ASYMPTOTIC CHARACTERIZATIONS OF INTEGRAL INPUT-OUTPUT TO STATE STABILITY

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Abstract

We present new characterizations of integral Input-Output-to-State Stability. This is a notion of dectectability formulated in the Input-to-State Stability framework. Equivalent properties are discussed in terms Lyapunov dissipation inequalities and asymptotic estimates of the state variables on the basis of external information provided by input and output signals.

1 Introduction

Detectability is a central notion in control theory. It plays a major role both in static state-feedback design (Lasalle's invariance principle, Jurdjevic-Quinn control, see [9]) as well as stabilization by means of dynamic output feedback or observers design. Several possibilities are available when formulating such a notion in the context of nonlinear control. According to the specific problem under consideration, they capture some or most of the useful features of its linear counterpart. One way of addressing the problem, which has proved to be especially powerful for systems subject to exogenous disturbances, is to define 0-detectability in terms of estimates involving (possibly nonlinear) gains of input and output norms. This is the so called input-output-to-state stability (IOSS), [10], and integral input-output-to-state-stability.Such notions not only allows one to extend Lasalle's type stability results to the case of non-autonomous systems, [1], it also provides a machinery, fully compatible with the small-gain and ISS formalisms, in order to understand relevant questions such as minimum-phase behaviour or certainty equivalence [11, 6].

Although general nonlinear systems may often exhibit an overwhelming variety of behaviours, it turns out that many of of the “reasonable” formulations, (meaning at least compatible with the linear notion of detectability), for such a property are in the end equivalent to each other. Hereby we discuss characterizations of IOSS in terms of asymptotic behaviour of systems solutions. This leads to several useful decompositions of the property in terms of seemingly weaker notions.

2 Basic definitions

Consider systems in the following general form:

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x(t)), \tag{1}$$

where, for each $t \geq 0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{U}$, a subset of $\mathbb{R}^m$. We assume that the maps $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h: \mathbb{R}^n \to \mathbb{R}^p$ are locally Lipschitz continuous, with $f(0,0) = 0$ and $h(0) = 0$. The symbol $|\cdot|$ denotes the Euclidean norms in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^p$.

By an *input* we mean a measurable and locally essentially bounded function $u: \mathcal{I} \to \mathbb{U}$, where $\mathcal{I}$ is a subinterval of $\mathbb{R}$ which contains the origin. Whenever the domain $\mathcal{I}$ of an input u is not specified, it will be understood that $\mathcal{I} = \mathbb{R}_{\geq 0}$. The L_∞^m-norm (possibly infinite) of an input u is denoted by $\|u\|$, i.e.,$\|u\| = (\text{ess})\sup\{|u(t)|, t \in \mathcal{I}\}$.

Given any input u and any $\xi \in \mathbb{R}^n$, the unique maximal solution of the initial value problem $\dot{x} = f(x,u)$, $x(0) = \xi$ (defined on some maximal open subinterval of $\mathcal{I}$) is denoted by $x(\cdot, \xi, u)$. When $\mathcal{I} = \mathbb{R}_{\geq 0}$, this maximal subinterval has the form $[0, T_{\xi,u})$. The system is said to be *forward complete* if for every initial state ξ and for every input u defined on $\mathbb{R}_{\geq 0}$, $T_{\xi,u} = +\infty$. The corresponding output is denoted by $y(\cdot, \xi, u)$, that is, $y(t, \xi, u) = h(x(t, \xi, u))$ on the domain of definition of the solution.

We use standard terminology: $\mathcal{N}$ is the class of of continuous, increasing functions from $[0, \infty)$ to $[0, \infty)$; $\mathcal{K}$ is the subset of $\mathcal{N}$ functions that are zero at zero and strictly increasing; $\mathcal{K}_\infty$ is the subset of $\mathcal{K}$ functions that are unbounded; $\mathcal{L}$ is the set of functions $[0, +\infty) \to [0, +\infty)$ which are continuous, decreasing, and converging to 0 as their argument tends to $+\infty$; $\mathcal{KL}$ is the class of functions $[0, \infty)^2 \to [0, \infty)$ which are class $\mathcal{K}$ on the first argument and class $\mathcal{L}$ on the second argument. A positive definite function $[0, \infty) \to [0, \infty)$ is one that is zero at 0 and positive otherwise.

The following notion was introduced in [12]:

Definition 2.1 The system (1) satisfies the *unboundedness observability* (UO) property if, for each state ξ and control u such that $T_{\xi,u} < \infty$, it follows that

$$\limsup_{t \to T_{\xi,u}} |y(t, \xi, u)| = +\infty.$$

The detectability notion that will be investigated throughout this note was introduced in [2].

Definition 2.2 The system (1) is *integral input-output-to-state stable* (iIOSS) if there exist some $\beta \in \mathcal{KL}$, σ_u, σ_y and $\alpha \in \mathcal{K}_\infty$ such that

$$\alpha(|x(t,\xi,u)|) \leq \beta(|\xi|, t) + \int_0^t \sigma_u(|u(s)|) + \sigma_y(|y(s)|)\, ds \tag{2}$$

for all $t \in [0, T_{\xi,u})$, all $\xi \in \mathbb{R}^n$ and all u. □

In order to provide asymptotic characterizations of iIOSS we will show that the property is equivalent to a different detectability notion, the so called Input-Output-to-State Stability, of a suitably augmented system. For the sake of completeness we recall the following definition (see for instance [15] and [10]).

Definition 2.3 The system (1) is *input-output-to-state stable* (IOSS) if there exist some $\beta \in \mathcal{KL}$, $\gamma_u \in \mathcal{K}$ and $\gamma_y \in \mathcal{K}$ such that

$$|x(t,\xi,u)| \leq \beta(|\xi|, t) + \gamma_u(\|u\|_\infty) + \gamma_y\left(\left\|y_{[0,t]}\right\|\right) \tag{3}$$

for all $t \in [0, T_{\xi,u})$, all $\xi \in \mathbb{R}^n$ and all u. □

Clearly, the IOSS property implies the UO property.

3 A preliminary result on iIOSS

The following result of independent interest will be needed in order to prove the asymptotic characterizations in Theorem 2. The proof is only sketched and deferred to a forthcoming paper.

Proposition 3.1 Assume that system (1) satisfies, for some $\beta \in \mathcal{KL}$, $\sigma_u, \gamma_u, \sigma_y$ and $\alpha \in \mathcal{K}_\infty$ the estimate

$$\begin{aligned} \alpha(|x(t,\xi,u)|) \leq \beta(|\xi|, t) + \int_0^t \sigma_u(|u(s)|) + \\ + \int_0^t \sigma_y(|y(s)|)\, ds + \gamma_u(\|u_{[0,t]}\|_\infty) \end{aligned} \tag{4}$$

for all $t \in [0, T_{\xi,u})$, all $\xi \in \mathbb{R}^n$ and all inputs u. Then the system is integral IOSS. □

Proof. Let us assume without loss of generality that $\sigma_u = \gamma_u$ in (4) (if this is not the case just consider $\gamma := \max\{\sigma_u, \gamma_u\}$). Choose any smooth $\varphi \in \mathcal{K}_\infty$ with the property that $\gamma \circ \varphi \leq \alpha/2$. We first look at trajectories of the following auxiliary system:

$$\dot{x} = f(x, \varphi(|x|)d), \qquad y = h(x) \tag{5}$$

where the input signal d is assumed to take values in the unit ball of $\mathbb{R}^m$. Let us denote by $x^\varphi(t,\xi,d)$ the solution of (5). Since trajectories of (5) can be interpreted as solutions of (1) corresponding to the input $u = \varphi(|x^\varphi|)d$, we have by virtue of (4):

$$\begin{aligned} & \alpha(|x^\varphi(t,\xi,d)|) \\ \leq\ & \beta(|\xi|, t) + \int_0^t \gamma(|u(s)|) + \sigma_y(|h(x^\varphi(s))|)\, ds \\ & + \gamma(\|u_{[0,t]}\|_\infty) \\ \leq\ & \beta(|\xi|, t) + \int_0^t \gamma(|u(s)|) + \sigma_y(|h(x^\varphi(s))|)\, ds \\ & + \gamma \circ \varphi(\|x^\varphi_{[0,t]}\|_\infty) \\ \leq\ & \beta(|\xi|, t) + \int_0^t \gamma(|u(s)|) + \sigma_y(|h(x^\varphi(s))|)\, ds \\ & + \alpha(\|x^\varphi_{[0,t]}\|_\infty)/2 \end{aligned} \tag{6}$$

Now taking supremums over $t \in [0, \tau]$, $\tau < T_{\xi,d}$ yields

$$\begin{aligned} \alpha(\|x^\varphi_{[0,\tau]}\|_\infty) \leq \beta(|\xi|, 0) + \alpha(\|x^\varphi_{[0,\tau]}\|_\infty)/2 + \\ + \int_0^\tau \sigma_u(|u(s)|) + \sigma_y(|h(x^\varphi(s))|)\, ds. \end{aligned} \tag{7}$$

Thus, subtracting $\alpha(\|x^\varphi_{[0,\tau]}\|_\infty)/2$ in both sides of (7) we can conclude

$$\begin{aligned} \alpha(|x^\varphi(\tau,\xi,d)|)/2 \leq \alpha(\|x^\varphi_{[0,\tau]}\|_\infty)/2 \qquad (8) \\ \leq \beta(|\xi|, 0) + \int_0^\tau \gamma(|u(s)|) + \sigma_y(|h(x^\varphi(s))|)\, ds. \end{aligned}$$

Let us denote by

$$\begin{aligned} z(\tau,\xi,d) := \alpha(|x^\varphi(\tau,\xi,d)|)/2 \\ - 2\int_0^\tau \gamma(|\varphi(|x^\varphi|)d|) + \sigma_y(|h(x^\varphi(s))|)\, ds. \end{aligned} \tag{9}$$

Then, consider the following definition of $g(\cdot) : \mathbb{R}^n \to \mathbb{R}$

$$g(\xi) := \sup\{z(\tau,\xi,d) : \tau \geq 0,\ d \in \mathcal{L}_\infty(1)\} \tag{10}$$

Thanks to (8) we are able to show that the $g(\xi)$ is well defined; in fact it is radially unbounded and decrescent:

$$\alpha(|\xi|)/2 \leq g(\xi) \leq \beta(|\xi|, 0). \tag{11}$$

Local Lipschitzianity of g can be shown along the same lines as in [3]. The next step in the proof is to show that g cannot increase too fast along trajectories. This follows by a technique similar to [2] by exploiting the definition of g. We now follow an argument as in [2] in order to show that a suitable locally Lipschitz iIOSS function $V(x)$ exists and, consequently, that the system is iIOSS as claimed. Notice that an estimate as in (4) implies, for $u \equiv 0$, the so called integral Output-to-State Stability. We already know that this is equivalent (see Lemma 3.3 in [7]) to the existence of a smooth function $V_0(x)$, positive definite and radially unbounded, such that:

$$DV_0(x)f(x,0) \leq -\kappa(|x|) + \sigma_1(|h(x)|) \qquad \forall\, x \in \mathbb{R}^n. \tag{12}$$

where κ is positive definite and $\sigma_1 \in \mathcal{K}_\infty$. By using a similar argument as in [2], Proposition 2.5, we can show that it is possible to rescale V_0 (and relaxing properness to *semi-properness*) in order to satisfy the following dissipation inequality

$$D\tilde{V}_0 f(x,u) \leq -\tilde{\kappa}(|x|) + \tilde{\sigma}_1(|h(x)|) + \tilde{\sigma}_2(|u|) \qquad (13)$$

with $\tilde{\kappa}$ positive definite and $\tilde{\sigma}_1$ of class $\mathcal{K}_\infty$. Then, it is straightforward following the same lines as in [2] to show that $V(x) := \tilde{V}_0(x) + g(x)$ is an iIOSS-Lyapunov function. Then a system satisfying (4) is iIOSS.

■

The proof of Proposition 3.1 relies on a (locally Lipschitz) converse Lyapunov Theorem for integral Input-Output to State Stability. Since the result deserves attention in itself we state it separately :

Theorem 1 *A system as in (1) is integral Input-Output to State Stable if and only if it admits a smooth iIOSS-Lyapunov function, viz. there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\sigma_u, \sigma_y \in \mathcal{K}$, ρ positive definite and $V(x) : \mathbb{R}^n \to \mathbb{R}$ such that $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$ and for any x in $\mathbb{R}^n$ and any u in $\mathbb{R}^m$*

$$DV(x)f(x,u) \leq -\rho(|x|) + \sigma_u(|u|) + \sigma_y(|h(x)|) \qquad (14)$$

■

4 An IOSS formulation of integral IOSS

The main result in this section is to estabilish equivalent formulations of integral IOSS in terms of asymptotic detectability notions. We first introduce the following auxiliary system:

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x,u) \\ \dot{e}_1 & = & \sigma_u(|u|) \\ \dot{e}_2 & = & \sigma_y(|h(x)|) \\ & & \\ e & = & [e_1, e_2]' \end{array} \right. \qquad (15)$$

The following Proposition is central for the proof of our Main Result.

Proposition 4.1 A system as in (1) is integral IOSS if and only if the auxiliary system (15) is IOSS with respect to the output e. □

Proof. Let us first show that IOSS of (15) implies integral IOSS. We let $z = [x', e_1, e_2]'$. By hypothesis we know that the following estimate holds along trajectories of (15):

$$|z(t,\eta,u)| \leq \beta(|\eta|,t) + \gamma_u(\|u_{[0,t]}\|) + \gamma_e(\|e_{[0,t]}\|) \quad (16)$$

where β is of class $\mathcal{KL}$ and γ_u, γ_e are of class $\mathcal{K}_\infty$. In particular, for $e_1(0) = 0$ and $e_2(0) = 0$, equation (16) yields

$$\begin{array}{rcl} |x(t,\xi,u)| & \leq & |z(t,[\xi',0,0]',u)| \\ & \leq & \beta(|\xi|,t) + \gamma_u(\|u_{[0,t]}\|_\infty) + \gamma_e(\|e_{[0,t]}\|_\infty) \\ & \leq & \beta(|\xi|,t) + \gamma_u(\|u_{[0,t]}\|) \\ & + & \gamma_e\left(\int_0^t \sigma_u(|u(s)|) + \sigma_y(|h(x(s,\xi,u))|)\,ds\right) \end{array} \qquad (17)$$

Letting $\alpha = (3\gamma_e)^{-1}$ we have

$$\begin{array}{rl} & \alpha(|x(t,\xi,u)|) \leq \alpha\left(\beta(|\xi|,t) + \gamma_u(\|u_{[0,t]}\|)\right. \\ + & \left.\gamma_e\left(\int_0^t \sigma_u(|u(s)|) + \sigma_y(|h(x(s,\xi,u))|)\,ds\right)\right) \\ \leq & \alpha(3\beta(|\xi|,t)) + \alpha(3\gamma_u(\|u_{[0,t]}\|)) \\ + & \alpha \circ 3\gamma_e\left(\int_0^t \sigma_u(|u(s)|) + \sigma_y(|h(x(s,\xi,u))|)\,ds\right) \\ = & \tilde{\beta}(|\xi|\,,t) + \tilde{\gamma}(\|u_{[0,t]}\|_\infty) \\ + & \int_0^t \sigma_u(|u(s)|) + \sigma_y(|h(x(s,\xi,u))|)\,ds \end{array} \qquad (18)$$

Application of Proposition 3.1 to the estimate (18) is enough to conclude integral IOSS of system (1).

Let us look now at the converse implication. Clearly:

$$\begin{array}{rcl} |e_1(t)| & \leq & \|e_{1[0,t]}\|_\infty \leq \|e_{[0,t]}\|_\infty \\ |e_2(t)| & \leq & \|e_{2[0,t]}\|_\infty \leq \|e_{[0,t]}\|_\infty \end{array}$$

Thus, in order to show IOSS of the auxiliary system (15) we only need to find suitable estimates for the x-component of the extended state z. Now, by virtue of integral IOSS, there exist class $\mathcal{K}_\infty$ functions $\alpha, \sigma_u, \sigma_y$ and β of class $\mathcal{KL}$ so that

$$\begin{array}{rl} & \alpha(|x(t,\xi,u)|) \\ \leq & \beta(|\xi|\,,t) + \int_0^t \sigma_u(|u(s)|) + \sigma_y(|h(x(s,\xi,u))|)\,ds \\ \leq & \beta(|z(0)|\,,t) + |e_1(t) - e_1(0)| + |e_2(t) - e_2(0)| \\ \leq & \beta(|z(0)|\,,t) + |e_1(t)| + |e_1(0)| + |e_2(t)| + |e_2(0)| \\ \leq & \beta(|z(0)|\,,t) + 4\|e_{[0,t]}\|_\infty \end{array} \qquad (19)$$

It is straightforward from (19) and by exploiting the weak triangular inequality to show that IOSS of the auxiliary system holds. ■

5 Asymptotic characterizations of integral IOSS

Before stating our main results we need the following definitions:

Definition 5.1 A system as in (1) is zero-input *locally stable modulo output* (O-LS) if for for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for any ξ satisfying $\max\{|\xi|, \|y_{[0,t]}\|_\infty\} \leq \delta_\varepsilon$, it holds that

$$|x(t,\xi,0)| \leq \varepsilon \qquad \forall\, t \in [0, T_{\xi,u\equiv 0});\qquad (20)$$

□

Definition 5.2 A system as in (1) is zero-input *locally stable modulo integral output* (O-iLS) if for for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for any ξ satisfying $\max\{|\xi|, \int_0^t \sigma_y(|y(s)|)\, ds\} \leq \delta_\varepsilon$, it holds that

$$|x(t,\xi,0)| \leq \varepsilon \qquad \forall\, t \in [0, T_{\xi,u\equiv 0}); \tag{21}$$

□

Definition 5.3 A system as in (1) enjoys the *IO-LIM property* if for some $\gamma_u, \gamma_y \in \mathcal{K}$, all $\xi \in \mathbb{R}^n$ and all measurable $u(\cdot)$

$$\inf_{t\in[0,T_{\xi,u})} |x(t,\xi,u)| \leq \max\{\gamma_u(\|u\|_\infty), \gamma_y(\|y\|_\infty)\}, \tag{22}$$

where the $\|\cdot\|_\infty$ norms are taken over $[0, T_{\xi,u})$ □

It was one of the main results in [4] that:

IOSS $\quad\Leftrightarrow\quad$ [IO-LIM & zero-input O-LS].

Therefore, by means of the above equivalence and exploiting Proposition 4.1, it is possible to derive asymptotic characterizations of integral IOSS.

Definition 5.4 A system as in (1) enjoys the *bounded input-output energy converging state property* if there exists σ_u and σ_y such that for all $\xi \in \mathbb{R}^n$ and all measurable inputs u the following implication holds:

$$\begin{aligned} &\int_0^{+\infty} \sigma_u(|u(s)|) + \sigma_y(|y(s)|)\, ds < +\infty \\ \Rightarrow\ &\liminf_{t\to+\infty} |x(t,\xi,u)| = 0 \end{aligned} \tag{23}$$

□

Definition 5.5 A system as in (1) enjoys the *bounded input-output energy* frequently *bounded state property* if there exists σ_u and σ_y such that for all $\xi \in \mathbb{R}^n$ and all measurable inputs u the following implication holds:

$$\begin{aligned} &\int_0^{+\infty} \sigma_u(|u(s)|) + \sigma_y(|y(s)|)\, ds < +\infty \\ \Rightarrow\ &\liminf_{t\to+\infty} |x(t,\xi,u)| < +\infty \end{aligned} \tag{24}$$

□

Our main result is as follows:

Theorem 2 *Given a system as in (1), the following facts are equivalent:*

1. *the system is integral IOSS*
2. *the system is zero-input O-iLS and BIOE-CS*
3. *the system is zero-input globally iOSS and BIOE-FBS*

∎

Proof. Implication 1 ⇒ 3 follows immediately from the definition of iIOSS. We show next 3 ⇒ 2.

According to Theorem 1 applied to the zero-input system, iOSS implies the existence of a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, such that $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$ for some α_1, α_2 of class $\mathcal{K}_\infty$ and along trajectories

$$DV(x)f(x,0) \leq -\rho(|x|) + \tilde{\sigma}_y(|h(x)|) \tag{25}$$

for some $\mathcal{K}_\infty$ function $\tilde{\sigma}_y$ and some positive definite ρ. By exploiting the class $\mathcal{KK}$ function lemma in [2] it follows from (25) that:

$$\begin{aligned} & DV(x)f(x,u) \\ \leq\ & DV(x)f(x,0) + |DV(x)(f(x,u) - f(x,0))| \\ \leq\ & -\rho(|x|) + \tilde{\sigma}_y(|h(x)|) + \gamma(|x|)\tilde{\sigma}_u(|u|) \end{aligned} \tag{26}$$

with $\tilde{\sigma}_u$ of class $\mathcal{K}_\infty$ and γ of class $\mathcal{N}$. Along the same lines as in Lemma 4.10 of [2], equation (25) is equivalent to the existence of a *semi-proper* function $U(x)$ (viz. resulting from the composition of a proper function $V(x)$ with a class $\mathcal{K}$ function) which satisfies the following dissipation inequality:

$$DU(x)f(x,u) \leq -\tilde{\rho}(|x|) + \tilde{\sigma}_u(|u|) + \tilde{\sigma}_y(|h(x)|). \tag{27}$$

with $\tilde{\rho}$ positive definite. Let $\hat{\sigma}_u = \max\{\tilde{\sigma}_u, \sigma_u\}$ and $\hat{\sigma}_y = \max\{\tilde{\sigma}_y, \sigma_y\}$ where σ_u, σ_y are as in definition 5.5. Pick $\xi \in \mathbb{R}^n$ and u with $\int_0^\infty \hat{\sigma}_u(|u(s)|) + \hat{\sigma}_y(|y(s)|)\, ds < +\infty$. By the BIOE-FBS assumption,

$$m := \liminf_{t\to+\infty} |x(t,\xi,u)| < +\infty.$$

We want to show $m = 0$. By contradiction, assume $m > 0$. For any $r > 0$, let $w(r) = \max_{|x|\leq r} U(x)$. By semi-properness of $U(x)$ there exists $M \geq 3m$ such that $w(M) - w(2m) > 0$. We let T be such $\int_T^{+\infty} \hat{\sigma}_u(|u(s)|) + \hat{\sigma}_y(|y(s)|)\, ds < w(M) - w(2m)$. By the BIOE-FBS assumption there exists $\tau \geq T$ such that $|x(\tau,\xi,u)| < 2m$. By virtue of (25), for all $t \geq \tau$

$$\begin{aligned} & V(x(t,\xi,u)) - V(x(\tau,\xi,u)) \\ \leq\ & \int_\tau^t \tilde{\sigma}_u(|u(s)|) + \tilde{\sigma}_y(|y(s)|)\, ds \\ <\ & \int_T^{+\infty} \tilde{\sigma}(|u(s)|) + \tilde{\sigma}_y(|y(s)|)\, ds \\ <\ & w(M) - w(2m) \end{aligned} \tag{28}$$

Hence $V(x(t,\xi,u)) < w(M) < w(\infty)$ for all $t \geq \tau$ and hence $x(t,\xi,u)$ is uniformly bounded, because semi-proper Lyapunov function have compact sublevel sets. Let $\tilde{M} > 0$

be such that $|x(t,\xi,u)| < \tilde{M}$ for all t. Therefore we have by virtue of (26)

$$DV(x)f(x,u) \leq -\rho(|x|) + \tilde{\sigma}_y(|h(x)|) + \gamma(\tilde{M})\tilde{\sigma}_u(|u|)$$

This is enough to conclude that, for the considered trajectory, a $\mathcal{KL}$ function β and α of class $\mathcal{K}_\infty$ exist so that

$$\alpha(|x(t,\xi,u)|) \leq \beta(|\xi|,t) + \int_0^t \tilde{\sigma}(|u(s)|) + \tilde{\sigma}_y(|y(s)|)\,ds$$

and therefore, along the same lines as in Proposition 6 of [13], $|x(t,\xi,u)| \to 0$. This implies $m = 0$ which is clearly a contradiction.

We are only left to show $2 \Rightarrow 1$. Let (1) enjoy the BIOE-CS property and consider the auxiliary system (15), where σ_u and σ_y are the energy supply functions, as in definition 5.4. By virtue of BIOE-CS we have that (15) satisfies

$$\begin{aligned} \|e\|_\infty < +\infty \quad &\Rightarrow \quad \int_0^\infty \sigma_u(|u(s)|) + \sigma_y(|y(s)|)\,ds < +\infty \\ &\Rightarrow \quad \liminf_{t\to+\infty} |x(t,\xi,u)| = 0. \end{aligned} \tag{29}$$

Therefore the following asymptotic property is true for any choice of γ_1 and γ_2 in $\mathcal{K}_\infty$:

$$\liminf_{t\to+\infty} |x(t,\xi,u)| \leq \max\{\gamma_1(\|u\|_\infty), \gamma_2(\|e\|_\infty)\}. \tag{30}$$

for any $\xi \in \mathbb{R}^n$ and any measurable $u(\cdot)$. Since $|e(t)| \leq \|e\|_\infty$ for all $t \geq 0$ system (15) satisfies the IO-LIM property. We show next that zero-input iO-LS of system (1) implies zero-input O-LS of the augmented system (15). Before going ahead though, we remark that this completes the proof of Theorem 2; in fact by the main result in [4], IO-LIM and zero-input O-LS imply Input-Output to State Stability of 15, and this is equivalent (by virtue of Proposition 4.1) to integral IOSS of system (1). Let $\varepsilon > 0$ be arbitrary. We define $\tilde{\delta}_\varepsilon := \delta_{\varepsilon/2}/2$, where δ_ε is generated as in definition 5.2. Then we have the following implications:

$$\begin{aligned} & \max\{|[\xi', e_1(0), e_2(0)]|\,, \|e_{[0,t]}\|_\infty\} \leq \tilde{\delta}_\varepsilon \\ \Rightarrow \quad & \max\{|\xi|, |e(0)|\} \leq \tilde{\delta}_\varepsilon \text{ and} \\ & \left|\int_0^t \sigma_y(|y(s)|)\,ds\right| = |e_1(t) - e_1(0)| \leq 2\tilde{\delta}_\varepsilon = \delta_{\varepsilon/2} \\ \Rightarrow \quad & |x(t,\xi,0)| \leq \varepsilon/2 \;\Rightarrow\; |[x(t,\xi,0)', e(t)']| \leq \varepsilon \end{aligned} \tag{31}$$

that is the auxiliary system (15) is zero-input locally stable modulo output. ∎

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