SOLVABILITY OF NORM-TYPE DISCRETE ALGEBRAIC RICCATI EQUATION

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Abstract

This paper proposes a norm-type discrete algebraic Riccati Equation, which is a generalized version of the well-known standard discrete algebraic Riccati Equation, and has additional norm terms. Under stabilizability and assumption that the additional terms are not too large, the existence of a positive semi-definite solution is guaranteed. Application to guaranteed cost control is given.

1 Introduction

In this paper, we propose the following equation with an unknown symmetric matrix P.

$$\begin{aligned} P \ &= \ A_0^T P A_0 - A_0^T PB(B^T PB + R)^{-1} B^T PA_0 \\ &\quad + C^T C + \Upsilon(P) \end{aligned} \tag{1}$$

where A_0, B and C are matrices with appropriate size and R is positive definite. Further,

$$\Upsilon(P) = (a\|P - PB(B^T PB + R)^{-1} B^T P\| + b\|P\|) \cdot I \tag{2}$$

where a and b are positive real numbers and $\|A\| = \sigma_{max}(A)$ (σ_{max} denotes the maximum singular value). We call equation (1) a norm-type discrete algebraic Riccati equation. In [1], for a continuous-time system, the authors have proposed the similar algebraic equation which is called a norm-type continuous-time algebraic Riccati equation and given a sufficient condition for it to be solvable. The main purpose of this paper is to extend the result of [1] to the discrete-time case.

2 Main Result

Notice that the equation (1) is equivalent to

$$\begin{aligned} P \ &= \ C^T C + (A_0 - B\Omega(P)A_0)^T P(A_0 - B\Omega(P)A_0) \\ &\quad + A_0^T \Omega(P)^T R\Omega(P)A_0 + \Upsilon(P), \end{aligned} \tag{3}$$

$$\Omega(P) = (B^T PB + R)^{-1} B^T P. \tag{4}$$

In order to prove a main result we shall require:

(H1) $\quad (a+b)\inf_{\Psi} \| \sum_{j=0}^{\infty} ((A_0 - B\Psi)^T)^j (A_0 - B\Psi)^j \| < 1.$

We need the following preliminary results.

Lemma 1 *Suppose that (H1) holds and further*

(H2) $\quad (A_0, B)$ *is stabilizable.*

i) From hypotheses, there exists Ψ such that $A_0 - B\Psi$ is asymptotically stable and

$$(a+b)\| \sum_{j=0}^{\infty} ((A_0 - B\Psi)^T)^j (A_0 - B\Psi)^j \| < 1. \tag{5}$$

For such Ψ, the equation

$$\begin{aligned} P \ &= \ C^T C + (A_0 - B\Psi)^T P(A_0 - B\Psi) + \Psi^T R\Psi \\ &\quad + \Upsilon(P) \end{aligned} \tag{6}$$

has a unique positive semi-definite solution P and is equivalent to

$$P = g(P, \Psi), \tag{7}$$

where

$$\begin{aligned} g(P,\Psi) \ &= \ \sum_{j=0}^{\infty} ((A_0 - B\Psi)^T)^j (\Psi^T R\Psi \\ &\quad + \Upsilon(P) + C^T C)(A_0 - B\Psi)^j. \end{aligned} \tag{8}$$

ii) For the solution P of the equation (6), $A_0 - B\Omega(P)A_0$ is asymptotically stable.

Lemma 2 *Let K and L be positive definite matrices. If $K \geq L$, we have the following relation*

$$\|(K^{-1} + R_c)^{-1}\| \geq \|(L^{-1} + R_c)^{-1}\| \tag{9}$$

for any positive semi-definite matrix R_c.

Proof Since $\lambda_{max}(K) \geq \lambda_{max}(L)$, it follows that $\lambda_{min}(K^{-1}+R_c) \leq \lambda_{min}(L^{-1}+R_c)$, where λ_{max} denotes the maximum eigenvalues and λ_{min} denotes the minimum one. Thus, we have $1/\lambda_{min}(K^{-1}+R_c) \geq 1/\lambda_{min}(L^{-1}+R_c)$. This implies $\|(K^{-1}+R_c)^{-1}\| \geq \|(L^{-1}+R_c)^{-1}\|$.

Q.E.D

Theorem 1 *Assume that (H1) and (H2) hold. Then there exists a positive semi-definite solution P for equation (1).*

Remark 1 *Instead of solving (1), P can be determined by limiting solution of the norm-type Riccati difference equation*

$$\begin{aligned}P(k) &= C^TC+\Upsilon(P(k+1))+A_0^TP(k+1)A_0\\ &-A_0^TP(k+1)B(B^TP(k+1)B+R)^{-1}B^TP(k+1)A_0\end{aligned}$$

Remark 2 [12] *Consider solvability of the stochastic discrete algebraic Riccati equation (SDARE),*

$$\begin{aligned}P &= C^TC+A_0^TPA_0\\ &+A_0^TPB(B^TPB+R)^{-1}B^TPA_0+\sum_{i=1}^{p}A_i^TPA_i \quad (10)\end{aligned}$$

where R is a positive-definite matrix. And in addition to (H1) and (H2), it is required that (C, A_0) is observable. That is, the condition of Theorem 1 is weaker than the one of SDARE.

Proof From (H1) and (H2), there exists Ψ_1 such hat $A_0-B\Psi_1$ is asymptotically stable and

$$(a+b)\|\sum_{j=0}^{\infty}((A_0-B\Psi_1)^T)^j(A_0-B\Psi_1)^j\|<1. \quad (11)$$

Let P_1 be the solution of $P=g(P,\Psi_1)$ then define

$$\Psi_2=\Omega(P_1)A_0. \quad (12)$$

In virtue of Lemma 1 i), P_1 is positive semi-definite and satisfies

$$\begin{aligned}P_1 &= C^TC+(A_0-B\Psi_1)^TP_1(A_0-B\Psi_1)\\ &+\Psi_1^TR\Psi_1+\Upsilon(P_1). \quad (13)\end{aligned}$$

Since $A_0-B\Psi_1$ is asymptotically stable in virtue of Lemma 1 ii), $g(P,\Psi_2)$ can be defined.

Now set

$$P_2^{(1)}=0, P_2^{(\kappa+1)}=g(P_2^{(\kappa)},\Psi_2), \kappa=1,2,3,\ldots$$

First, we shall show that $P_2^{(\kappa)}$ is monotonically non-decreasing. It is clear that $P_2^{(2)} \geq P_2^{(1)}$.

It follows from (8) that

$$\begin{aligned}&P_2^{(\kappa+1)}-P_2^{(\kappa)}\\ &= g(P_2^{(\kappa)},\Psi_2)-g(P_2^{(\kappa-1)},\Psi_2)\\ &= \sum_{j=0}^{\infty}((A_0-B\Psi_2)^T)^j(\Upsilon(P_2^{(\kappa)})-\Upsilon(P_2^{(\kappa-1)}))\\ &\qquad\cdot(A_0-B\Psi_2)^j\\ &= \sum_{j=0}^{\infty}((A_0-B\Psi_2)^T)^j\\ &\quad\cdot[a\{\|P_2^{(\kappa)}-P_2^{(\kappa)}B(B^TP_2^{(\kappa)}B+R)^{-1}B^TP_2^{(\kappa)}\|\\ &-\|P_2^{(\kappa-1)}-P_2^{(\kappa-1)}B(B^TP_2^{(\kappa-1)}B+R)^{-1}B^TP_2^{(\kappa-1)}\|\}\\ &\qquad+b\{\|P_2^{(\kappa)}\|-\|P_2^{(\kappa-1)}\|\}](A_0-B\Psi_2)^j\\ &= \sum_{j=0}^{\infty}((A_0-B\Psi_2)^T)^j[a\{\|((P_2^{(\kappa)})^{-1}+BR^{-1}B^T)\|\\ &\quad-\|((P_2^{(\kappa-1)})^{-1}+BR^{-1}B^T)^{-1}\|\}\\ &\qquad+b\{\|P_2^{(\kappa)}\|-\|P_2^{(\kappa-1)}\|\}](A_0-B\Psi_2)^j.\end{aligned}$$

Assume that $P_2^{(\kappa)} \geq P_2^{(\kappa-1)}$. Then, since $\|P_2^{(\kappa)}\| \geq \|P_2^{(\kappa-1)}\|$ and, in virtue of Lemma 2, $\|((P_2^{(\kappa)})^{-1}+BR^{-1}B^T)^{-1}\| \geq \|((P_2^{(\kappa-1)})^{-1}+BR^{-1}B^T)^{-1}\|$, we have $P_2^{(\kappa+1)} \geq P_2^{(\kappa)}$, so that $\{P_2^{(\kappa)}\}$ is monotone non-decreasing.

Next, we shall show that $P_2^{(\kappa)} \leq P_1$. It is clear that $P_2^{(1)} \leq P_1$. For arbitrary Ψ, P and Ω defined by (4), we have

$$\begin{aligned}&(A_0-B\Omega A_0)^TP(A_0-B\Omega A_0)+A_0^T\Omega^TR\Omega A_0\\ &= (A_0-B\Psi)^TP(A_0-B\Psi)+\Psi^TR\Psi\\ &\quad-(\Psi-\Omega A_0)^T(B^TPB+R)(\Psi-\Omega A_0). \quad (14)\end{aligned}$$

Substituting $P=P_1$ into (4) and use (12) to obtain

$$\Omega A_0=\Psi_2. \quad (15)$$

Since substituting $\Omega A_0=\Psi_2, \Psi=\Psi_1$ into (14) and noting (13) gives

$$\begin{aligned}&(A_0-B\Psi_2)^TP_1(A_0-B\Psi_2)+\Psi_2^TR\Psi_2\\ &\leq (A_0-B\Psi_1)^TP_1(A_0-B\Psi_1)+\Psi_1^TR\Psi_1\\ &= P_1-C^TC-\Upsilon(P_1). \quad (16)\end{aligned}$$

we have

$$\begin{aligned}&\Psi_2^TR\Psi_2+C^TC\\ &\leq P_1-(A_0-B\Psi_2)^TP_1(A_0-B\Psi_2)-\Upsilon(P_1). \quad (17)\end{aligned}$$

Thus,

$$\begin{aligned}&g(P_2^{(\kappa)},\Psi_2)\\ &= \sum_{j=0}^{\infty}((A_0-B\Psi_2)^T)^j\{\Psi_2^TR\Psi_2+\Upsilon(P_2^{(\kappa)})\\ &\quad+C^TC\}(A_0-B\Psi_2)^j\end{aligned}$$

$$\leq \sum_{j=0}^{\infty}((A_0 - B\Psi_2)^T)^j(P_1 - (A_0 - B\Psi_2)^T \cdot P_1(A_0 - B\Psi_2) + \Upsilon(P_2^{(\kappa)}) - \Upsilon(P_1))(A_0 - B\Psi_2)^j$$
$$= \sum_{j=0}^{\infty}((A_0 - B\Psi_2)^T)^j[a\|((P_2^{(\kappa)})^{-1} + BR^{-1}B^T)^{-1}\| \cdot I - (\|(P_1^{-1} + BR^{-1}B^T)^{-1}\| \cdot I + b(\|P_2^{(\kappa)}\| - \|P_1\|) \cdot I] \cdot(A_0 - B\Psi_2)^j + P_1. \quad (18)$$

Assume that $P_2^{(\kappa)} \leq P_1$. Then, since $\|P_2^{(\kappa)}\| \leq \|P_1\|$ and, in virtue of Lemma 2,

$$\|((P_2^{(\kappa)})^{-1} + BR^{-1}B^T)^{-1}\| \leq \|(P_1^{-1} + BR^{-1}B^T)^{-1}\|,$$

we have

$$P_2^{(\kappa+1)} = g(P_2^{(\kappa)}, \Psi_2) \leq P_1$$

If $P_2^{(\kappa)} \leq P_1, P_2^{(\kappa+1)} = g(P_2^{(\kappa)}, \Psi_2) \leq P_1$. Since $P_2^{(1)} = 0 \leq P_1$, we have $P_2^{(\kappa)} \leq P_1$, that is, $P_2^{(\kappa)}$ is bounded. Thus, it follows that there exists $P_2 = \lim_{\kappa\to\infty} P_2^{(\kappa)}$, and $P_2 \leq P_1$. It is shown by repeating this procedure that P_k is monotone non-increasing.

Since $P_k \geq 0$, it follows that there exists $P = \lim_{k\to\infty} P_k$ and P satisfies (1).

Q.E.D

3 APPLICATION TO GUARANTEED COST CONTROL

In the sequel, we consider guaranteed cost control in the linear quadratic case. Consider the following uncertain plant

$$x(t+1) = A(\xi)x(t) + Bu(t), \quad \xi = [\xi_1, \xi_2, \ldots, \xi_p] \quad (19)$$

where $\xi_i (i = 1, 2, \ldots, p)$ is an uncertain parameter and $A(\xi)$ is assumed to be expressed as

$$A(\xi) = A_0 + \Delta A(\xi) = A_0 + \sum_{i=1}^{p} \xi_i A_i, \quad |\xi_i| \leq 1. \quad (20)$$

Further, the quadratic cost function to be minimized is

$$J(x(0), u, \xi) = \sum_{t=0}^{\infty}(x^T(t)C^TCx(t) + u^T(t)Ru(t)), \quad (21)$$

where R is positive definite. In the following, we use a linear feedback control law

$$u(t) = -Fx(t). \quad (22)$$

If there exist a positive real number V and control $u(\cdot)$ such that

$$J(x(0), u, \xi) \leq V, \quad (23)$$

V and $u(\cdot)$ are said to be a guaranteed cost and guaranteed cost control, respectively. The following fact follows from Appendix A.2. Let

$$T_1(P, \xi) = \Delta A^T P(I - B\Omega(P))A_0 + A_0^T(I - B\Omega(P))^T P\Delta A + \Delta A^T P\Delta A, \quad (24)$$

where $\Omega(P)$ is defined by (4). When $T_1(P, \xi) \leq U_1(P)$ for any $\xi(|\xi_i| \leq 1)$, we call $U_1(P)$ an upper bound matrix of T_1. Then, in order to calculate the guaranteed cost control law, we have to solve a discrete algebraic Riccati equation with the upper bound term

$$P = C^TC + T_0(P) + U_1(P), \quad (25)$$

where

$$T_0(P) = A_0^T PA_0 - A_0^T PB(B^T PB + R)^{-1}B^T PA_0. \quad (26)$$

Using the solution P of (25), we have guaranteed cost control law

$$F(P) = \Omega(P)A_0 \quad (27)$$

The following theorem gives an upper bound.

Theorem 2 *In (2), let*

$$a = 2\|A_0\| \cdot \sum_{i=1}^{p} \|A_i\|, \quad (28)$$
$$b = \left(\sum_{i=1}^{p} \|A_i\|\right)^2. \quad (29)$$

Then, an upper bound of $T_1(P, \xi)$ is given by $\Upsilon(P)$.

Proof Since

$$T_1(P, \xi) = \sum_{i=1}^{p} \xi_i[A_i^T P(I - B\Omega(P))A_0 + A_0^T(I - B\Omega(P))^T PA_i] + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} \xi_i\xi_j D_{ij} \quad (30)$$

where

$$D_{ij} = A_i^T PA_j + A_j^T PA_i,$$

we have

$$\begin{aligned} T_1(P, \xi) &\leq 2\sum_{i=1}^{p} \|A_i^T P(I - B\Omega(P))A_0\| \cdot I + \sum_{i=1}^{p}\sum_{j=1}^{p} \|A_i^T PA_j\| \cdot I \\ &\leq 2\sum_{i=1}^{p} \|A_i^T\| \cdot \|P - PB\Omega(P)\| \cdot \|A_0\| \cdot I + \sum_{i=1}^{p}\sum_{j=1}^{p} \|A_i\| \cdot \|A_j\| \cdot \|P\| \cdot I \\ &= \Upsilon(P) \end{aligned} \quad (31)$$

Q.E.D

Remark 3 [13]*The guaranteed cost control law (27) guarantees robust stability of the closed-loop system for any admissible perturbation $\xi \in \Xi$.*

4 NUMERICAL EXAMPLE

In this section, we shall give a numerical example and show the effectiveness of our method in contrast to the existence method. Let

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

For this system, let us try to apply the methods of [8], [9], which are based on the quadratic bound. Then, we have to solve a discrete algebraic Riccati equation with additional terms.

$$\begin{aligned} P &= A_0^T(P^{-1} - \epsilon DD^T + BR^{-1}B)^{-1}A_0 + Q + (1/\epsilon)E^T E, \\ A_1 &= E^T D. \end{aligned}$$

However, since for any ϵ the pair $(C + (1/\epsilon)E, A_0)$ is not detectable, there is no stabilizing positive semi-definite solution, we cannot apply guaranteed cost control based on the quadratic bound.

Next, We shall apply our method to this system. The Hypothesis (H1) and (H2) are satisfied. By solving (1), we have the following guaranteed cost control law.

$$u(t) = [\,0 \quad 0.3304\,]x(t).$$

Then the eigenvalues of the nominal closed-loop system become 0.0 and 0.66960.

For the case $\xi_1 = 1.0$, the eigenvalues of the closed-loop system are 0.165548 and 0.604054. Thus for this perturbation, the closed-loop system is asymptotically stable.

5 CONCLUSION

In this paper, we have proposed the norm-type discrete algebraic Riccati equation, and discussed the existence of a positive semi-definite solution. And we have shown that this equation appears in guaranteed cost control in discrete time systems. It is important to develop an numerical algorithm in order to solve the equation (1). One of the methods is extend the approach of [10], [11].

The problem for future study is to extend our result on guaranteed cost control to the case of parameter variations in an input matrix.

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A Appendix

A.1 Proof of Lemma 1

i)
Since

$$\begin{aligned}
&\|P - PB(B^TPB + R)^{-1}B^TP\| \\
&\quad = \|(P^{-1} - BR^{-1}B^T)^{-1}\| \\
&\quad = \frac{1}{\sigma_{min}(P^{-1} - BR^{-1}B^T)} \\
&\quad \leq \frac{1}{\sigma_{min}(P^{-1})} = \sigma_{max}(P) \\
&\quad = \|P\|, \qquad (32)
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|\sum_{j=0}^{\infty}((A_0 - B\Psi)^T)^j(\Upsilon(P))(A_0 - B\Psi)^j\| \\
&\leq a\sum_{j=0}^{\infty}\|((A_0 - B\Psi)^T)^j\| \cdot \|P - PB(B^TPB \\
&\quad +R)^{-1}B^TP\| \cdot \|(A_0 - B\Psi)^j\| \\
&\quad +b\sum_{j=0}^{\infty}\|((A_0 - B\Psi)^T)^j\| \cdot \|P\| \cdot \|(A_0 - B\Psi)^j\| \\
&= a\|P\| \cdot \|\sum_{j=0}^{\infty}((A_0 - B\Psi)^T)^j(A_0 - B\Psi)^j\| \\
&\quad +b\|P\| \cdot \|\sum_{j=0}^{\infty}((A_0 - B\Psi)^T)^j(A_0 - B\Psi)^j\| \\
&= (a+b)\|P\| \cdot \|\sum_{j=0}^{\infty}((A_0 - B\Psi)^T)^j(A_0 - B\Psi)^j\|. \qquad (33)
\end{aligned}$$

In virtue of (H1), we have

$$\begin{aligned}
&\|\sum_{j=0}^{\infty}((A_0 - B\Psi)^T)^j\Upsilon(P)(A_0 - B\Psi)^j\| \\
&\quad \leq \delta\|P\|, \qquad (34)
\end{aligned}$$

where $\delta \in (0, 1)$ is independent on P. Thus, since for this Ψ the function $g(P, \Psi)$ is a contraction mapping in P, (7) has a unique solution. From (8), it is clear that P is positive semi-definite.
ii)
It follows from Lemma 1 of [7] that

$$\begin{aligned}
&(A_0 - B\Omega(P)A_0)^TP(A_0 - B\Omega(P)A_0) \\
&\quad \leq (A_0 - B\Psi)^TP(A_0 - B\Psi) + \Psi^TR\Psi. \qquad (35)
\end{aligned}$$

Since in virtue of (6), the right hand side of (35) coincides with $P - C^TC - \Upsilon(P)$,

$$\begin{aligned}
&(A_0 - B\Omega(P)A_0)^TP(A_0 - B\Omega(P)A_0) - P \\
&\quad \leq -C^TC - \Upsilon(P). \qquad (36)
\end{aligned}$$

Assume that $A_0 - B\Omega(P)A_0$ is not asymptotically stable. Let λ be unstable eigenvalue and w be the corresponding eigenvector. Premultiplying (36) by w^* and postmultiplying it by w shows that the left hand side of (36) is positive semi-definite, where w^* denotes the conjugate transpose of w. However, since $\Upsilon(P)$ is positive definite, this is contradiction.

Q.E.D.

A.2 Guaranteed cost control

We consider an uncertain system described as equation (19). For this system, we define performance index given as

$$\begin{aligned}
J(x(t_1), u, \xi) &= \sum_{t=t_1}^{t_f-1}\{x^T(t)C^TCx(t) + u^T(t)Ru(t)\} \\
&\quad +x^T(t_f)P_fx(t_f) \qquad (37)
\end{aligned}$$

where R and P_f are positive definite and positive semi-definite, respectively. t_1 and t_f are initial time and final time, respectively.

Definition 1 *If there exist a positive real number $V(x(t_1), t_1)$ and $u(\cdot)$ such that*

$$J(x(t_1), u(\cdot), \xi) \leq V(x(t_1), t_1)$$

for any admissible perturbation $\xi \in \Xi$, $V(x(t_1), t_1)$ is said to be a guaranteed cost for the system staring from $x(t_1)$ at time t_1, and $u(\cdot)$ is said to be a guaranteed cost control.

Remark 4 *Let*

$$\begin{aligned}
H(V, x, \eta, t, \xi) &= x^TC^TCx + u^TRu \\
&\quad +V(x(l+1), l+1) - V(x(l), l) \qquad (38)
\end{aligned}$$

Then, it follows from optimal principle that if there exists a scalar function $V(x, t)$ and an m-vector valued function $\eta(x(t), t)$ such that

$$\begin{aligned}
&H(V, x, \eta, t, \xi) \leq 0, \quad t < t_f \\
&V(x(t_f), t_f) = x^T(t_f)P_fx(t_f)
\end{aligned}$$

for any $\xi \in \Xi$, $V(x(t_1), t_1)$ is a guaranteed cost for the system starting from $x(t_1)$ at any $t_1 < t_f$ and $\eta(x(t), t)$ $(t_1 \leq t < t_f)$ is a guaranteed cost control.

Definition 2 *Let $T(\xi)$ be an $n \times n$ matrix which depends on an uncertainty $\xi \in \Xi$. U is said to be an upper bound matrix of $T(\xi)$ if $U - T(\xi)$ is non-negative definite for all $\xi \in \Xi$.*

$A(\xi)$ is assumed to be represented as

$$A(\xi) = A_0 + \Delta A(\xi)$$

where A_0 denotes nominal matrix, $\Delta A(\xi)$ denotes their perturbation matrix. We shall seek a guaranteed cost solution of the form

$$V(x,t) = x^T P(t) x,$$

where $P(t)$ is a positive semi-definite, and is called a guaranteed cost matrix. Then (38) becomes

$$\begin{aligned}
& H(V,x,u,t,\xi) \\
= \; & x^T C^T C x + u^T R u \\
& + (A(\xi)x + Bu)^T P(t+1)(A(\xi)x + Bu) - x^T P(t) x \\
= \; & x^T C^T C x + u^T R u + x^T A^T(\xi) P(t+1) A(\xi) x \\
& + 2u^T B^T P(t+1) A(\xi) x + u^T B^T P(t+1) B u - x^T P(t) x \\
= \; & x^T C^T C x + u^T R u + x^T (A_0 + \Delta A)^T P(t+1)(A_0 + \Delta A) x \\
& + 2u^T B^T P(t+1)(A_0 + \Delta A) x \\
& + u^T B^T P(t+1) B u - x^T P(t) x
\end{aligned}$$

Some simple calculation leads to the order representation of $H(V,x,u,t,\xi)$ as

$$\begin{aligned}
& H(V,x,u,t,\xi) \\
= \; & x^T C^T C x + H_0(P(t+1),x,u) \\
& + H_1(P(t+1),x,u,\xi) - x^T P(t) x,
\end{aligned} \tag{39}$$

where $H_0(P(t+1),x,u)$ is a certain term and $H_1(P(t+1),x,u,\xi)$ is an uncertain term represented as follows.

$$\begin{aligned}
& H_0(P(t+1),x,u) \\
= \; & x^T A_0^T P(t+1) A_0 x + 2u^T B^T P(t+1) A_0 x \\
& + u^T \{B^T P(t+1) B + R\} u \\
& H_1(P(t+1),x,u,\xi) \\
= \; & 2x^T \Delta A^T P(t+1) A_0 x \\
& + x^T \Delta A^T P(t+1) \Delta A x + 2u^T B^T P(t+1) \Delta A x
\end{aligned} \tag{40}$$

Since $H_1(P(t+1),x,u,\xi)$ can not be evaluated directly, we request u^* which minimize only $H_0(P(t+1),x,u)$.

Thus,

$$\begin{aligned}
& u^*(t) = \eta^*(x(t),t) \\
= \; & (B^T P(t+1) B + R)^{-1} B^T P(t+1) A_0 x(t) \\
= \; & -\Omega(P(t+1)) A_0 x(t)
\end{aligned} \tag{41}$$

is obtained.

If $H(V,x,\eta^*,t,\xi)$ is non-positive and

$$P(t_f) = P_f, \tag{42}$$

$V(x,t) = x^T P(t) x$ is a guaranteed cost in virtue of Remark 4, it follows from non-positivity of $H(V,x,\eta^*,t,\xi)$ that

$$P(t) \geq C^T C + T_0(P(t+1)) + T_1(\xi, P(t+1)) \tag{43}$$

where $T_0(P(t+1)), T_1(\xi, P(t+1))$ are symmetric and

$$\begin{aligned}
& T_0(P(t+1)) \\
= \; & A_0^T \{I - B\Omega(P(t+1))\}^T P(t+1) \\
& \cdot \{I - B\Omega(P(t+1))\} A_0 \\
& + A_0^T \Omega(P(t+1))^T R \Omega(P(t+1)) A_0 \\
& T_1(\xi, P(t+1)) \\
= \; & \Delta A^T P(t+1)\{I - B\Omega(P(t+1))\} A_0 \\
& + A_0^T \{I - B\Omega(P(t+1))\}^T P(t+1) \Delta A \\
& + \Delta A^T P(t+1) \Delta A
\end{aligned} \tag{44}$$

Let the upper bound matrix of $T_1(\xi, P(t+1))$ be $U_1(P(t+1))$. Then if

$$P(t) = C^T C + T_0(P(t+1)) + U_1(P(t+1)) \tag{45}$$

the condition (43) is satisfied, that is, if there exists $P(t)$ which satisfies (42) and (45), $V(x,t) = x^T P(t) x$ becomes a guaranteed cost. Furthermore, if the difference equation (45) has a stationary solution P, P satisfies the following Riccati-like equation

$$P = C^T C + T_0(P) + U_1(P) \tag{46}$$

Provided that P is nonsingular, in virtue of matrix inversion lemma, equation (46) can be transformed into

$$P = C^T C + A_0^T (P^{-1} + R_c)^{-1} A_0 + U(P) \tag{47}$$

where $R_c = BR^{-1}B^T$.

Remark 5 *For the case $U(P) = 0$, (46) coincides with the standard discrete-time Algebraic Riccati Equation.*