NEW RESULTS FOR H_2 STATE FEEDBACK CONTROL OF LARGE–SCALE SYSTEMS

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Abstract

In this paper, H_2 state feedback control for large–scale systems is studied in a different approach from the existing methods. The attention is focused on the design of a near–optimal H_2 controller which does not depend on the values of the small unknown weak coupling parameter. It is newly shown that the resulting controller in fact achieves $O(\varepsilon^2)$ approximation of the optimal H_2 cost.

1 Introduction

The stability analysis and control for large–scale systems have been investigated extensively (see e.g., [5]). These control problem situations in practice are illustrated by the multiarea power system [6]. In order to obtain the optimal solution to the Linear Quadratic Regulator (LQR) problem, the Algebraic Riccati Equation (ARE), which is parameterized by the small positive weak coupling parameter ε must be solved. Various reliable approaches for solving the ARE have been well documented in literatures (see e.g., [3]). However, a limitation of these approaches are that the small parameter is assumed to be known. In practice, the small perturbation parameter ε is often not known. Thus, it is not applicable to a large class of the control problem when the weak coupling parameter represents small unknown perturbation whose value is not known exactly. Moreover, in case where the Schur method is used, the computing need two times dimension of the ARE. Therefore, the reduction of the algebraic manipulation must be needed because the large–scale systems include numerous subsystems.

Recently, the optimal control problem for the large–scale systems via the recursive approach has been investigated [4]. When ε is small or known the previously used technique is very efficient. However, when the small weak coupling parameter is unknown, the recursive algorithm approach cannot also apply. Furthermore, so far, the loss of performance between the optimal control and the resulting controller which is based on the recursive technique has not been investigated.

In this paper, H_2 state feedback control for large–scale systems is investigated. The considered large–scale systems are connected by the small weak coupling parameter for the other subsystems. The main contribution in this paper is as follows. Firstly, the unique and bounded solution of the ARE and its asymptotic structure are established using the different manner compared with the existing result [1]. That is, the proof is done by using the implicit function theorem [2]. Using the asymptotic structure, a new near–optimal H_2 controller which does not depend on the values of the small weak parameter is obtained. This is done by eliminating the parameters ε for the optimal controller. Secondly, it is newly shown that the resulting controller achieves $O(\varepsilon^2)$ approximation of the optimal H_2 cost. It should be noted that there exists no result of the loss of the cost performance via the near–optimal control so far. Even if the parameter is unknown, when the parameter is sufficiently small, the new near–optimal H_2 controller can be used reliably for the large scale systems.

Notation: The superscript T denotes matrix transpose. Trace denotes the trace for any square matrix. $\det$ denotes the determinant for any square matrix. I_n denotes the $n \times n$ identity matrix. $\|\cdot\|_2$ denotes its 2–norm for any matrix. $\|\cdot\|$ denotes its Euclidean norm for any matrix. vec denotes the column vector for any square matrix. $\mathrm{block}-\mathrm{diag}$ denotes the block diagonal matrix. $\otimes$ denotes the Kronecker product.

2 Problem formulation

Consider the linear time–invariant large–scale systems

$$\begin{aligned} \dot{x}_i(t) &= A_{ii}x_i(t) + B_{ii}^1 w_i(t) + B_{ii}^2 u_i(t) \\ &\quad +\varepsilon \sum_{j=1,\ j\neq i}^{N} A_{ij}x_j(t) + \varepsilon \sum_{j=1,\ j\neq i}^{N} B_{ij}^1 w_j(t) \\ &\quad +\varepsilon \sum_{j=1,\ j\neq i}^{N} B_{ij}^2 u_j(t), \qquad (1a) \\ z_i(t) &= C_{ii}x_i(t) + D_{ii}u_i(t), \qquad (1b) \\ y_i(t) &= x_i(t),\ \ i = 1,\ 2,\ \cdots, N \qquad (1c) \end{aligned}$$

where $x_i \in \mathbf{R}^{n_i}$, $i = 1,\ 2,\ \cdots, N$ are the state vectors, $w_i \in \mathbf{R}^{p_i}$, $j = 1,\ 2,\ \cdots, N$ are the disturbance inputs, $u_i \in \mathbf{R}^{m_i}$, $j = 1,\ 2,\ \cdots, N$ are the control inputs, $z_i \in \mathbf{R}^{q_i}$, $j = 1,\ 2,\ \cdots, N$ are the controlled inputs, $y_i \in \mathbf{R}^{r_i}$, $j = 1,\ 2,\ \cdots, N$ are the outputs, ε denotes a small positive weak coupling parameter which connect the other subsystems.

Let us introduce the partitioned matrices

$$A_\varepsilon := \begin{bmatrix} A_{11} & \varepsilon A_{12} & \varepsilon A_{13} & \cdots & \varepsilon A_{1(N-1)} & \varepsilon A_{1N} \\ \varepsilon A_{21} & A_{22} & \varepsilon A_{23} & \cdots & \varepsilon A_{2(N-1)} & \varepsilon A_{2N} \\ \varepsilon A_{31} & \varepsilon A_{32} & A_{33} & \cdots & \varepsilon A_{3(N-1)} & \varepsilon A_{3N} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \varepsilon A_{N1} & \varepsilon A_{N2} & \varepsilon A_{N3} & \cdots & \varepsilon A_{N(N-1)} & A_{NN} \end{bmatrix},$$

$$B_\varepsilon^1 := \begin{bmatrix} B_{11}^1 & \varepsilon B_{12}^1 & \varepsilon B_{13}^1 & \cdots & \varepsilon B_{1(N-1)}^1 & \varepsilon B_{1N}^1 \\ \varepsilon B_{21}^1 & B_{22}^1 & \varepsilon B_{23}^1 & \cdots & \varepsilon B_{2(N-1)}^1 & \varepsilon B_{2N}^1 \\ \varepsilon B_{31}^1 & \varepsilon B_{32}^1 & B_{33}^1 & \cdots & \varepsilon B_{3(N-1)}^1 & \varepsilon B_{3N}^1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \varepsilon B_{N1}^1 & \varepsilon B_{N2}^1 & \varepsilon B_{N3}^1 & \cdots & \varepsilon B_{N(N-1)}^1 & B_{NN}^1 \end{bmatrix},$$

$$B_\varepsilon^2 := \begin{bmatrix} B_{11}^2 & \varepsilon B_{12}^2 & \varepsilon B_{13}^2 & \cdots & \varepsilon B_{1(N-1)}^2 & \varepsilon B_{1N}^2 \\ \varepsilon B_{21}^2 & B_{22}^2 & \varepsilon B_{23}^2 & \cdots & \varepsilon B_{2(N-1)}^2 & \varepsilon B_{2N}^2 \\ \varepsilon B_{31}^2 & \varepsilon B_{32}^2 & B_{33}^2 & \cdots & \varepsilon B_{3(N-1)}^2 & \varepsilon B_{3N}^2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \varepsilon B_{N1}^2 & \varepsilon B_{N2}^2 & \varepsilon B_{N3}^2 & \cdots & \varepsilon B_{N(N-1)}^2 & B_{NN}^2 \end{bmatrix},$$

$$C := \text{block} - \text{diag}\begin{pmatrix} C_{11} & \cdots & C_{NN} \end{pmatrix},$$
$$D := \text{block} - \text{diag}\begin{pmatrix} D_{11} & \cdots & D_{NN} \end{pmatrix}.$$

Using the state feedback control

$$u(t) = K_\varepsilon x(t), \tag{2}$$

where

$$u(t)^T := \begin{bmatrix} u_1(t)^T \cdots u_N(t)^T \end{bmatrix}^T \in \mathbf{R}^{\bar{m}},\ \bar{m} := \sum_{i=1}^N m_i,$$
$$x(t)^T := \begin{bmatrix} x_1(t)^T \cdots x_N(t)^T \end{bmatrix}^T \in \mathbf{R}^{\bar{n}},\ \bar{n} := \sum_{i=1}^N n_i,$$

the H_2–norm of the closed–loop transfer function matrix $G(s)$ is given by

$$\begin{aligned} \|G(s)\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}[G(-j\omega)^T G(j\omega)]d\omega \\ &= \text{Trace}[B_\varepsilon^{1T} L_\varepsilon B_\varepsilon^1], \end{aligned} \tag{3}$$

where

$$G(s) = (C + DK_\varepsilon)(sI_{\bar{n}} - A_\varepsilon - B_\varepsilon^2 K_\varepsilon)^{-1} B_\varepsilon^1,$$
$$L_\varepsilon(A_\varepsilon + B_\varepsilon^2 K_\varepsilon) + (A_\varepsilon + B_\varepsilon^2 K_\varepsilon)^T L_\varepsilon + (C + DK_\varepsilon)^T (C + DK_\varepsilon) = 0.$$

Without loss of generality, $D^T D = I_{\bar{m}}$ is assumed. H_2 control problem is to find a control $u(t)$ which minimizes the H_2–norm of the closed–loop transfer function matrix (3).

It is well–known from the existing results (see e.g., [8]) that such controller that minimizes the H_2–norm (3) is given by

$$u_{\text{opt}}(t) = K_{\text{opt}\varepsilon} x(t) = -(B_\varepsilon^{2T} P_\varepsilon + D^T C)x(t), \tag{4}$$

where P_ε is the positive semidefinite stabilizing solution which satisfies the ARE

$$P_\varepsilon \mathcal{A}_\varepsilon + \mathcal{A}_\varepsilon^T P_\varepsilon - P_\varepsilon \mathcal{S}_\varepsilon P_\varepsilon + \mathcal{Q} = 0, \tag{5}$$

with

$$\mathcal{A}_\varepsilon := A_\varepsilon - B_\varepsilon^2 D^T C = \begin{bmatrix} \bar{A}_{11} & \varepsilon \bar{A}_{12} & \varepsilon \bar{A}_{13} & \cdots & \varepsilon \bar{A}_{1(N-1)} & \varepsilon \bar{A}_{1N} \\ \varepsilon \bar{A}_{21} & \bar{A}_{22} & \varepsilon \bar{A}_{23} & \cdots & \varepsilon \bar{A}_{2(N-1)} & \varepsilon \bar{A}_{2N} \\ \varepsilon \bar{A}_{31} & \varepsilon \bar{A}_{32} & \bar{A}_{33} & \cdots & \varepsilon \bar{A}_{3(N-1)} & \varepsilon \bar{A}_{3N} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \varepsilon \bar{A}_{N1} & \varepsilon \bar{A}_{N2} & \varepsilon \bar{A}_{N3} & \cdots & \varepsilon \bar{A}_{N(N-1)} & \bar{A}_{NN} \end{bmatrix},$$

$$\mathcal{S}_\varepsilon := B_\varepsilon^2 B_\varepsilon^{2T} = \begin{bmatrix} \bar{S}_{11} & \varepsilon \bar{S}_{12} & \varepsilon \bar{S}_{13} & \cdots & \varepsilon \bar{S}_{1(N-1)} & \varepsilon \bar{S}_{1N} \\ \varepsilon \bar{S}_{12}^T & \bar{S}_{22} & \varepsilon \bar{S}_{23} & \cdots & \varepsilon \bar{S}_{2(N-1)} & \varepsilon \bar{S}_{2N} \\ \varepsilon \bar{S}_{13}^T & \varepsilon \bar{S}_{23}^T & \bar{S}_{33} & \cdots & \varepsilon \bar{S}_{3(N-1)} & \varepsilon \bar{S}_{3N} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \varepsilon \bar{S}_{1N}^T & \varepsilon \bar{S}_{2N}^T & \varepsilon \bar{S}_{3N}^T & \cdots & \varepsilon \bar{S}_{(N-1)N}^T & \bar{S}_{NN} \end{bmatrix},$$

$$\mathcal{Q} := C^T(I_{\bar{q}} - DD^T)C = \text{block} - \text{diag}\begin{pmatrix} Q_{11} & \cdots & Q_{NN} \end{pmatrix},$$
$$\bar{q} := \sum_{i=1}^N q_i,\ \bar{S}_{ii} = \bar{S}_{ii}^T,\ Q_{ii} = C_{ii}^T(I_{q_i} - D_{ii}D_{ii}^T)C_{ii},$$
$$i = 1,\ 2,\ \cdots, N.$$

Moreover, the minimum value of the H_2–norm (3) is given by

$$\min \|G(s)\|_2^2 = \text{Trace}[B_\varepsilon^{1T} P_\varepsilon B_\varepsilon^1]. \tag{6}$$

Since A_ε, B_ε^1 and B_ε^2 include the term of the small weak coupling parameter ε, a solution P_ε of the ARE (5), if it exists, must contain terms of order ε. Taking this fact into account, the solution P_ε of the ARE (5) with the following structure is considered [4]

$$P_\varepsilon := \begin{bmatrix} P_{11} & \varepsilon P_{12} & \varepsilon P_{13} & \cdots & \varepsilon P_{1(N-1)} & \varepsilon P_{1N} \\ \varepsilon P_{12}^T & P_{22} & \varepsilon P_{23} & \cdots & \varepsilon P_{2(N-1)} & \varepsilon P_{2N} \\ \varepsilon P_{13}^T & \varepsilon P_{23}^T & P_{33} & \cdots & \varepsilon P_{3(N-1)} & \varepsilon P_{3N} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \varepsilon P_{1N}^T & \varepsilon P_{2N}^T & \varepsilon P_{3N}^T & \cdots & \varepsilon P_{(N-1)N}^T & P_{NN} \end{bmatrix}.$$

In the following analysis, the basic assumptions are needed.

Assumption 1 *The triples* $(A_{ii},\ B_{ii}^2,\ C_{ii}),\ i = 1,\ 2,\ \cdots, N$ *are stabilizable and detectable.*

Assumption 2 D_{ii} *has full column rank.*

Assumption 3 $\begin{bmatrix} A_{ii} - j\omega I_{n_i} & B_{ii}^2 \\ C_{ii} & D_{ii} \end{bmatrix}$ *has full column rank for any* ω.

3 Asymptotic Structure of the ARE

Substituting the matrices $\mathcal{A}_\varepsilon$, $\mathcal{S}_\varepsilon$, Q and P_ε into the ARE (5), setting $\varepsilon = 0$ and partitioning the ARE (5), the following reduced–order AREs are obtained, where $\bar{P}_{ii},\ i = 1,\ \cdots, N$ be the limiting solutions of the ARE (5) as $\varepsilon \to +0$.

$$\bar{P}_{ii}\bar{A}_{ii} + \bar{A}_{ii}^T\bar{P}_{ii} - \bar{P}_{ii}S_{ii}\bar{P}_{ii} + Q_{ii} = 0, \tag{7}$$

where $S_{ii} := B_{ii}^2 B_{ii}^{2T}$.

The limiting behavior of P_ε as the parameter $\varepsilon \to +0$ is described by the following lemma.

Lemma 1 *Under Assumptions 1–3, there exists a small* σ^* *such that for all* $\varepsilon \in (0,\ \sigma^*)$ *the ARE* (5) *admits a positive semidefinite stabilizing solution* P_ε *which can be written as*

$$P_\varepsilon = \bar{P} + O(\varepsilon) = \mathrm{block-diag}\left(\ \bar{P}_{11}\ \cdots\ \bar{P}_{NN}\ \right) + O(\varepsilon), \tag{8}$$

Proof: The proof can be done by using the implicit function theorem [2] to the ARE (5). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\varepsilon = 0$. It can be shown, after some algebra, that the Jacobian of the ARE (5) in the limit as $\varepsilon \to +0$ is given by

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{11} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \mathbf{J}_{NN} & 0 & \cdots & 0 \\ \vdots & \cdots & \cdots & \mathbf{J}_{12} & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \ddots & 0 \\ * & \cdots & \cdots & \cdots & \cdots & \mathbf{J}_{(N-1)N} \end{bmatrix}, \tag{9}$$

where

$$\mathbf{J}_{ii} = (\bar{A}_{ii} - S_{ii}\bar{P}_{ii}) \otimes I_{n_i} + I_{n_i} \otimes (\bar{A}_{ii} - S_{ii}\bar{P}_{ii}),$$
$$\mathbf{J}_{ij} = (\bar{A}_{ii} - S_{ii}\bar{P}_{ii}) \otimes I_{n_i} + I_{n_j} \otimes (\bar{A}_{jj} - S_{jj}\bar{P}_{jj}).$$

The Jacobian (9) can be expressed as

$$\det\mathbf{J} = \left[\prod_{i=1}^{N} \det\mathbf{J}_{ii}\right] \cdot \left[\prod_{i=1,\ j=2,\ i<j}^{N} \det\mathbf{J}_{ij}\right] \tag{10}$$

Obviously, $\mathbf{J}_{ii}$, $\mathbf{J}_{ij}$ are nonsingular because the matrix $\bar{A}_{ii} - S_{ii}\bar{P}_{ii}$ is stable under Assumptions 1–3. Thus, $\det\mathbf{J} \neq 0$, i.e., $\mathbf{J}$ is nonsingular at $\varepsilon = 0$. The conclusion of Lemma 1 is obtained directly by using the implicit function theorem.

The remainder of the proof is to show that P_ε is the positive semidefinite stabilizing solution. For sufficiently small parameter ε, $P_\varepsilon \geq 0$ because the solution $\bar{P}_{ii}$ is the positive semidefinite matrix. Moreover, using (8), the following relation holds

$$\begin{aligned} &\mathcal{A}_\varepsilon - \mathcal{S}_\varepsilon P_\varepsilon \\ &= \mathrm{block-diag}\left(\ \bar{A}_{11} - S_{11}\bar{P}_{11}\ \cdots\ \bar{A}_{NN} - S_{NN}\bar{P}_{NN}\ \right) \\ &\quad + O(\varepsilon), \end{aligned} \tag{11}$$

because the matrices $\bar{A}_{ii} - S_{ii}\bar{P}_{ii}$ are stable under Assumptions 1–3. Therefore, if the parameter ε is very small, $\mathcal{A}_\varepsilon - \mathcal{S}_\varepsilon P_\varepsilon$ is stable also. ■

4 Kleinman Algorithm for Solving ARE

In order to obtain the near–optimal H_2 controller, the following useful result is obtained.

Lemma 2 *Consider the iterative algorithm which is based on the Kleinman algorithm*

$$\begin{aligned} &P_\varepsilon^{(i+1)}(\mathcal{A}_\varepsilon - \mathcal{S}_\varepsilon P_\varepsilon^{(i)}) + (\mathcal{A}_\varepsilon - \mathcal{S}_\varepsilon P_\varepsilon^{(i)})^T P_\varepsilon^{(i+1)} \\ &\quad + P_\varepsilon^{(i)}\mathcal{S}_\varepsilon P_\varepsilon^{(i)} + Q = 0,\ \ i = 0,\ 1,\ \cdots, \end{aligned} \tag{12a}$$

$$P_\varepsilon^{(i)} := \begin{bmatrix} P_{11}^{(i)} & \varepsilon P_{12}^{(i)} & \varepsilon P_{13}^{(i)} & \cdots & \varepsilon P_{1(N-1)}^{(i)} & \varepsilon P_{1N}^{(i)} \\ \varepsilon P_{12}^{(i)T} & P_{22}^{(i)} & \varepsilon P_{23}^{(i)} & \cdots & \varepsilon P_{2(N-1)}^{(i)} & \varepsilon P_{2N}^{(i)} \\ \varepsilon P_{13}^{(i)T} & \varepsilon P_{23}^{(i)T} & P_{33}^{(i)} & \cdots & \varepsilon P_{3(N-1)}^{(i)} & \varepsilon P_{3N}^{(i)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \varepsilon P_{1N}^{(i)T} & \varepsilon P_{2N}^{(i)T} & \varepsilon P_{3N}^{(i)T} & \cdots & \varepsilon P_{(N-1)N}^{(i)T} & P_{NN} \end{bmatrix}, \tag{12b}$$

with the initial condition obtained from

$$P_\varepsilon^{(0)} = \bar{P} = \mathrm{block-diag}\left(\ \bar{P}_{11}\ \ \cdots\ \ \bar{P}_{NN}\ \right). \tag{13}$$

Under Assumptions 1–3, there exists a small $\bar{\sigma}$ *such that for all* $\varepsilon \in (0,\ \bar{\sigma})$, $\bar{\sigma} \leq \sigma^*$ *the iterative algorithm* (12) *converges to the exact solution of* P_ε *with the rate of quadratic convergence, where* $P_\varepsilon^{(i)}$ *is positive semidefinite. That is, the following conditions are satisfied.*

$$\|P_\varepsilon^{(i)} - P_\varepsilon\| = O(\varepsilon^{2^i}),\ i = 0,\ 1,\ \cdots, \tag{14}$$

where

$$\gamma = 2\|\mathcal{S}_\varepsilon\| < \infty,\ \beta = \|[\nabla\mathcal{G}(P_\varepsilon^{(0)})]^{-1}\|,$$
$$\eta = \beta \cdot \|\mathcal{G}(P_\varepsilon^{(0)})\|,\ \theta = \beta\eta\gamma,$$
$$\nabla\mathcal{G}(P_\varepsilon) = \frac{\partial \mathrm{vec}\mathcal{G}(P_\varepsilon)}{\partial(\mathrm{vec}P_\varepsilon)^T},$$
$$\mathcal{G}(P_\varepsilon) = P_\varepsilon\mathcal{A}_\varepsilon + \mathcal{A}_\varepsilon^T P_\varepsilon - P_\varepsilon\mathcal{S}_\varepsilon P_\varepsilon + \mathcal{Q}.$$

Proof: The proof follows directly by applying Newton–Kantorovich theorem [7] for the ARE (5). It is easy to verify that function $\mathcal{G}(P_\varepsilon)$ is differentiable. Using the fact that

$$\begin{aligned} \nabla\mathcal{G}(P_\varepsilon) &:= \frac{\partial \mathrm{vec}\mathcal{G}(P_\varepsilon)}{\partial(\mathrm{vec}P_\varepsilon)^T} \\ &= (\mathcal{A}_\varepsilon - \mathcal{S}P_\varepsilon)^T \otimes I_{\bar{n}} + I_{\bar{n}} \otimes (\mathcal{A}_\varepsilon - \mathcal{S}P_\varepsilon)^T, \end{aligned} \tag{15}$$

the following inequality holds

$$\|\nabla\mathcal{G}(P_{1\varepsilon}) - \nabla\mathcal{G}(P_{2\varepsilon})\| \leq \gamma\|P_{1\varepsilon} - P_{2\varepsilon}\|, \tag{16}$$

where $\gamma = 2\|\mathcal{S}_\varepsilon\|$. Moreover, using the result of the stability established by (11), it is shown that there exists a small $\bar{\sigma}$ such that for sufficiently small parameter $\varepsilon \in (0,\ \bar{\sigma}),\ \bar{\sigma} \le \sigma^*$, $\nabla\mathcal{G}(P_\varepsilon)$ is nonsingular. Therefore, there exists β such that $\|[\nabla\mathcal{G}(P_\varepsilon)]^{-1}\| \equiv \beta$. On the other hand, using the Lemma1, it is easy to show that $\|\mathcal{G}(P_\varepsilon)\| = O(\varepsilon)$. Hence, there exists η such that $\|[\nabla\mathcal{G}(P_\varepsilon)]^{-1}\| \cdot \|\mathcal{G}(P_\varepsilon)\| \equiv \eta = O(\varepsilon)$. Thus, there exists θ such that $\theta \equiv \beta\gamma\eta < 2^{-1}$ because $\eta = O(\varepsilon)$. Using Newton–Kantorovich theorem, the strict error estimate is given by (14). ■

5 Near–optimal H_2 control

The required solution of the ARE (5) exists under Assumptions 1–3. The attention is focused on the specific linear state feedback controller which does not depend on the values of the small parameter. Such a linear state feedback controller is obtained by eliminating $O(\varepsilon)$ item of the linear state feedback controller (4). If ε is very small, it is obvious that the linear state feedback controller (4) can be approximated as

$$\begin{aligned} & u_{\text{app}}(t) \\ &= \text{block}-\text{diag}\begin{pmatrix} u_{1\text{app}}(t) & \cdots & u_{N\text{app}}(t) \end{pmatrix} \\ &= \bar{K}x(t) = -(\bar{B}^{2T}\bar{P} + D^TC)x(t) \\ &= -\text{block}-\text{diag}\left(B_{11}^{2T}\bar{P}_{11} + D_{11}^TC_{11} \cdots \right. \\ &\qquad \left. B_{NN}^{2T}\bar{P}_{NN} + D_{NN}^TC_{NN} \right) x(t), \end{aligned} \tag{17}$$

where $\bar{B}^{2T} := \text{block}-\text{diag}\begin{pmatrix} B_{11}^{2T} & \cdots & B_{NN}^{2T} \end{pmatrix}$.

It should be noted that the proposed control design is quite different from the multi–level computation design approach [1]. When ε is sufficiently small, it is known from Lemma 1 that the resulting controller (17) will be close to the optimal controller (4). In an optimization problem it is of interest to check whether the resulting value of the cost function will be near to its optimal value.

The main result for the degradation of the H_2–norm via the new H_2 controller (17) is given as follows.

Theorem 1 *Under Assumptions 1–3, the use of the reduced–order controller* (17) *results in* (18)

$$\|\bar{G}(s)\|_2^2 = \|G(s)\|_2^2 + O(\varepsilon^2), \tag{18}$$

where

$$\begin{aligned} & \|\bar{G}(s)\|_2^2 = \text{Trace}[B_\varepsilon^{1T}\bar{L}_\varepsilon B_\varepsilon^1], \\ & \bar{G}(s) := (C + D\bar{K})(sI_{\bar{n}} - A_\varepsilon - B_\varepsilon^2\bar{K})^{-1}B_\varepsilon^1, \\ & \bar{L}_\varepsilon(A_\varepsilon + B_\varepsilon^2\bar{K}) + (A_\varepsilon + B_\varepsilon^2\bar{K})^T\bar{L}_\varepsilon \\ &\qquad +(C + D\bar{K})^T(C + D\bar{K}) = 0, \end{aligned}$$

and the optimal value $\|G(s)\|_2^2$ is obtained with the controller (4) *which optimizes H_2 cost for the actual system* (1).

Proof: When $u_{\text{app}}(t)$ is used, the value of the norm is

$$\|\bar{G}(s)\|_2^2 = \text{Trace}[B_\varepsilon^{1T}W_\varepsilon B_\varepsilon^1], \tag{19}$$

where W_ε is a positive semidefinite solution of the algebraic Lyapunov equation (ALE)

$$(\mathcal{A}_\varepsilon - \mathcal{S}_\varepsilon\bar{P})^TW_\varepsilon + W_\varepsilon(\mathcal{A}_\varepsilon - \mathcal{S}_\varepsilon\bar{P}) + \bar{P}\mathcal{S}_\varepsilon\bar{P} + \mathcal{Q} = 0 \tag{20}$$

with $A_\varepsilon - B_\varepsilon^2\bar{K} = \mathcal{A}_\varepsilon - \mathcal{S}_\varepsilon\bar{P}$ and $(C + D\bar{K})^T(C + D\bar{K}) = \bar{P}\mathcal{S}_\varepsilon\bar{P} + \mathcal{Q}$.

Subtracting (5) from (20), $V_\varepsilon = W_\varepsilon - P_\varepsilon$ satisfies the following ALE

$$\begin{aligned} & (\mathcal{A}_\varepsilon - \mathcal{S}_\varepsilon\bar{P})^TV_\varepsilon + V_\varepsilon(\mathcal{A}_\varepsilon - \mathcal{S}_\varepsilon\bar{P}) \\ &\qquad +(P_\varepsilon - \bar{P})\mathcal{S}_\varepsilon(P_\varepsilon - \bar{P}) = 0. \end{aligned} \tag{21}$$

Similarly, subtracting (5) from (12a), the following ALE holds.

$$\begin{aligned} & (\mathcal{A}_\varepsilon - \mathcal{S}_\varepsilon P_\varepsilon^{(i)})^T(P_\varepsilon^{(i+1)} - P_\varepsilon) \\ &\quad +(P_\varepsilon^{(i+1)} - P_\varepsilon)(\mathcal{A}_\varepsilon - \mathcal{S}_\varepsilon P_\varepsilon^{(i)}) \\ &\quad +(P_\varepsilon - P_\varepsilon^{(i)})\mathcal{S}_\varepsilon(P_\varepsilon - P_\varepsilon^{(i)}) = 0. \end{aligned} \tag{22}$$

When $i = 0$, the following relation is satisfied.

$$\begin{aligned} & (\mathcal{A}_\varepsilon - \mathcal{S}_\varepsilon P_\varepsilon^{(0)})^T(P_\varepsilon^{(1)} - P_\varepsilon) + (P_\varepsilon^{(1)} - P_\varepsilon)(\mathcal{A}_\varepsilon - \mathcal{S}_\varepsilon P_\varepsilon^{(0)}) \\ &\quad +(P_\varepsilon - P_\varepsilon^{(0)})\mathcal{S}_\varepsilon(P_\varepsilon - P_\varepsilon^{(0)}) \\ &= (\mathcal{A}_\varepsilon - \mathcal{S}_\varepsilon\bar{P})^T(P_\varepsilon^{(1)} - P_\varepsilon) + (P_\varepsilon^{(1)} - P_\varepsilon)(\mathcal{A}_\varepsilon - \mathcal{S}_\varepsilon\bar{P}) \\ &\quad +(P_\varepsilon - \bar{P})\mathcal{S}_\varepsilon(P_\varepsilon - \bar{P}) = 0. \end{aligned}$$

Therefore, it is easy to verify that $V_\varepsilon = P_\varepsilon^{(1)} - P_\varepsilon$ because $\mathcal{A}_\varepsilon - \mathcal{S}_\varepsilon\bar{P}$ is stable. Using Lemma 2 it is easy to show that

$$\|V_\varepsilon\| = \|W_\varepsilon - P_\varepsilon\| = \|P_\varepsilon^{(1)} - P_\varepsilon\| = O(\varepsilon^2). \tag{23}$$

Hence

$$V_\varepsilon = W_\varepsilon - P_\varepsilon = O(\varepsilon^2), \tag{24}$$

which implies (18). ■

6 Numerical Example

In order to demonstrate the efficiency of the proposed algorithm, a numerical example is tested. Consider the interconnected large–scale system (1) composed of three four–dimensional subsystems. The system matrix is given as a modification of [1].

$$A_{11} = \begin{bmatrix} 0 & 1 & -0.266 & -0.009 \\ -2.75 & -2.78 & -1.36 & -0.037 \\ 0 & 0 & 0 & 1 \\ -4.95 & 0 & -55.5 & -0.039 \end{bmatrix},$$

$$\varepsilon A_{12} = \begin{bmatrix} 0.0024 & 0 & -0.087 & 0.002 \\ -0.185 & 0 & 1.11 & -0.011 \\ 0 & 0 & 0 & 0 \\ 0.222 & 0 & 8.17 & 0.004 \end{bmatrix},$$

$$\varepsilon A_{13} = \begin{bmatrix} 0.073 & 0 & -0.25 & 0.003 \\ -0.46 & 0 & 2.8 & -0.02 \\ 0 & 0 & 0 & 0 \\ 0.924 & 0 & 17.5 & 0.02 \end{bmatrix},$$

$$\varepsilon A_{21} = \begin{bmatrix} 0.021 & 0 & 0.121 & 0.003 \\ -1.1 & 0 & -1.62 & -0.015 \\ 0 & 0 & 0 & 0 \\ -2.43 & 0 & 1.37 & -0.034 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} -0.21 & 1 & -1.6 & -0.005 \\ -1.9 & -1.8 & 9.3 & -0.12 \\ 0 & 0 & 0 & 1 \\ -3.1 & 0 & -56 & 0.032 \end{bmatrix},$$

$$\varepsilon A_{23} = \begin{bmatrix} 0.06 & 0 & 0.46 & 0.002 \\ -1 & 0 & 1.49 & -0.04 \\ 0 & 0 & 0 & 0 \\ 0.12 & 0 & 29.8 & -0.028 \end{bmatrix},$$

$$A_{31} = \begin{bmatrix} -0.002 & 0 & 0.83 & 0 \\ -6.78 & 0 & -10.1 & 0.09 \\ 0 & 0 & 0 & 0 \\ -1.24 & 0 & 0.498 & -0.017 \end{bmatrix},$$

$$\varepsilon A_{32} = \begin{bmatrix} 0.011 & 0 & 0.22 & 0 \\ -2.1 & 0 & 1.7 & -0.123 \\ 0 & 0 & 0 & 0 \\ -0.07 & 0 & 6.38 & -0.011 \end{bmatrix},$$

$$\varepsilon A_{33} = \begin{bmatrix} -0.197 & 1 & -1.2 & -0.003 \\ -54.5 & -20 & 70.1 & -2.37 \\ 0 & 0 & 0 & 1 \\ -3.4 & 0 & -21.0 & -0.017 \end{bmatrix},$$

$$B_\varepsilon^1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \; B_\varepsilon^2 = \begin{bmatrix} 0 & 0 & 0 \\ 36.1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 78.9 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1000 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The small parameter is chosen as $\varepsilon = 0.5065$. Referring the proposed design procedure, the near–optimal H_2 control is given by

$$u_{\rm app}(t) = \text{block} - \text{diag}\left(u_{1\rm app}(t)\ u_{2\rm app}(t)\ u_{3\rm app}(t) \right), \quad (25)$$

$$u_{1\rm app}(t) = \left[-1.6073\ -1.0155\ -6.7626\ -1.6343 \times 10^{-2} \right] x(t),$$

$$u_{2\rm app}(t) = \left[-1.3703\ -1.0046\ -6.6746\ 8.3537 \times 10^{-2} \right] x(t),$$

$$u_{3\rm app}(t) = \left[-1.5716\ -1.0006\ -3.4550\ 2.5050 \times 10^{-1} \right] x(t).$$

Now, let us evaluate the costs using the near–optimal controller (25). The values of the H_2–norm is $\|\bar{G}(s)\|_2^2 = \text{Trace}[B_\varepsilon^{1T} W_\varepsilon B_\varepsilon^1] = 2.4136 \times 10$. Hence, the loss of $\|\bar{G}(s)\|_2^2$ is less than 12.3075% compared with the optimal value $\|G(s)\|_2^2 = 2.1491 \times 10$. The values of the H_2–norm for various ε are given in Table 1, where $\phi = \dfrac{\|\bar{G}(s)\|_2^2 - \|G(s)\|_2^2}{\varepsilon^2}$.

Table 1.

ε	$\|G(s)\|_2^2$	$\|G(s)\|_2^2$	ϕ
0.5	2.1441×10	2.4019×10	1.0311×10
10^{-1}	1.9056×10	1.9162×10	1.0608×10
10^{-2}	1.8919×10	1.8920×10	1.0968×10
10^{-3}	1.8918×10	1.8918×10	1.1018×10
10^{-4}	1.8918×10	1.8918×10	1.1053×10
10^{-5}	1.8918×10	1.8918×10	1.0200×10
10^{-6}	1.8918×10	1.8918×10	2.9161×10

It is easy to verify that $\|\bar{G}(s)\|_2^2 - \|G(s)\|_2^2 = O(\varepsilon^2)$ because of $\phi < \infty$. Therefore, the new result for the H_2–norm property which is indicated by (18) is correct.

As a result, from the point of view of the numerical example, if the small positive weak coupling parameter which connect the other subsystems is sufficiently small, H_2 controller can be constructed by using the information only of the subsystems.

7 Conclusion

In this paper, H_2 state feedback control of the large–scale systems which are connected by the weak small parameter has been studied. The main contribution of this paper is to propose the new design method of the ε–independent reduced–order controller. It should be noted that the proposed design method is quite different from the existing method such as the multi–level computation design method [1]. Furthermore, it has been newly shown that the resulting controller achieves $O(\varepsilon^2)$ approximation of the optimal solution. Thus, the proposed H_2 controller design is very useful and reliable because such controller can be obtained without information of other connected subsystem and calculated in the same dimension compared with the subsystems.

It is expected that the proposed approach is also applied to the output feedback case. Such problem is more realistic than state feedback case. This problem will be addressed in future investigations.

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