K-S-Φ ITERATION FOR ROBUST H_2 CONTROLLER SYNTHESIS

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Abstract

An iterative procedure is proposed for robust H_2 controller design. This method improves a previously reported technique, where optimization over two variables - the controller and a scaling matrix - was carried out by keeping one fixed at a time and minimizing the worst-case H_2 norm over the other. In this paper, it is shown how optimization over both parameters at the same time can be formulated as an LMI problem with an additional constraint. Application to a well-known benchmark problem illustrates that this approach leads to a significantly larger range of feasible solutions, thus allowing the synthesis of controllers that guarantee robust performance over a larger range of uncertain parameters.

1 Introduction

Analysis techniques for estimating the worst-case H_2 norm of a control system have received considerable attention. Recently reported results include an approach based on parameter-dependent Lyapunov functions and an LFT representation of both the uncertainty and the Lyapunov function [9]. In the same paper a variety of objectives has been formulated in a general framework. Related results had been presented earlier in [6]. One of the strongest results - based on a multiplier approach - on estimating the worst-case H_2 norm was reported in [4]. This work was an extension to robust H_2 performance of results on robust stability in [10].

In contrast, synthesis techniques for controllers that minimize the worst-case H_2 norm are less well developed. An approach using Popov multipliers and an iterative technique was used in [8] and [7]. The results presented here are closely related to the work reported in [5], where affine parameter-dependent Lyapunov functions are used for both analysis and synthesis. The analysis results in [5] are translated into synthesis techniques by using a transformation of controller variables originally proposed in [1]. This leads to a technique referred to as K-D iteration where alternatingly scaling parameters and controller parameters are optimized.

Most methods for analyzing the worst-case H_2 norm are based either on multiplier techniques or on parameter-dependent Lyapunov functions. When translated into synthesis techniques, both approaches lead to terms that are nonlinear in the controller variables so that they cannot be solved as convex problems. Transforming these techniques into convex problems is only possible by introducing conservatism, and the challenge in developing robust H_2 synthesis techniques is to reduce this conservatism as far as possible, while still maintaining a tractable problem that can be solved as a convex optimization problem. It should be kept in mind that what is required in industrial practice is controller synthesis, with user-friendly and reliable tuning interfaces.

An iterative design procedure - referred to as K-S iteration - was proposed recently in [2], together with an efficient tuning strategy. This method uses a fixed quadratic Lyapunov function, and the S-procedure for transforming a constrained optimization problem into an unconstrained one. Both imply that the resulting design is conservative, but it could be demonstrated on a benchmark problem that this approach still outperforms other robust design techniques.

In this paper we propose a refined version of the approach in [2], where additional degrees of freedom are introduced into the design. The K-S iteration technique proposed in [2] is based on alternatingly minimizing a bound on the worst-case H_2 norm over the controller $K(s)$ and over a scaling matrix S. Here we present an algorithm that allows the simultaneous search for controller and scaling matrix, subject to a norm constraint. This method can lead to a significant increase in achievable robustness; this is illustrated by application to a benchmark problem for robust control.

An alternative approach to minimizing the worst case H_2 norm that allows for non-square LFT representations of the model uncertainty is presented in [3].

The paper is organized as follows. Section 2 gives the problem description and a brief review of the K-S iteration technique proposed in [2]. An improved algorithm for robust H_2 synthesis is presented in section 3. In section 4 the proposed method is applied to the ACC benchmark problem, and the achievable robustness is compared with the results from [2]. Conclusions are drawn in section 5.

2 Problem Description and Review of K-S Iteration for Robust H_2 Control

Consider the control system shown in Figure 1. The generalized plant P has a state space representation

$$\begin{aligned} \dot{x} &= A_0 x + B_1 w_1 + B_2 w_2 + Bu \\ z_1 &= C_1 x \\ z_2 &= C_2 x + D_{2u} u \\ y &= Cx + D_{2w} w_2 \end{aligned} \quad (1)$$

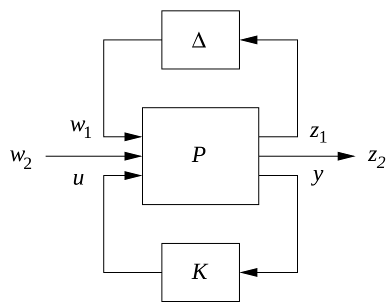


Figure 1: Closed-loop system with model uncertainty

Here (A_0, B, C) represents the physical plant with control input u and measured output y. Perturbations of the nominal plant dynamics (A_0) are expressed via fictitious inputs through B_1 and fictitious outputs through C_1: Introducing feedback $w_1 = \Delta z_1$, where the matrix $\Delta(t)$ represents perturbations and is assumed to satisfy $\|\Delta\| < 1$ at all times, leads to

$$\dot{x} = (A_0 + B_1 \Delta C_1)x + B_2 w_2 + Bu$$

The input w_2 is a white noise process with unit variance. If the matrices $C_2,\ D_{2u},\ B_2$ and D_{2w} are chosen as

$$\begin{aligned} C_2 &= \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix}, \qquad D_{2u} = \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} \\ B_2 &= [Q_e^{1/2}\ \ 0], \qquad D_{2w} = [0\ \ R_e^{1/2}] \end{aligned} \quad (2)$$

then

$$J = E\|z_2(t)\|_2^2 = E\left[\lim_{t\to\infty} \frac{1}{T}\int_0^T z_2^T z_2\, dt\right] \quad (3)$$

represents a LQG cost function with the usual weight matrices $Q,\ R$ and noise covariances $Q_e,\ R_e$. Note that the cost J is equal to the square of the H_2 norm of the closed-loop transfer function from w_2 to z_2.

The problem considered in this paper is to find a strictly proper controller $K(s)$ with state space realization

$$\begin{aligned} \dot{\zeta}(t) &= A_K \zeta(t) + B_K y(t) \\ u(t) &= C_K \zeta(t) \end{aligned} \quad (4)$$

such that

$$J \le \nu^2 \quad \forall \Delta : \|\Delta\| < 1 \quad (5)$$

for a given constant $\nu > 0$.

This problem can be expressed in the form of a matrix inequality as follows. Consider the closed-loop system

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \bar{A} \begin{bmatrix} x \\ \zeta \end{bmatrix} + [\bar{B}_1\ \ \bar{B}_2] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \end{bmatrix} \begin{bmatrix} x \\ \zeta \end{bmatrix}$$

where

$$\bar{A} = \begin{bmatrix} A_0 & BC_K \\ B_K C & A_K \end{bmatrix},\ \bar{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},\ \bar{B}_2 = \begin{bmatrix} B_2 \\ B_K D_{2w} \end{bmatrix}$$

$$\bar{C}_1 = [C_1\ \ 0], \quad \bar{C}_2 = [C_2\ \ D_{2u} C_K]$$

The work reported in [2] is based on the following result.

Theorem 2.1

In the control system in Figure 1, the performance index satisfies $J \le \nu^2$ for all $\|\Delta\| < 1$, if there exist a positive definite matrix P and a matrix W such that

$$\text{trace}\ W < \nu^2$$

and

$$\begin{bmatrix} P\bar{A}^T + \bar{A}P & (*) & (*) & (*) \\ \bar{C}_2 P & -I & 0 & 0 \\ \bar{C}_1 P & 0 & -S & 0 \\ S\bar{B}_1^T & 0 & 0 & -S \end{bmatrix} < 0, \quad \begin{bmatrix} W & \bar{B}_2^T \\ \bar{B}_2 & P \end{bmatrix} > 0 \quad (6)$$

The variables in the constraint (6) are the controller $K(s)$ (the closed-loop matrices $(\bar{A}, \bar{B}_2, \bar{C}_2)$ depend on the controller), and a scaling matrix S. This constraint can be transformed into an LMI condition in the controller variables and used for controller synthesis, by employing a linearizing change of variables [1]. Unfortunately, this change of variables introduces a product term between controller variables and S, thus preventing the optimization over K and S simultaneously. In [2] the following algorithm was proposed to find a sub-optimal solution to the problem of minimizing the worst-case H_2 norm.

Algorithm 1 (K-S Iteration)

Take $S = S_0$ as start value, e.g. $S_0 = I$. Repeat the following two steps until no further reduction in the worst-case H_2 norm is observed.

K-step: Fix S and find the controller $K(s)$ that minimizes the worst-case H_2 norm.

S-step: Fix $K(s)$ and find the scaling matrix S that minimizes the worst-case H_2 norm.

Fixing either the controller or the scaling matrix during optimization makes it possible to solve the minimization problem in each step as an LMI problem; however this restriction is a source of conservatism in the design. It was nevertheless possible to demonstrate on a well-known benchmark problem that K-S iteration outperforms other robust design strategies [2]. The algorithm presented in the next section was motivated by the desire to reduce the conservatism of the K-S iteration, by allowing a simultaneous optimization over K and S.

3 K-S-Φ Iteration for Robust H_2 Control

The motivation for the K-S iteration technique was the fact that the linearizing change of variables that is required to convert the design problem into an LMI problem, introduces a term that is nonlinear in S and K. This term arises because of the presence of S in the same row (or column) in the first block matrix in (6) as the matrix variable P. To remove S from the first column, apply the congruence transformation diag(I, I, I, Ψ), where $\Psi = S^{-1}$, to obtain

$$\begin{bmatrix} P\bar{A}_0^T + \bar{A}_0 P & (*) & (*) & (*) \\ \bar{C}_2 P & -I & 0 & 0 \\ \bar{C}_1 P & 0 & -S & 0 \\ \bar{B}_1^T & 0 & 0 & -\Psi \end{bmatrix} < 0, \quad \begin{bmatrix} W & (*) \\ \bar{B}_2 & P \end{bmatrix} > 0 \tag{7}$$

The condition (7) is equivalent to (6) if we impose the additional constraint $\Psi S = I$. It is in fact straightforward to show that (7) will still guarantee the bound (5) on the worst-case H_2 norm when this constraint is relaxed to

$$\Psi S < I \tag{8}$$

After a linearizing change of variables, (7) can be solved as an LMI problem with variables S, Ψ and the transformed controller variables. The problem is now that the additional constraint (8) is nonlinear in S and Ψ. To replace (8) by a linear constraint, introduce the matrix

$$G = \begin{bmatrix} 0 & \Psi \\ S & 0 \end{bmatrix}$$

Note that if λ is an eigenvalues of G, then λ^2 is an eigenvalue of ΨS. Thus, the constraint (8) is equivalent to the constraint that G has its eigenvalues inside the unit disc, or that there exists a symmetric matrix Φ that satisfies

$$\begin{bmatrix} -\Phi & G^T\Phi \\ \Phi G & -\Phi \end{bmatrix} < 0, \qquad \Phi = \Phi^T > 0 \tag{9}$$

We can summarize the above as follows.

Theorem 3.1

$J < \nu^2$ for all $\Delta : \ \|\Delta\| < 1$ if there exist symmetric matrices $P > 0$, W, S and Ψ that satisfy trace $W < \nu^2$ and conditions (7) and (9).

Based on Theorem 3.1, a sub-optimal solution to the robust H_2 design problem can be obtained by an iterative algorithm involving three steps.

Algorithm 2 (K-S-Φ Iteration)

Take $S = S_0$ and $\Psi = (1-\epsilon)S_0^{-1}$, where S_0 is a suitable start value and $\epsilon > 0$ is a small constant (e.g. $\epsilon = 10^{-3}$). Compute Φ from (9), and repeat the following three steps until no further reduction in the worst-case H_2 norm is observed.

K-Step: Fix Φ and minimize ν over K, S, Ψ and P, subject to trace $W < \nu^2$ and conditions (7) and (9).

S-Step: Fix K and minimize ν over S and P subject to trace $W < \nu^2$ and (6).

Φ-Step: Fix S and $\Psi = (1-\epsilon)S^{-1}$ (this fixes G), and determine Φ from (9).

In the K-step, the search for the controller K is carried out with less conservatism than in Algorithm 1, because instead of fixing S, S and Ψ are variables constrained only by (9). The objective of this step is to find the best controller K. In order to be able to solve the minimization problem in the K-step as an LMI problem, a congruence transformation must be applied to the constraint (6). This transformation and the resulting LMI constraints are given in the Appendix.

The purpose of the S-step is then - with K fixed - to find the best scaling matrix; this step is the same as in Algorithm 1. With S obtained in the S-step, and Ψ fixed as the (slightly reduced for feasibility) inverse of S, in the Φ-step a matrix Φ is determined for the constraint (9) in the next K-step. To obtain a matrix Φ that allows maximum freedom when searching for K in the K-step, the following problem is solved

$$\max_{\Phi} \alpha \ \text{ subject to } \quad \alpha I + \begin{bmatrix} -\Phi & G^T\Phi \\ \Phi G & -\Phi \end{bmatrix} < 0, \ \Phi = \Phi^T > 0 \tag{10}$$

4 Application to the ACC Benchmark Problem

This problem was proposed as a benchmark problem for robust control at the American Control Conference [12]. Two bodies with masses m_1 and m_2 are connected by a spring with stiffness k, as shown in Fig. 2. A state space model of the system is

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_1 & 0 & 0 \\ k/m_2 & -k/m_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m_1 & 0 \\ 0 & 1/m_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix},$$

$$y = x_2 \tag{11}$$

We consider the following two problems.

Problem 1: Spring Constant k Uncertain

We use the following model to represent the parameter uncertainty

$$A = A_0 + \rho B_1 \Delta C_1 \quad \text{with} \quad -1 < \Delta < 1$$

where

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, B_1 = \rho \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, C_1^T = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

An additional tuning parameter ρ has been introduced that can be used to scale the uncertainty. This tuning parameter is absorbed into the model by replacing B_1 with ρB_1.

We choose the following values of LQG tuning parameters in (2)

$$Q = 100 \cdot I, \quad Q_e = 100 \cdot I, \quad R = 1, \quad R_e = 1$$

The aim is to push the tuning parameter ρ as high as possible to achieve the maximum possible degree of robustness against variation in the spring constant k. Note that Δ is just a scalar in this case, thus both Ψ and S are scalar. A comparison of the worst-case H_2 norm achieved with controllers obtained from Algorithm 1 and 2 are shown in Figure 3. The worst-case H_2 norm is plotted versus the scaling parameter S for different values of the tuning parameter ρ. The endpoints of the curves indicate the points where the LMI problem becomes infeasible.

Problem 2: Spring Constant k and Mass m_2 Uncertain

A systematic way of constructing the uncertainty representation in (1) was proposed in [11]. Here the model uncertainty is represented by

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1.25 & 1.25 & 0 & 0 \\ 2.12 & -2.12 & 0 & 0 \end{bmatrix}, B_1 = \rho \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.75 & 0 \\ 0 & 1.875 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad \Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$$

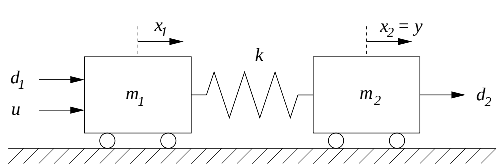
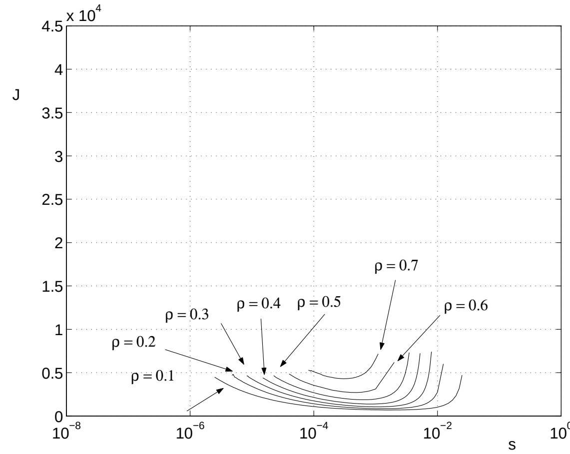
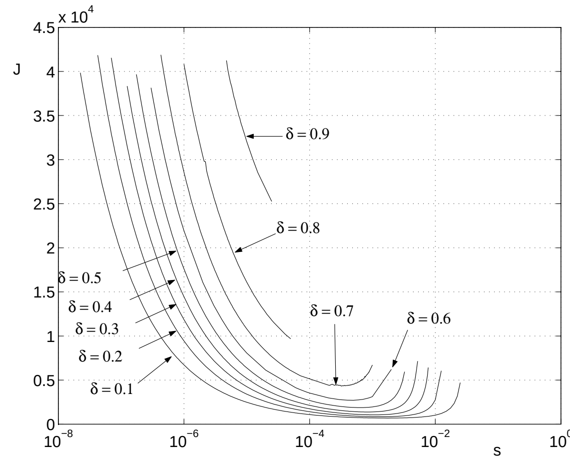


Figure 2: Two-mass-spring system



Algorithm 1



Algorithm 2

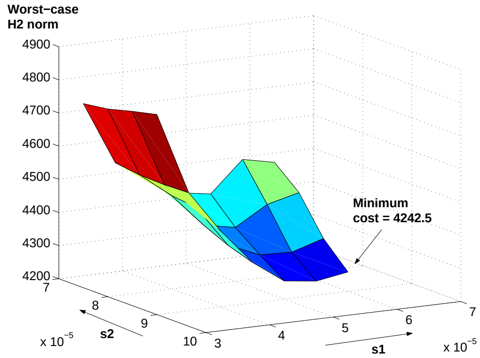
Figure 3: Comparison of the worst-case H_2 norm in the K-step achieved with Algorithm 1 and 2, for different values of ρ

A comparison between the controllers obtained from Algorithm 1 and 2 for different parameter values are shown in Figures 4 - 6. Note that in problem 2 we have two scaling parameters s_1 and s_2 - the non-zero diagonal entries of S. Again, the edges of the cost surfaces indicate the boundaries of feasibility.

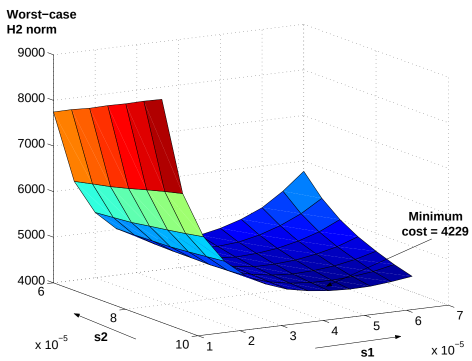
The following observations can be made:

- If a feasible solution exists, Algorithm 2 gives in controllers that are identical to those obtained from Algorithm 1, as is illustrated clearly in Figure 6. This is enforced by the constraint $\Psi S < I$; note that maximizing α in (10) pushes this towards $\Psi = S^{-1}$.

- Figure 3 shows that although the same worst-case cost is achieved with both approaches, at low values of ρ ($\rho \leq 0.7$), the range of feasible solutions is much larger when using Algorithm 2 - Algorithm 1 gives no solution for any scaling S less than $2 \cdot 10^{-6}$. Figure 4 confirms this observation for the case of a two-dimensional uncertainty.

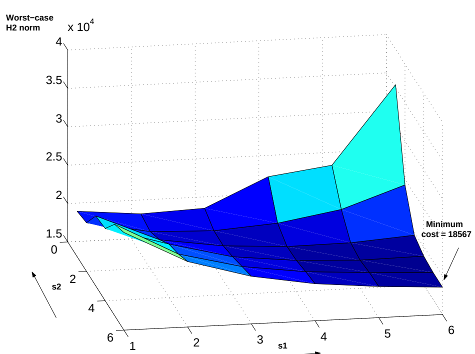


Algorithm 1, $\rho = 0.6$



Algorithm 2, $\rho = 0.6$

Figure 4: Worst-case H_2 norm vs. scaling parameters s_1, s_2 in problem 2

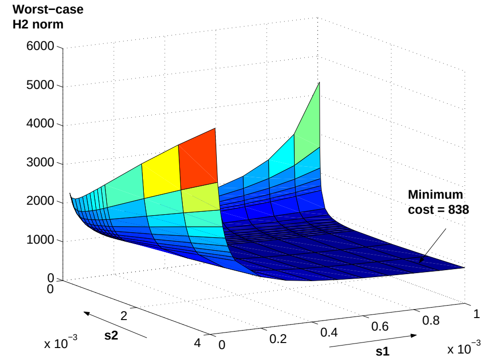


Algorithm 2, $\rho = 0.8$

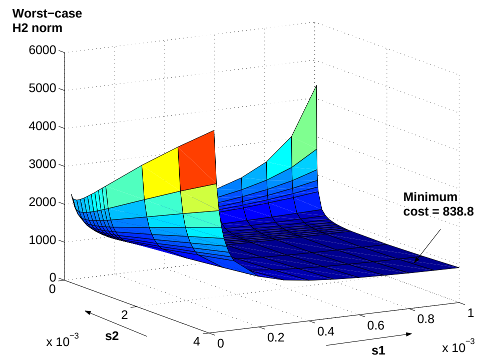
Figure 5: Worst-case H_2 norm versus scaling parameters s_1, s_2 in problem 2

- Algorithm 1 gives no feasible solution for $\rho > 0.7$, while Algorithm 2 still finds a solution when $\rho = 0.9$ for a scalar uncertainty. A similar observation can be made in the case of a two dimensional uncertainty: Algorithm 1 leads to infeasible problems if $\rho > 0.6$, while Algorithm 2 leads to infeasible problems when $\rho > 0.8$

- The larger range of feasible values of S achieved with Algorithm 2, makes it easier to find feasible starting values when searching for the minimum over S.



Algorithm 1



Algorithm 2

Figure 6: Worst-case H_2 norm versus scaling parameters s_1, s_2 at $\rho = 0.2$

A practical difficulty is to find a feasible start value S_0 for the K-step when searching for a controller that guarantees the worst-case H_2 bound for a large range of uncertain parameters. When ρ is large, the range of feasible values tends to be small. Here, the following strategy was used to find a feasible S_0 for large ρ: carry out K-S-Φ iteration with a smaller ρ and use the optimal scaling S as start value with an increased ρ.

5 Conclusion

An improved method for computing robust H_2 controllers has been proposed. The conservatism in the K-step of a previously proposed K-S iteration technique can be significantly reduced by introducing additional degrees of freedom. This reduction in conservatism does not involve much additional computational cost, because the additional Φ-step in the iteration procedure is relatively cheap.

There is still considerable conservatism in the S-step of this procedure, especially when the range of admissible parameter uncertainty is large. Further reduction of conservatism, for example by exploiting the non-uniqueness of the factorization in (17) below, and by employing parameter-dependent Lyapunov functions, are currently under investigation.

A Appendix

A congruence transformation proposed in [1] can be used to transform the constraints (7) into equivalent LMI constraints. Theorem 2.1 can then be reformulated as follows.

Theorem A.1

The cost in (3) satisfies

$$J \le \nu^2 \tag{12}$$

in all admissible operating conditions, i.e. for all $\|\Delta\| < 1$, if there exist matrices $X, Y, W, S, \Phi, \tilde{A}_K, \tilde{B}_K, \tilde{C}_K$ that satisfy

$$\text{trace}\, W < \nu^2 \tag{13}$$

$$\begin{bmatrix} \Omega(Y) & * & * & * & * \\ \tilde{A}_K^T + A_0 & \Omega(X) & * & * & * \\ C_2 & C_2X + D_{2u}\tilde{C}_K & -I & 0 & 0 \\ C_1 & C_1X & 0 & -S & 0 \\ B_1^TY & B_1^T & 0 & 0 & -\Psi \end{bmatrix} < 0 \tag{14}$$

$$\begin{bmatrix} W & B_2^TY + D_{2w}^T\tilde{B}_K^T & B_2^T \\ YB_2 + \tilde{B}_KD_{2w} & Y & I \\ B_2 & I & X \end{bmatrix} > 0 \tag{15}$$

$$\begin{bmatrix} -\Phi_{11} & * & * & * \\ -\Phi_{12} & -\Phi_{22} & * & * \\ \Phi_{12}S & \Phi_{11}\Psi & -\Phi_{11} & * \\ \Phi_{22}S & \Phi_{12}\Psi & -\Phi_{12} & -\Phi_{22} \end{bmatrix} < 0 \tag{16}$$

where

$$\Omega(Y) = YA_0 + \tilde{B}_KC_2 + (*)$$
$$\Omega(X) = A_0X + B\tilde{C}_K + (*)$$

Φ has been partitioned compatible with the partition of G.

Having found solutions $\tilde{A}_K,\ \tilde{B}_K,\ \tilde{C}_K,\ \ Y$ and X to the above LMI problem, a controller that satisfies the constraint (12) can be computed as follows. Compute via singular value decomposition nonsingular square matrices M and N that satisfy

$$MN^T = I - YX \tag{17}$$

then the controller (4) is given by

$$C_K = \tilde{C}_KM^{-T}$$
$$B_K = N^{-1}\tilde{B}_K$$
$$A_K = N^{-1}(\tilde{A}_K - NB_KCY - XBC_KM^T - XAY)M^{-1}$$

Algorithm 2 can be reformulated in terms of the transformed controller variables as

Algorithm 2 (K-S-Φ Iteration)

Take $S = S_0$ and $\Psi = (1-\epsilon)S_0$ as start values. Compute Φ from (10) and repeat the following three steps until no further reduction in the worst-case H_2 norm is observed.

K-Step: Fix Φ and minimize ν over $\tilde{A}_K,\ \tilde{B}_K,\ \tilde{C}_K,\ X,\ Y,$ S and Ψ, subject to (13) - (16)

S-Step: Fix $\tilde{A}_K,\ \tilde{B}_K,\ \tilde{C}_K$ and minimize ν over $S,\ X$ and Y subject to (13) - (15)

Φ-Step: Fix S and $\Psi = (1-\epsilon)S^{-1}$ and determine Φ by solving (10)

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