LOW ORDER MODELING AND OPTIMAL CONTROL DESIGN OF A HEATED PLATE

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Abstract

A CFD (computational fluid dynamics model) of a heated two-dimensional plate is presented. The model is finely discretised and the resulted order, which accounts to more than 1,000 states is too high for control and optimization design. A model reduction strategy is applied to the model, which allows significant reduction of the original order as well as control and optimization designs. The plant successfully follows the desired trajectory when controlled by an optimal controller based on the reduced order model.

1 Introduction

Heat transfer is one of the most encountered physical phenomena in daily life. In process control, manipulating heat transfer becomes an integral part of reaching the desired temperature distribution. Heat transfer as for example conduction is described by a set of partial differential equations. In the simulation tools, these equations are finely discretised over time and spatial coordinates, and the discretization procedure increases the model order significantly. The numerical model is adequate for simulation, but control design requires a much more lower order model.

To reduce the model order, Proper Orthogonal Decomposition or also known as Karhunen-Loève expansions are getting more commonly applied in many physical systems governed by partial differential equations. The advantage of applying this technique is the incorporation of simulation or experimental data as well as the existing physical relationships in the original model. The resulting reduced order model is of state-space form and this is very convenient for control design.

In this paper, we present the application of a method of Proper Orthogonal Decomposition (POD) to a 2D heated plate. The control problem amount to reaching to a desired temperature distribution along the plate based on the reduced order model derived by POD reduction technique. The designed controller is an LQR controller with reference signal(s) unequal to zero(s).

The paper is organized as follows. First, the model of heat transfer by conduction on a 2D plate is presented. Thereafter, the reduced order modeling strategy by POD is discussed and some simulation results are shown. Further, the optimal control design is presented and finally the controlled temperature distribution of the heated plate is shown.

2 Heated plate

The model that we are using in this paper is a thin two dimensional plate. The sketch of the plate is given in Figure 1. The dimension of the plate is $0.3\mathtt{m} \times 0.4\mathtt{m}$ in length (x) and height(y), respectively. This corresponds to a Cartesian spatial area $\mathbb{X} = [0, 0.3] \times [0, 0.4]$. The temperature along the north side is kept at $100^{\diamond}C$ and there is incoming heat flux at the west side of $500\mathtt{kW/m^2}$. The eastern and the southern boundaries are insulated.

Thermal conductivity of the plate material is $k = 1000\mathtt{W/m.K}$. The plate is very thin, only $1\mathtt{cm}$. The computational domain is divided into 44 grid cells along the y direction and 33 grid cells along the x direction. Thus in total, the number of equations solved are 1452 equations.

The model of this heated plate is given by:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + S \qquad (1)$$

In the numerical simulation, Eq. 1 is discretised by employing The Finite Volume Method [5], where Eq. 1 is integrated over a unit volume $\Delta V = \Delta x \times \Delta y \times 1\mathtt{cm}$. The term S denotes the source terms, in this case the west heat flux and the constant temperature at the north boundary.

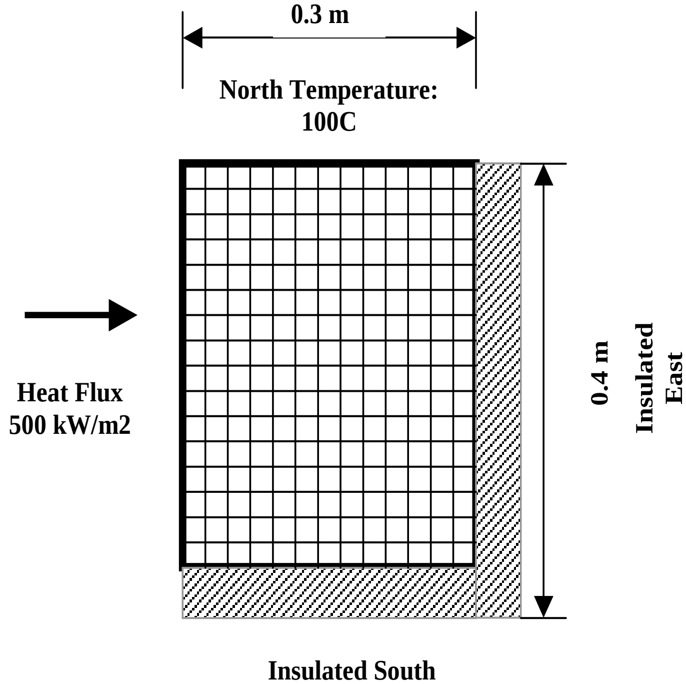

Figure 1: Sketch of the Heated Plate

$$\int_t^{t+\Delta t}\int_{\Delta V}\rho c\frac{\partial T}{\partial t}dV\,dt = \int_t^{t+\Delta t}\int_{\Delta V}\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dV\,dt + \int_t^{t+\Delta t}\int_{\Delta V}\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}dV\,dt\right) + \int_t^{t+\Delta t}\int_{\Delta V}S dV\,dt \tag{2}$$

The integration of Eq. 2 for a specific grid point x_g results in Eq. 3 with $\tilde{T}(x_g, k+1)$ denotes the temperature of a specific grid point x_g at time step $k+1$ and $\tilde{T}(x_{gw}, k+1), \tilde{T}(x_{ge}, k+1), \tilde{T}(x_{gn}(k+1), \tilde{T}(x_{gs}, k+1)$ denote the temperatures of the west, east, north, and south neighboring grid points, respectively. The temperature at the current time step is denoted by $T(x_g, k)$.

$$\begin{aligned}\rho c\frac{\Delta x\Delta y}{\Delta t}\left(\tilde{T}(x_g,k+1\right) &= \rho c\frac{\Delta x\Delta y}{\Delta t}\left(\tilde{T}\right)(x_g,k))\\ &+ \left(kA\frac{T(x_{ge},k+1)-\tilde{T}(x_g,k+1)}{\delta x}\right)\\ &- \left(kA\frac{\tilde{T}(x_g,k+1)-\tilde{T}(x_g,k+1)}{\delta x}\right)\\ &+ \left(kA\frac{\tilde{T}(x_{gn},k+1)-\tilde{T}(x_g,k+1)}{\delta y}\right)\\ &- \left(kA\frac{\tilde{T}(x_g,k+1)-\tilde{T}(x_{gs},k+1)}{\delta}y\right)\\ &+ S\Delta V\,(x_g,k) = 0\end{aligned} \tag{3}$$

Rearranging Eq. 3 gives:

$$\begin{aligned}a_P\tilde{T}(x_g,k+1) &= a_P^0\tilde{T}(x_g,k) + a_W\tilde{T}(x_{gw}(k+1)\\ &+ a_E\tilde{T}(x_{ge},k+1) + a_N\tilde{T}(x_{gn},k+1)\\ &+ a_S\tilde{T}(x_{gs},k+1) + S(x_g,k)\end{aligned} \tag{4}$$

Calculating Eq. 4 for all grid points and collecting all grid points into $\mathbf{T}(k+1) = \mathtt{col}[\tilde{T}(x_g,k+1)]$ and $\mathbf{S}(k) = \mathtt{col}[\tilde{S}(x_g,k)]$ give the recursive linear system of equations:

$$A\mathbf{T}(k+1) = A_0\mathbf{T}(k) + \mathbf{S}(k) \tag{5}$$

Eq. 5 is the equation solved by the simulation tools employing CFD Finite Volume Method. Due to the high dimension of the matrices, iteration procedure is common in CFD calculation before a solution is determined. In the case of non-linear PDEs, for example when the conductivity constant is temperature-dependent, the matrices in Eq. 5 are updated at every time step.

With initial condition of $\mathbf{T}\,(0) = \mathbf{0}$ along the plate and subjection to north temperature boundary at $100\mathtt{C}$ and west heat flux of $k = 1000\mathtt{W/m.K}$, the steady-state temperature distribution of the full order model is shown in Fig. 2.

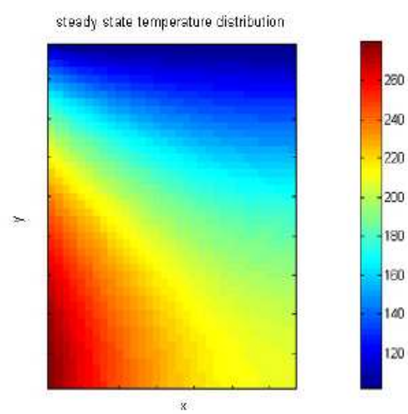

Figure 2: Steady state Response of the full order model 1452 grid cells)

3 Model reduction strategy

The temperature distribution $T(x, y)$ of the heated plate is described by Eq. 1 over over a (gridded) domain $\mathbb{X} \times \mathbb{T}$ of spatial and temporal coordinates.

Eq. 1 can be written as

$$\frac{\partial T}{\partial t}(x,t) = D(T(x,t))$$

with $D(\cdot)$ an operator that involve spatial derivatives and other functions [1], $x \in \mathbb{X}$ and $t \in \mathbb{T}$.

For any function $T : \mathbb{X} \times \mathbb{T} \longrightarrow \mathbb{R}$ we define the residual $R\left(T\left(x,t\right)\right)$:

$$R(T(x,t)) = \frac{\partial T}{\partial t}(x,t) - D(T(x,t)) \tag{6}$$

We postulate that the solutions T can be expanded as a Fourier series (or spectral form):

$$T(x,t) = \sum_{i \in \mathbb{I}} a_i(t)\varphi_i(x) \qquad x \in \mathbb{X}, \quad t \in \mathbb{T}$$

where $\varphi_i(\cdot)$ is a set of orthonormal basis functions in $\mathbb{X}$, with i ranging over a countable index set $\mathbb{I} \subseteq \mathbb{N}$, that is $\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$ where δ_{ij} is the Kronecker delta and $\langle \cdot, \cdot \rangle$ is an appropriate inner product defined on the spatial domain $\mathbb{X}$.

An nth order approximation of (7) is then given by the truncated sequence

$$T_n(x,t) = \sum_{i=1}^{n} a_i(t)\varphi_i(x) \qquad x \in \mathbb{X}, \quad t \in \mathbb{T}. \tag{7}$$

The method of Proper Orthogonal Decompositions requires that the Galerkin projection of the residual $R(T_n(x,t))$ on the space spanned by the basis functions $\varphi_i(\cdot)$, $i = 1, \ldots, n$ vanishes, cf. [3] [4].

$$\langle R(T(x,t)), \varphi_i(x) \rangle \; = \; 0 \qquad i = 1, \ldots, n \tag{8}$$

for all $t \in \mathbb{T}$.

3.1 Determination of time varying coefficients $a_i(t)$

For every choice of orthonormal basis functions, the requirement (8) defines a finite number of constraints on the time varying coefficients $a_i(t)$, $i \in \mathbb{I}$. The condition (8) is equivalent to setting

$$\dot{a}_i(t) = \langle D\left(\sum_{i \in \mathbb{I}} a_i(t)\varphi_i(x)\right), \varphi_i(x) \rangle \tag{9}$$

Since $D\left(.\right)$ is linear in our case, replacing the original sequence of $T\left(x,t\right)$ by its truncated sequence $T_n\left(x,t\right)$ in (8) will result in n constraints of coefficient functions $a_i\left(t\right)$.

For linear operator $D\left(.\right)$, (8) is equivalent to setting

$$\dot{a}_i(t) = \langle D\left(\sum_{i=1}^{n} a_i(t)\varphi_i(x)\right), \varphi_i(x) \rangle \tag{10}$$

Thus, the time-varying coefficients $a_i(t)$ associated with the approximation (7) can be found by solving a system of n ordinary differential equations. The initial conditions of (10) is determined from:

$$a_i(0) = \langle T\left(x,0\right), \varphi_i(x) \rangle \tag{11}$$

3.2 Determination of basis function $\varphi_i(x)$

The basis functions $\varphi_i(x)$ with $i \in \mathbb{I}$ are the *modes* of the spatial dynamics of $T(x,t)$. A characteristic feature of the reduction method that we employ here is that the basis functions $\varphi_i(x)$, with $i \in \mathbb{I}$, are obtained from *data*. To determine $\varphi_i(x)$, we consider the data

$$\tilde{T}(x,t) \qquad x \in \mathbb{X}, t \in \mathbb{T}$$

where $\mathbb{X}$ is a gridded space of dimension K and $\mathbb{T}$ is also gridded and has dimension L. The relation between the time step k_i and the time t_i is given by $t_i = k_i.\Delta t$. The data is stored in a matrix T_{snap} which is defined as $T_{\text{snap}} = (T_{\text{snap}})_{ij}$ with $i = 1, \ldots, K$ and $j = 1, \ldots, L$. With $\mathbf{T}(k) = \texttt{col}[\tilde{T}(x,k)]$ we can write T_{snap} as:

$$T_{\text{snap}} \;\; = \;\; \begin{pmatrix} \mathbf{T}(k_1) & \ldots & \mathbf{T}(k_{L-1}) & \mathbf{T}(k_L) \end{pmatrix}$$

We then calculate a singular value decomposition of T_{snap} as

$$T_{\text{snap}} = \Phi\Sigma\Psi^{\top} \qquad \Phi \in \mathbb{R}^{K \times K}, \quad \Psi \in \mathbb{R}^{L \times L} \text{ unitary matrices}$$

where Σ is a $K \times L$ matrix which contains the singular values of T_{snap} in *nonincreasing order* on its main diagonal. Let

$$\Phi = \begin{pmatrix} \varphi_1, \ldots, \varphi_K \end{pmatrix}$$

denote the column-partitioning of Φ and set, for all $i \in \mathbb{I}$, the basis functions

$$\varphi_i(x) = \langle \varphi_i, x \rangle$$

where $x \in \mathbb{X}$. Note that the calculation of the first n ($n \ll K$) basis functions requires only the first n left singular vectors of T_{snap}. We denote the set of basis vectors correspond to n largest singular values as Φ_n.

4 Application of POD

Proper Orthogonal Decomposition is then applied to the CFD numerical model which originally solves 1452 equations. The original model is simulated for $L = 400$ time steps and the results are collected in the in matrix T_{snap}. After performing SVD for T_{snap}, we pick 5 basis vectors which corresponds to the 5 largest singular value. The comparison between the original model (K=1452) and the reduced order model (n=5) is shown in Figure. 3. It is shown that the reduced order model can capture the dynamics of the original model quite accurately, with highest deviation is less than $1^{\diamond}\texttt{C}$. So with less than 0.5% of the original order, the reduced order model still performs very well.

4.1 State Space Form

From the derivation of the discrete CFD model (Eq. 5), we can derive the state space form of the reduced order model:

$$\begin{aligned} A\mathbf{T}(k+1) &= A_0\mathbf{T}(k) + \mathbf{S}(k) \\ \mathbf{T}(k+1) &= \Phi_n a\left(k+1\right) \end{aligned} \tag{12}$$

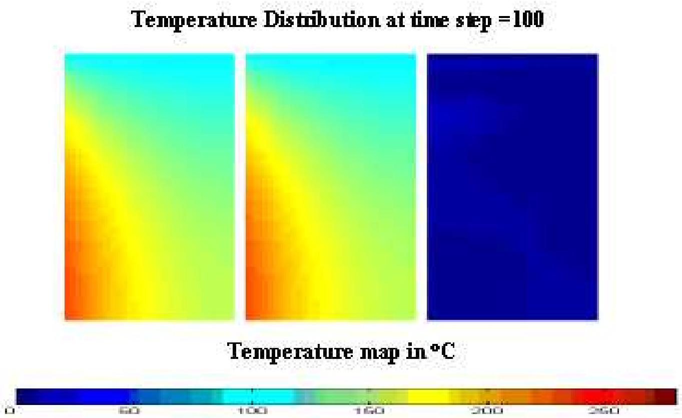

Figure 3: Original Model (left), Reduced Model (middle), Error (right)

By applying the inner product criterion on (12), the reduced oder model can be written as:

$$\begin{aligned}\underbrace{\Phi^T A \Phi_n}\, a(k+1) &= \underbrace{\Phi^T A_0 \Phi_n}\, a(k) + \underbrace{\Phi^T}\, \mathbf{S}(k) \\ A_{\text{red}} a(k+1) &= A_{0_{\text{red}}} a(k) + B_{\text{red}} S(k) \\ a(k+1) &= A_{\text{red}}^{-1} A_{0_{\text{red}}} a(k) + A_{\text{red}}^{-1} B_{\text{red}} S\,(k)\end{aligned} \tag{13}$$

The reduced order model can then be cast into standard state space form

$$a(k+1) = Aa(k) + Bu\,(k) \tag{14}$$

When some output points are chosen for measurements:

$$y(k) = \mathbb{T}_{\text{output}}(k) = \Phi_{\text{output}} a(k) = Ca(k) \tag{15}$$

with Φ_{output} is Φ with the row corresponding to non-measured temperatures set to zero.

With the state-space model for the reduced order model, we can easily design a controller.

5 Controller design

To design a controller, the original plant is adjusted such that there are four actuators (I_1, I_2, I_3, I_4) along the west side whose heat fluxes can be varied and the actuator on the north edge (I_5)whose temperature level can be manipulated to reach the desired temperature distribution. The east side and the south side are kept insulated. Let $\mathbb{X}_{\text{act}} \subset \mathbb{X}$ denote these 5 actuator positions in the spatial domain. The sketch of the plate with the actuator position is given in Fig. 4.

One important assumption when designing a controller based on POD-based model is that the closed-loop behavior can still be well approximated by the generated POD basis vectors. In general, POD basis are only **guaranteed** to be effective in describing the response where they are generated [6]. Thus, there is no guarantee that the POD-based model derived from an open loop situation will work well in closed loop situation.

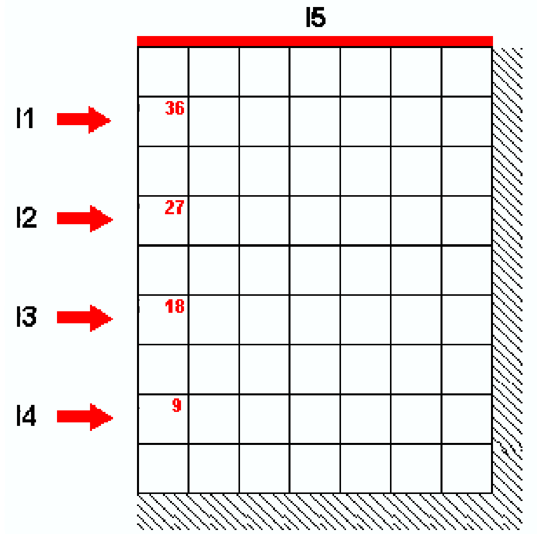

Figure 4: Actuator Positions

However, it is infeasible to generate snapshots on the basis of closed loop response since the order of the original model is too high that for optimal control design. Hence we rely on the open loop response only to derive a reduced model. For this particular plant, we controlled in the same operating range as where the POD basis vectors are generated.

5.1 Control Problem

The control problem for this model is how to track a reference temperature distribution as close as possible. Prior to the the mathematical translation of this problem, some formal definitions are given.

Definition 1 *Function $T\,(x, t, u)$ is defined as the temperature T at position $x \in \mathbb{X}$, time $t \in \mathbb{T}$ subject to the actuator input $u\,(\xi, \tau)$ with $\xi \in \mathbb{X}_{act}$, $\tau \in \mathbb{T}$.*

Definition 2 *The tracking error $J\,(u)$ is defined as:*

$$\begin{aligned} J\,(u) &:= \left\| T_{ref}\,(x,t) - T\,(x,t,u) \right\|^2 \\ &:= \sum_{x \in \mathbb{X}} \sum_{t \in \mathbb{T}_{control}} \left| T_{ref}\,(x,t) - T\,(x,t,u) \right| \end{aligned} \tag{16}$$

with the control horizon $\mathbb{T}_{control} \subseteq \mathbb{T}$.

With the given definitions, we can proceed to the control problem of the heated plate

Problem 3 *Find for each actuator position $\xi \in \mathbb{X}_{act} \subseteq \mathbb{X}$, a control sequence $u^*\,(\xi, \tau)$ with $\tau \in \mathbb{T}_{control} \subseteq \mathbb{T}$ such that*

$$J\,(u^*) \leq J\,(u)$$

for any other control sequence u of this type. If such u^* exists, it is called optimal.

The control problem is defined for the full order model. Since the order of the discretised original model is very high, i.e.1452, it is infeasible to incorporate all states into the control and optimization problem. Thus, the control problem has to be re-formulated based on the reduced model.

The recursive state space model of the reduced model is given in (13). Based on (7) we can translate the reference signals and the temperature $T\left(x,t,u\right)$ as truncated expansions.

$$\begin{aligned} T_{\text{ref}}\left(x,t\right) &\longleftrightarrow a_{\text{ref}}\left(t\right) \\ T\left(x,t,u\right) &\longleftrightarrow a\left(t,u\right) \\ T_n\left(x,t\right) &\longleftrightarrow a\left(t,u\right) \end{aligned}$$

where $a\left(t,u\right)$ denotes the state of the reduced order model subject to the input u (14).

Hence, the control problem for the discrete reduced model (14) is defined as:

Problem 4 *Find a control sequence $u^*\left(t\right)$ with $t \in \mathbb{T}_{\mathit{control}} \subseteq \mathbb{T}$ such that*

$$J_{\mathit{red}}\left(u^*\right) \leq J_{\mathit{red}}\left(u\right) \forall u \in \mathcal{U}$$

where

$$J_{\mathit{red}}\left(u\right) = \left\|a_{\mathit{ref}}\left(t\right) - a\left(t,u\right)\right\|^2$$

Since the controller is applied to the **discretised model**(5), in the next discussion, the notation t for the time is replaced to time step $t = k\Delta t$.

It is interesting to note that by incorporating the truncated approximation of the reference and the temperature signals in the tracking error, modeling error due to the truncation si incorporated as well in the objective function.

$$\begin{aligned} \left\|T_{\text{ref}}\left(x,t\right) - T\left(x,t,u\right)\right\|^2 &= \left\|T_{\text{ref}}^{\text{red}} + T_{\text{ref}}^{\text{tail}} - T^{\text{red}} - T^{\text{tail}}\right\|^2 \\ &= \underbrace{\left\|T_{\text{ref}}^{\text{red}} - T^{\text{red}}\right\|^2}_{\text{reduced problem}} + \underbrace{\left\|T_{\text{ref}}^{\text{tail}} - T^{\text{tail}}\right\|^2}_{\text{modeling error}} \end{aligned}$$

As long as this modeling error is small, then minimization of the tracking error will not deviate much from the minimization of the original tracking error.

5.2 LQR controller

Based on the control problem, we would like to optimize the control input $u(k)$ also on nonzero reference signals, **as opposed to the standard LQR problem** which only optimizes the input to track zero reference signals [2].

Consider a general **discrete** state equation:

$$x(k+1) = Ax(k) + Bu(k) \tag{17}$$

To obtain an expression for the standard state feedback which optimizes the states compared to the reference states, $(r(k) - x(k))^T Q(r(k) - x(k))$, the new cost function with the incorporation of the reference signal is given by:

$$\begin{aligned} J(x_0, u) = \sum_{k=0}^{N_e - 1} &\left[(r(k) - x(k))^T Q(r(k) - x(k))\right] \\ &+ u^T(k) R u(k) + x(N_e)]^T E[r(N_e) - x(N_e)] \end{aligned} \tag{18}$$

Control problem is defined as:

Problem 5 *Find $u^* \in \mathcal{U}$ such that*

$$J(x_0, u^*) \leq J(x_0, u) \qquad \forall u \in \mathcal{U}$$

where $\mathcal{U}$ is the set of all admissible discrete time input signals.

It can be shown [2] that the minimizing solution to the control problem is

$$u^*(k) = -F_k x(k) - G_k v(k+1)$$

where

$$\begin{aligned} F_k &= (R + B^T P(k+1) B)^{-1} B^T P(k+1) A \\ G_k &= (R + B^T P(k+1) B)^{-1} B^T \end{aligned}$$

In (19), let P:$\mathbb{T} \rightarrow \mathbb{R}^{n \times n}$ with $P(N_e) = E$ be the symmetric solution of

$$\begin{aligned} P(k) &= A^T P(k+1) A + Q \\ &- A^T P(k+1) B (R + B^T P(k+1) B)^{-1} B^T P(k+1) A \end{aligned}$$

and $v : \mathbb{T} \rightarrow \mathbb{R}^n$ is the unique solution of

$$\begin{aligned} v(k) &= A^T - A^T P(k+1) B (R + B P(k+1) B^T)^{-1} B^T \\ v(k+1) &= Q r(k) \end{aligned}$$

with $v(N_e)$ = 0.

The reduced model (13) is used as the basis to derive the optimal control signal $u^*(k)$ and the control signal is sent out to the full model (5) to reach the desired temperature distribution.

Schematic representation of the LQR controller with the reference signals unequal to zeros is given in Fig. 5.

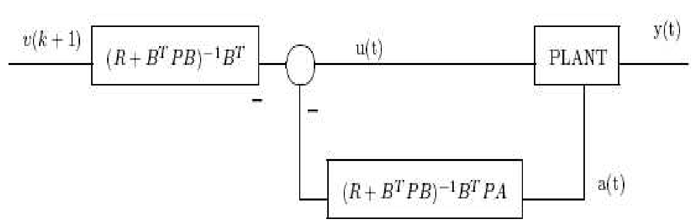

Figure 5: Schematic representation of the LQR Controller

6 Application of LQR Controller

The LQR controller as discussed in the previous section is applied to the full model of the heated plate. The control design is based upon the reduced model which only has 5 states instead of 1452 as in the original model. As the tracking trajectory, the temperature distribution at time step $k = 1000$ when the plant is excited from zero temperature distribution by the constant heat flux on the west side of $500\mathtt{kW}/\mathtt{m}^2$ and the constant north temperature boundary at $100^\diamond\mathtt{C}$.

Despite of the dramatic reduction of model order upon which the controller is based, the controller performs very well. In Figure. 7, it can be seen that the deviation from the desired temperature distribution is very small.

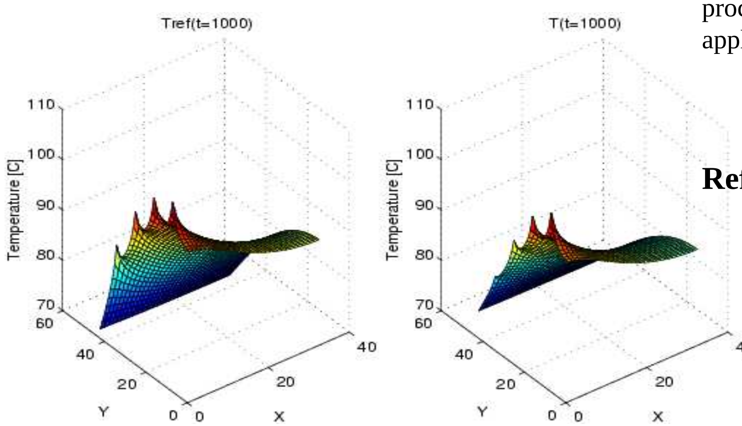

Figure 6: Reference Trajectory (Left) and Response of The Controlled Plant (Right)

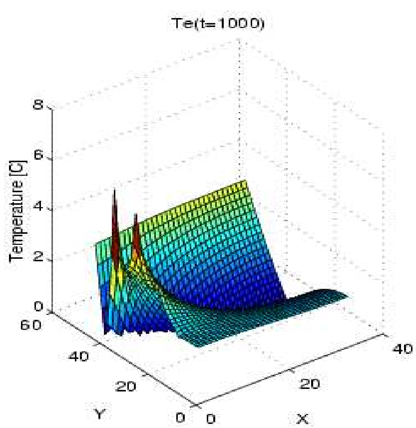

Figure 7: Deviation from The Reference Profile

The maximum deviation $4^\diamond\mathtt{C}$ for a temperature range of $70^\diamond\mathtt{C}$ to $90^\diamond\mathtt{C}$. The maximum deviations occur at the temperature near the north boundary, which is understandable since the temperature near this boundary is quite low compared to the temperature near the insulated boundaries (see Fig. 2). Due to the insulations, heat is preserved more in the area near the insulations because there is no heat exchanged with the surroundings. The dynamics of this area then predominates the dynamics of the whole systems because there are more temperature changes. Therefore, the dynamics captured by the basis vectors used to construct the reduced order model is more accurate in this area than elsewhere.

7 Conclusion

POD is a prominent model reduction technique to reduce significantly the order of the original model. The advantages of this technique are the use of data to determine the modes of the dynamics as well as the incorporation of the original equations. By employing the technique to large-scale numerical model, control and optimization designs becomes feasible. In particular for CFD models largely employed in many industrial processes, this technique will provide many possibilities in the application of more advanced controllers.

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