

$$\begin{array}{rcl} Y_1(x) & = & V_x(x)f(x) + \gamma^2 w_*^T(x)w_*(x) \\ & & + h_1^T(x)h_1(x) - u_*^T(x)u_*(x) \end{array} \tag{5}$$

is negative definite near $x = 0$, where

$$\begin{array}{rcl} w_*(x) & = & \frac{1}{2\gamma^2} g_1^T(x)V_x^T(x) \\ u_*(x) & = & -\frac{1}{2} g_2^T(x)V_x^T(x) \end{array} \tag{6}$$

(H2) *There exists a positive definite function $Q(x,t)$, locally defined on $\Psi_2 \times [0, \Gamma]$ with Ψ_2 a neighborhood of the origin in $I\!\!R^n$, which is T-periodic, piecewise differentiable with respect to t, and C^3 with respect to x, and is such that the Hessian matrix $Q_{xx}(0,T)$ is nonsingular, which satisfies*

$$\hat{g}^T(0)Q_{xx}(0,T)\hat{g}(0) - 2\gamma^2 I < 0, \tag{7}$$

and for all $t \in [0,T]$ the function $Y_2(x,t)$ is negative definite near $x = 0$ with nonsingular Hessian matrix at $x = 0$, where the function $Y_2(x,t)$ is piecewise continuously, defined as

$$\begin{array}{rcl} Y_2(x,t) & = & Q_t(x,t) + Q_x(x,t)(f(x) + g_1(x)w_* \\ & & + \dfrac{1}{4\gamma^2} Q_x(x,t)g_1(x)g_1^T(x)Q_x^T(x,t) \\ + u_*^T(x)u_*(x), & 0 & \leq \ t < T, \end{array}$$

and

$$Y_2(x,T) = Q(x,T) - Q(x,T^-) - \gamma^2 h_2^T(x)h_2(x). \tag{9}$$

Then the nonlinear H^∞ control problem is solved by the feedback law

$$\begin{array}{rclr} \dot{\hat{x}}\ (t) & = & \hat{f}(\hat{x}(t)), & t \neq iT, \\ \hat{x}(iT) & = & \hat{x}(iT^-) + \hat{g}(\hat{x}(iT^-))(y(iT) - h_2(\hat{x}(iT^-))), & \\ u & = & \hat{h}(\hat{x}), & \end{array} \tag{10}$$

where $\hat{x} \in I\!R^n$ is defined on a neighborhood of the origin,

$$\hat{f}(x) = f(x) + g_1(x)w_*(x) + g_2(x)u_*(x),$$
$$\hat{h}(x) = u_*(x),$$

and $\hat{g}(\hat{x})$ satisfies

$$Q_x(x,T)\hat{g}(x) = 2\gamma^2 h_2^T(x).$$

III. Main Results

As it has been seen, **Proposition 2** provides a feedback law of the same order as the plant (1), that solves the H^∞ problem in question. The objective of this section is to design a reduced-order H^∞ controller K, using the sampled measurement $y(iT)$, of the form

$$\begin{aligned} \dot{\xi}(t) &= F(\xi), & t \neq iT, \\ \xi(iT) &= \xi(iT^-) + G(\xi(iT^-))(y(iT) - H_2(\xi(iT^-))), \\ u(t) &= H(\xi), \end{aligned} \tag{11}$$

where $\xi \in I\!R^r$ $(r \leq n)$ is defined on a neighborhood of the origin, with $F(0) = 0$, $H_2(0) = 0$, and $H(0) = 0$, such that the resulting closed-loop system has a locally asymptotically stable equilibrium at the origin $(x,\xi) = (0,0)$, and has L^2-gain $\leq \gamma$, i.e. there exists a neighborhood of the origin $(x,\xi) = (0,0)$ such that for all $\Gamma > 0$ and for each noise input $w(\cdot) \in L^2[0,\Gamma]$ (and $v(\cdot) \in l^2[0,\Gamma]$), the state trajectory of the closed-loop system starting from the initial state $(x(0),\xi(0)) = (0,0)$ remains in the neighborhood for all $t \in [0,\Gamma]$, and the response $z(\cdot)$ of the closed-loop system satisfies the dissipativity inequality (2). For this purpose, we first assume that there exists a smooth function $\phi : I\!R^n \times I\!R \to I\!R^r$ defined around the origin $(x,t) = (0,0)$ in $I\!R^n \times I\!R$ with $\phi(0,0) = 0$ and rank$\frac{\partial \phi}{\partial x}(0,t) = r$ for all t around $t = 0$. The rank condition implies that the restriction of ϕ to some neighborhood of $(x,t) = (0,0)$ is a surjection, provided by the surjective mapping theorem [1]. Then, we make a change of variables

$$\hat{\xi} = \xi - \phi(x,t) \tag{12}$$

where $\hat{\xi} \in I\!R^r$ is also defined on a neighborhood of the origin. In terms of these variables the resulting closed-loop system is

$$\begin{aligned} \dot{x}_e &= F_e(x_e) + G_e(x_e)w, & t \neq iT, \\ x_e(iT) &= F_e^d(x_e(iT^-)) + G_e^d(x_e(iT^-))v(iT), & i \geq 1, \\ z &= H_e(x_e), \end{aligned} \tag{13}$$

where $x_e(t) := col(x(t), \hat{\xi}(t))$,

$$F_e(x_e) = \begin{bmatrix} f(x) + g_2(x)H(\hat{\xi} + \phi(x,t)) \\ \tilde{f} - \phi_x(x)g_2(x)H(\hat{\xi} + \phi(x,t)) \end{bmatrix},$$

$$G_e(x_e) = \begin{bmatrix} g_1(x) \\ -\phi_x(x,t)g_1(x) \end{bmatrix},$$

$$F_e^d(x_e) = \begin{bmatrix} x \\ \hat{\xi} + G(\hat{\xi} + \phi(x,t))\tilde{h}_2 \end{bmatrix},$$

$$G_e^d(x_e) = \begin{bmatrix} 0 \\ G(\phi(x,t) + \hat{\xi}) \end{bmatrix},$$

and

$$H_e(x_e) = h_1(x) + k_{12}(x)H(\hat{\xi} + \phi(x,t)),$$

where

$$\tilde{f} = -\phi_t(x,t) - \phi_x(x,t)f(x) + F(\hat{\xi} + \phi(x,t)),$$
$$\tilde{h}_2 = h_2(x) - H_2(\hat{\xi} + \phi(x,t)).$$

We will seek for a control law of the form (11), such that the closed-loop system satisfies the H^∞ performance criterion, i.e. the closed-loop system (13) is asymptotically stable and has L^2-gain $\leq \gamma$. We first observe that the problem in question can be cast as a two players, differential game problem, associated with which we define two Hamiltonian functions $\bar{M}_1 : I\!R^{n+r} \times I\!R^{m_1} \times I\!R \to I\!R$,

$$\begin{aligned} \bar{M}_1(x_e, w, t) \triangleq\ & W_{x_e}(x_e,t)(F_e(x_e) + G_e(x_e)w) \\ & + W_t(x_e,t) + H_e^T(x_e)H_e(x_e) - \gamma^2 w^T w \end{aligned} \tag{14}$$

for $t \neq iT$, and $\bar{M}_1^d : I\!R^{n+r} \times I\!R^p \to I\!R$,

$$\begin{aligned} \bar{M}_1^d(x_e, v) \triangleq\ & W(F_e^d(x_e) + G_e^d(x_e)v, T) - W(x_e, T^-) \\ & -\gamma^2 v^T v \end{aligned} \tag{15}$$

for $t = iT$. A preliminary lemma will be needed in the sequel.

Lemma 3 *: Consider system* (1)*. Suppose that* **Assumption (A1)** *is satisfied. Suppose also that there exists a smooth positive definite function $W(x_e,t)$, locally defined on $\Psi \times [0,\Gamma]$, with Ψ a neighborhood of the origin in $I\!R^{n+r}$, C^2 with respect to x_e, T-periodic (i.e., $W(x_e,t) = W(x_e,t+T)$), and piecewise differentiable with respect to t, such that $W(x_e,t)$ vanishes at $x_e = col(x,\phi(x,t))$ for all $t \in [0,\Gamma]$, and such that the following conditions are satisfied:*

(a) *the quantity*

$$G_e^{dT}(0)W_{x_e x_e}(0,T)G_e^d(0) - 2\gamma^2 I < 0, \tag{16}$$

(b) *the function*

$$\begin{aligned} J_e(x_e,t) =\ & W_t + \frac{1}{4\gamma^2}W_{x_e}G_e(x_e)G_e^T(x_e)W_{x_e}^T \\ & + H_e^T(x)H_e(x) + W_{x_e}F_e(x_e) \end{aligned} \tag{17}$$

vanishes at $x_e = col(x,\phi(x,t))$ and is negative elsewhere for all $t \in [0,\Gamma]$ with $t \neq iT$,

(c) *the function*

$$\begin{aligned} J_e^d(x_e) =\ & W(F_e^d(x_e,T) + G_e^d(x_e)v^*, T) \\ & -W(x_e,T^-) - \gamma^2 v^{*T} v^* \end{aligned} \tag{18}$$

is less than or equal to zero, where v^* is the unique solution, with $v^*(0) = 0$, of the implicit function

$$\frac{\partial W}{\partial \alpha}\Big|_{\alpha = F_e^d(x_e) + G_e^d(x_e)v} G_e^d(x_e) - 2\gamma^2 v^T = 0. \tag{19}$$

Then the reduced-order feedback law (11) *locally asymptotically stabilizes the resulting closed-loop system* (13) *and renders its L^2-gain $\leq \gamma$.*

Proof. With (14), (15), (17), and (18) in mind, and using the Taylor expansion theorem, it is easy to see that $\bar{M}_1$ can be rewritten as

$$\bar{M}_1(x_e, w, t) = J_e(x_e, t) - \gamma^2 \|w - w^*\|^2,$$

where $w^* := \frac{1}{2\gamma^2} G_e^T(x_e) W_{x_e}^T$ is the worst disturbance. Since $J_e(x_e, t) \leq 0$ by hypothesis, we have $\bar{M}_1(x_e, w, t) \leq 0$. Moreover, $\bar{M}_1(x_e, 0, t) < 0$. By a simple calculation, it can be shown that

$$\frac{\partial \bar{M}_1^d(x_e, v)}{\partial v} = \frac{\partial W}{\partial \alpha} \mid_{\alpha = F_e^d(x_e) + G_e^d(x_e) v} G_e^d(x_e) - 2\gamma^2 v^T, \tag{20}$$

and

$$\frac{\partial^2 \bar{M}_1^d(x_e, v)}{\partial v^2} = G_e^{dT} \frac{\partial^2 W}{\partial \alpha^2} \mid_{\alpha = F_e^d(x_e) + G_e^d(x_e) v} G_e^d - 2\gamma^2 I. \tag{21}$$

By (16), it follows from the implicit function theorem that there exists a unique solution $v^*(x_e)$, defined on a neighborhood of $x_e = 0$, satisfying

$$\frac{\partial \bar{M}_1^d(x_e, v)}{\partial v} \mid_{v = v^*} = 0,$$

and

$$v^*(x_e) \mid_{x_e = 0} = 0.$$

Hence $\bar{M}_1^d$ can be expressed as

$$\bar{M}_1^d(x_e, v) = J_e^d(x_e) + \frac{1}{2} \|v - v^*\|_{r_{11}}^2 + O(\|v - v^*\|^3),$$

where $r_{11} := \frac{\partial^2 \bar{M}_1^d(x_e, v)}{\partial v^2} \mid_{(x_e, v) = (0,0)}$ is negative definite. Since $J_e^d \leq 0$ by hypothesis, we have $\bar{M}_1^d \leq 0$. By **Proposition 1**, both $\bar{M}_1(x_e, 0, t) < 0$ and $\bar{M}_1^d \leq 0$ imply that the equilibrium $x_e = 0$ of the closed-loop system (13) is locally asymptotically stable. Furthermore, combination of (14) and (15) and integration on $[0, \Gamma]$ gives

$$\int_0^\Gamma \|z(t)\|^2 dt - \gamma^2 \left[\int_0^\Gamma \|w(t)\|^2 dt + \sum_{i=0}^{\lfloor \Gamma/T \rfloor} \|v(iT)\|^2 \right] + W(x_e(\Gamma), \Gamma) \leq 0.$$

Since $W(x_e(t), t) \geq 0$, it is concluded that the closed-loop system (13) has L^2-gain ≤ 0. This completes the proof. □

Remark 1 *: The reduced-order feedback law given above is a nonlinear system with finite jumps at discrete instants of time. At the sampling instants, the measurement $y(iT)$ is used to update the control law, and to guarantee the whole system being asymptotically stable and having L^2-gain $\leq \gamma$ even though there is no output injection between any two sampling instants. It is noted that the solution $x_e(t)$ to (13), if exists, is right continuous but may be left discontinuous with possibly finite jumps at $t = iT$. Also note that Hamiltonian function $J_e(x_e, t)$ solves the continuous-time H^∞ control problem while the Hamiltonian function $J_e^d(x_e)$, on the other hand, solves the discrete-time counterpart. In other word, the function $J_e^d(x_e)$ is only available at sampling instant iT, whereas $J_e(x_e, t)$ is available in other instants $t \neq iT$. The combination (17) and (18) can be regarded as a differential Hamiltonian function with jumps.* □

The conditions in **Lemma 3** for the solution of the H^∞ control problem can be further simplified by providing an alternative set of sufficient conditions, which involve a new Hamilton Jacobi inequality having fewer independent variables. This is summarized in the following statement.

Theorem 4 *: Suppose that* **Assumption (A1)** *is satisfied and that hypotheses* **(H1)** *and* **(H2)** *of* **Proposition 2** *hold. Suppose that there exists a smooth function $\phi : I\!R^n \times I\!R \to I\!R^r$, locally defined on a neighborhood of the origin $(x, t) = (0, 0)$ in $I\!R^n \times I\!R$, with $\phi(0, t) = 0$ and $\frac{\partial \phi}{\partial x}(0, t) (\frac{\partial \phi}{\partial x})^T (0, t) = I$. Suppose also that there exists a positive definite function $\hat{Q}(\hat{\xi}, t)$, locally defined on $\Psi_3 \times [0, \Gamma]$ with Ψ_3 a neighborhood of the origin in $I\!R^r$, which is T-periodic, piecewise differentiable with respect to t, C^3 with respect to $\hat{\xi}$, and satisfies $\frac{\partial^2 \hat{Q}}{\partial \hat{\xi}^2}(0, t) \frac{\partial \phi}{\partial x}(0, t) = \frac{\partial \phi}{\partial x}(0, t) \frac{\partial^2 Q}{\partial x^2}(0, t)$. Then, if F, H_2, G, and H satisfy*

$$\begin{array}{rcl} F(\phi(x, t)) & = & \phi_x(x, t) \hat{f}(x) + \phi_t(x, t), \\ H_2(\phi(x, t)) & = & h_2(x), \\ G(\phi(x, t)) & = & \phi_x(x, t) \hat{g}(x), \end{array} \tag{22}$$

and

$$H(\phi(x, t)) = \hat{h}(x), \tag{23}$$

the r-th order controller (11) *locally asymptotically stabilizes the resulting closed-loop system* (13) *and renders its L^2-gain $\leq \gamma$.*

Proof. Let $W(x_e, t) = V(x) + \hat{Q}(\hat{\xi}, t)$, which is positive definite since both $V(x)$ and $\hat{Q}(\hat{\xi}, t)$ are positive definite. The proof is divided into two parts, namely continuous part and discrete part.

(a) Continuous part, *i.e.* $t \neq iT$.

Take Taylor expansion around $\hat{\xi} = 0$ to $J_e(x, \hat{\xi}, t)$ to obtain

$$J_e(x, \hat{\xi}, t) = J_e(x, 0, t) + \frac{\partial J_e}{\partial \hat{\xi}} |_{\hat{\xi}=0} \hat{\xi} + \frac{1}{2} \hat{\xi}^T \frac{\partial^2 J_e}{\partial \hat{\xi}^2} |_{\hat{\xi}=0} \hat{\xi} + h.o.t., \tag{24}$$

where "*h.o.t.*" means higher order terms. With (22) and (23) in hand, a routine manipulation shows that

$$\begin{array}{rcl} J_e(x, 0, t) & = & V_x f(x) + V_x g_2(x) \hat{h}(x) + h_1^T(x) h_1(x) \\ & & + \hat{h}^T(x) \hat{h}(x) + \frac{1}{4\gamma^2} V_x g_1(x) g_1^T(x) V_x^T \\ & = & Y_1(x), \end{array}$$

$$\frac{\partial J_e}{\partial \hat{\xi}} |_{(x,0,t)} = 0,$$

and at $(0, 0, t)$

$$\frac{\partial^2 J_e}{\partial \hat{\xi}^2} = \phi_x(0, t)((\hat{f}_x^T(0) + \hat{h}_x(0) g_2^T(0)) Q_{xx}(0, t)$$

$$
\begin{aligned}
&+Q_{xx}(0,t)(\hat{f}_x(0)-g_2(0)\hat{h}_x(0))\\
&+\frac{1}{2\gamma^2}Q_{xx}(0,t)g_1(0)g_1(0)Q_{xx}^T(0,t)\\
&+2\hat{h}_x^T(0)\hat{h}_x(0)Q_{xxt}(0,t))\phi_x^T(0,t)\\
=\;& \phi_x(0,t)Y_{2_{xx}}(0,t)\phi_x(0,t).
\end{aligned}
$$

Since $Y_{2_{xx}}(0,t)$ is negative definite for all $t\in[0,\Gamma]$ by hypothesis (**H2**), and $Y_1(x)$ is also negative definite by hypothesis (**H1**), it is concluded that for all $t\in[0,\Gamma]$ the function $J_e(x_e,t)$ vanishes at $\phi(x,t)=\xi$ and is negative elsewhere. Thus, condition (b) of **Lemma 3** holds.

(b) Discrete part, *i.e.* $t=iT$.

Using (22), equation (16) can be rewritten as

$$\hat{g}^T(0)Q_{xx}(0,T)\hat{g}(0)-2\gamma^2 I<0,$$

which is the same as (7) of **Proposition 2**; thus, condition (a) of **Lemma 3** holds. Using (22) again, it is straightforward to show that

$$v^*(0)=0, \qquad \text{and} \qquad v^*_{\hat{\xi}}(0)=H_{2_{\hat{\xi}}}(0)$$

by noting that

$$G^T(0)\hat{Q}_{\hat{\xi}\hat{\xi}}(0,T)=2\gamma^2 H_{2_{\hat{\xi}}}(0).$$

Applying $v^*(0)$ and $v^*_{\hat{\xi}}(0)$ to (18) gets

$$J_e^d|_{\hat{\xi}=0}=0, \qquad \frac{\partial J_e^d}{\partial\hat{\xi}}|_{\hat{\xi}=0}=0,$$

and

$$\frac{\partial^2 J_e^d}{\partial\hat{\xi}^2}=\hat{Q}_{\hat{\xi}\hat{\xi}}(0,T)-\hat{Q}_{\hat{\xi}\hat{\xi}}(0,T^-)-2\gamma^2 v^{*T}_{\hat{\xi}}(0)v^*_{\hat{\xi}}(0)$$

at $(x,\hat{\xi})=(0,0)$. Using Taylor expansion again, it can be shown that

$$
\begin{aligned}
J_e^d(x_e) \;=\;& \frac{1}{2}\hat{\xi}^T(\phi_x(0,T)\;\frac{\partial^2 Y_2(x,T)}{\partial x^2}|_{x=0}\;\phi_x^T(0,T))\hat{\xi}\\
&+h.o.t.,
\end{aligned}
$$

which is less than or equal to zero since $Y_2(x,T)\le 0$. Thus, condition (c) of **Lemma 3** is also satisfied. This completes the proof. □

Remark 2 *: It has been shown in* **Theorem 4** *that the achievement of closed-loop asymptotic stability is implied by the fulfillment of the negative definiteness of $Y_1(x)$. If $Y_1(x)$ is only negative semidefinite near $x=0$, then closed-loop asymptotic stability can still be achieved if the equilibrium $\xi=0$ of the controller (11) is locally asymptotically stable and system (1) satisfies the following standard assumption usually considered in H^∞ controller design (see [12], [34] for details). The proof can be established with a similar argument used in [34], and thus is omitted here.*

Assumption (A2): *Any bounded trajectory $x(t)$ of the system*

$$\dot{x}(t)=f(x(t))+g_2(x(t))u(t)$$

satisfying

$$h_1(x(t))+k_{12}(x(t))u(t)=0$$

for all $t\ge 0$, is such that $lim_{t\to\infty}x(t)=0$. □

IV. Conclusions

Controller reduction is often desirable to reduce the complexity and computational burden in real-time control process, especially when fast data processing is actually required. In this paper, the reduced-order H^∞ control problem for nonlinear sampled-data systems has been extensively addressed. It has been shown that reduced-order H^∞ controllers can be constructed from the solutions of two standard Hamilton-Jacobi inequalities (with jumps), and four auxiliary equations.

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